ON THE SPECTRAL RADIUS OF CACTUSES WITH PERFECT MATCHINGS

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Let $\mathcal{C}(2m, k)$ be the set of all cactuses on $2m$ vertices, k cycles, and with perfect matchings. In this paper, we identify in $\mathcal{C}(2m, k)$ the unique graph with the largest spectral radius.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$, and edge set $E(G)$. $A(G)$ is the adjacency matrix of G, while $P(G; \lambda) = \det(\lambda I - A(G))$ is the characteristic polynomial of G. Since $A(G)$ is symmetric, its eigenvalues (or roots of the characteristic polynomial) are real. They are also called the eigenvalues of G. The largest eigenvalue of G, denoted by $\rho(G)$, is called the *spectral radius* (or the *index*) of G. If G is connected, or equivalently, if $A(G)$ is irreducible, then $\rho(G)$ is a simple eigenvalue of G, i.e. its multiplicity is one. By the Perron-Frobenius theory of non-negative matrices there exists a unique positive unit eigenvector corresponding to $\rho(G)$, called the *Perron vector* of G. Further on we will use to suppress the graph names whenever being understood from the context.

Spectral radius of a graph is an important graph invariant, which recently gained much attention with researchers (see [**10**] in the context of various applications of the spectral graph theory). Most of the early results can be found in [**7**] (see also [**6**, **8**, **9**], and results therein). BRUALDI and HOFFMAN (see [**2**]) put forward the problem of identifying (in some classes of graphs) those graphs whose spectral radius is extremal (either maximal, or minimal). This problem turns to be too hard for some classes of graphs, especially in the minimization variant. In contrast, tools

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for solving maximization variant are much more developed (see also [**8**, **9**]). Here we will mention only a few relevant results, addressing trees and unicyclic graphs. Basic observations on connected graphs with small cyclomatic number are given in [**3**]; in particular, for trees, the readers are referred to [**16**]. For trees with some constraints on the size of matchings see [**4**, **11**, **13**, **15**, **17**, **18**]; the corresponding results for unicyclic can be found in [**5** ,**12**, **19**]. (Recall, two edges are said to be *independent* in a graph G if they have no common vertices. A *perfect matching* of G is a set of mutually independent edges which cover every vertex of G.)

In this paper we will consider cactuses. Recall, a *cactus* is a connected graph in which any two cycles have at most one common vertex. Clearly, any tree, or unicyclic graph, is a cactus as well. The maximization problem for cactuses with fixed number of cycles is resolved in [**1**]. Here we consider the same problem restricted to cactuses with perfect matchings.

The paper is organized as follows: in Section 2, we give some further definitions and basic tools; in Section 3, we give our main result, i.e. we identify (in the observed class), the unique cactus with the maximal spectral radius.

2. PRELIMINARIES

$G - v$ $(G - uv)$ denotes the subgraph of G obtained by deleting a vertex $v \in V(G)$ (resp. an edge $uv \in E(G)$). Similarly, $G + uv$ is a graph that arises from G by adding an edge $uv \in E(G)$, where $u, v \in V(G)$. This notation is naturally extended if more than one edge is deleted from, or added to G. Given a vertex $v \in V(G)$, $d(v)$ denotes its degree, and $N(v)$ denotes the set of all neighbors of v.

The following lemmas will be helpful in proving our main result.

Lemma 2.1 ([**9**, Theorem 8.1.5, p. 230]). *Let $G = (V, E)$ be a connected graph, and $\rho(G)$ its spectral radius. Suppose that $u, v \in V$ and $W \subseteq N(v) \setminus N(u)$ $(1 \leq |W| \leq d(v))$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of G, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G' be the graph obtained from G by deleting the edges vw, and adding the edges uw $(w \in W)$. If $x_u \geq x_v$ then $\rho(G) < \rho(G')$.*

Lemma 2.2 ([**14**; 9, Theorem 8.1.20, p. 238]). *Let $G(k, \ell)$ $(k, \ell \geq 0)$ be the graph obtained from a non-trivial connected graph G by attaching hanging paths of lengths k and l at the same vertex of G. If $k \geq \ell \geq 1$ then*

$$\rho(G(k, \ell)) > \rho(G(k + 1, \ell - 1)).$$

It is well-known that if G' is a proper spanning subgraph of a connected graph G, then $\rho(G) > \rho(G')$. Note that any proper subgraph of G, if not being a spanning subgraph of G, can be extended to a proper spanning subgraph of G by adding isolated vertices. So we have the following lemma (see, for example, [**6**, Theorem 0.6, p. 19]):

Lemma 2.3. *Let G be a connected graph, and let G' be a proper subgraph of G. Then $\rho(G) > \rho(G')$.*

Lemma 2.4 ([**14**; 9, Lemma 8.1.19, p. 237]). *If H is a proper spanning subgraph of the connected graph G, then*

$$P(H; \lambda) > P(G; \lambda)$$

for all $\lambda > \rho(G)$.

Let $L = K_{1,3} + e$. Denote by L_a (L_b) the rooted variant of L, where the root (i.e. distinguished vertex) is of degree 2 (resp. 3) in L. Let H ($= H_r$) be any non-trivial connected graph having r as its root. Let $G = H_r \cdot L_a$, while $G' = H_r \cdot L_b$, where $\cdot$ denote the *dot-product* (or *coalescence*) of two graphs (see, for example, [**6**, pp. 158–159]). Then we have:

Lemma 2.5. *Let G and G' be the graphs described as above. Then $\rho(G') > \rho(G)$.*

Proof. Let v' (v) be a pendant vertex of G' (resp. G) originating from the pendant vertex of L. Using Heilbronner's formula (see, [**6**, Theorem 2.11, p. 59]) at v' (resp. v), we easily obtain

$$P(G'; \lambda) - P(G; \lambda) = -[P(B; \lambda) - P(A; \lambda)],$$

where $B = (H - r) \cup K_2$ and $A = H \cdot K_2$. Since B is a spanning subgraph of A, $P(B; \lambda) > P(A; \lambda)$ for $\lambda > \rho(A)$ (by Lemma 2.4). So $P(G'; \lambda) - P(G; \lambda) < 0$ for $\lambda > \rho(A)$, and consequently for $\lambda > \rho(G')$ (by Lemma 2.3). Therefore, we have $\rho(G') > \rho(G)$, as required. □

3. MAIN RESULTS

Denote by $\mathcal{C}(2m, k)$ the set of all cactuses on $2m$ vertices and k cycles, having perfect matchings. If all cycles of some cactus have exactly one vertex in common, we say that they form a *bundle*, or that the graph itself is a *bundle*. The graph $G(m, k, i) \in \mathcal{C}(2m, k)$ is obtained as follows:

- its cycles form the bundle consisting of k triangles, v_0 being their common vertex;

- a pendant edge together with $m - k - i - 1$ hanging paths of length two are attached at v_0;

- all other pendant edges, $2i$ in total, are attached at $2i$ vertices of degree 2 of i ($0 \leq i \leq k$) triangles from the bundle.

So $G(m, k, i)$ is a bundle. In particular, for $m = 4$ and $k = 3$ the cactuses $G(8, 3, 0)$, $G(8, 3, 1)$, $G(8, 3, 2)$ and $G(8, 3, 3)$ are depicted in Fig. 1.
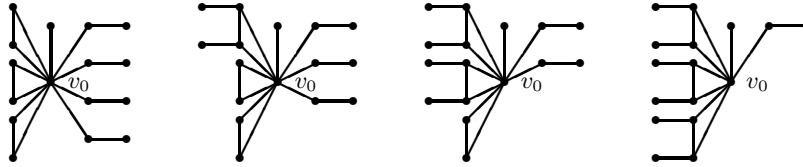

Fig. 1. Graphs $G(8, 3, 0)$, $G(8, 3, 1)$, $G(8, 3, 2)$, $G(8, 3, 3)$.

Our main result reads:

Theorem 3.1. *Let $G \in \mathcal{C}(2m, k)$. If $m \geq 4$ and $k \geq 2$ then*

$$\rho(G) \leq \rho(G(m, k, 0)),$$

with equality if and only if $G \cong G(m, k, 0)$.

To prove the theorem, we will prove six lemmas. In each of them, let $\hat{G} \in \mathcal{C}(2m, k)$ be a graph with maximal spectral radius. Let $V(\hat{G}) = \{v_1, v_2, \ldots, v_n\}$, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $\hat{G}$. The quantity x_i will be regarded as the *weight* of the vertex v_i. Finally, let M be a fixed perfect matching of G.

Lemma 3.1. *The length of any cycle in $\hat{G}$ is equal to* 3.

Proof. Assume, on the contrary, that there is a cycle C with length $p > 3$. For short, let $C = u_1 u_2 \cdots u_p u_1$. Without loss of generality, we may assume that u_1 is a vertex of C with minimal weight. Consider next the edges $u_p u_1$ and $u_2 u_1$ (of C). One of them, say $u_p u_1$, does not belong to M. Let $G = \hat{G} - u_p u_1 + u_p u_2$. Then $G \in \mathcal{C}(2m, k)$ (it is a cactus having M as a matching). By Lemma 2.1 we have $\rho(G) > \rho(\hat{G})$, a contradiction. $\square$

Lemma 3.2. *Any two cycles in $\hat{G}$ have one common vertex.*

Proof. Assume, on the contrary, that there are two disjoint cycles C_1 and C_2. Then, there exists a path $P = u_1 u_2 \cdots u_p$ of length $p - 1$ ($p \geq 2$) joining the cycles C_1 and C_2, where $u_1 \in V(C_1)$, $u_p \in V(C_2)$ and $u_i \notin (V(C_1) \cup V(C_2))$ for $i \neq 1, p$. By Lemma 3.1, C_1 (and C_2) are triangles. We write $C_1 = u_1 v w u_1$.

We now prove that neither $u_1 v$, nor $u_1 w$, is in M. Assume, on the contrary, that $u_1 v \in M$. If so, there is a vertex $w' \in V(\hat{G})$ such that $ww' \in M$.

(i) $x_{u_1} \geq x_w$. Let $G = \hat{G} - \{ww'\} + \{u_1 w'\}$. Note that $M' = M - \{u_1 v, ww'\} + \{vw, u_1 w'\}$ is a perfect matching in G.

(ii) $x_{u_1} < x_w$. Let $G = \hat{G} - \{u_1 y \mid y \in N(u_1) \setminus \{v, w\}\} + \{wy \mid y \in N(u_1) \setminus \{v, w\}\}$. Now M is a perfect matching also in G.

Then, in either case, $G \in \mathcal{C}(2m, k)$ and by Lemma 2.1 we have $\rho(G) > \rho(\hat{G})$, a contradiction. So, $u_1 v \notin M$, and similarly, $u_1 w \notin M$.

Without loss of generality, assume next that $x_{u_p} \geq x_{u_1}$. Let $G = \hat{G} - \{u_1 v, u_1 w\} + \{u_p v, u_p w\}$. Then $G \in \mathcal{C}(2m, k)$, since G is a cactus having M as a matching. By Lemma 2.1 we have $\rho(G) > \rho(\hat{G})$, a contradiction. $\square$

Lemma 3.3. *Any three cycles in G have exactly one common vertex.*

Proof. Assume, on the contrary, that cycles C_1, C_2 and C_3 have no common vertices. Having in mind Lemma 3.2, let u, v, w be the common vertices of C_1 and C_2, C_2 and C_3, C_1 and C_3, respectively. But then the shortest paths between u and v, v and w, w and u form a cycle which has at least one edge in common with any of observed cycles, a contradiction. $\square$

By Lemmas 3.1–3.3, all cycles of G are triangles, and they have exactly one vertex in common, i.e. they form a bundle having v_0 as the common vertex. We next consider hanging trees attached at vertices belonging to cycles.

Lemma 3.4. *Any tree T attached at vertex u of some cycles (belonging to $\hat{G}$) consists of hanging paths of lengths 2 and possibly one pendant edge, all of them attached at u. In addition, any two hanging paths of length 2 are attached at the same vertex.*

Proof. Let T be a tree attached at the vertex u. Assume that v is a vertex of T of degree at least 3 being at the largest distance from u. Then all hanging paths at v, but possibly one, have an even length (otherwise, $\hat{G}$ has no perfect matchings). Then we can substitute any hanging path of length ℓ (≥ 3), by two hanging paths attached also at v whose lengths are 2 and $\ell - 2$, to get a graph $G \in \mathcal{C}(2m, k)$. But then $\rho(G) > \rho(\hat{G})$, a contradiction (by Lemma 2.2). Therefore, we can assume that all paths but possibly one are of length 2, while the remaining (if any) of length 1. Suppose now that the distance between u and v is greater than 1. If $x_u \geq x_v$, let vv' be an edge of one of the aforementioned hanging paths of length 2, and let $G = \hat{G} - v'v + v'u$. Otherwise, if $x_u < x_v$, let uu' be an edge of the cycle which does not belong to M, and let $G = \hat{G} - u'u + u'v$. Clearly, in both cases $G \in \mathcal{C}(2m, k)$, and by Lemma 2.1, $\rho(G) > \rho(\hat{G})$, a contradiction. So $v = u$, and the proof of the first claim follows. To prove the second claim, assume on the contrary, that vertices u_1 and u_2, in the role of u, have attached hanging paths $P_1 = u_1v_1w_1$ and $P_2 = u_2v_2w_2$, respectively. Then u_1v_1 and u_2v_2 are not in M. Without loss of generality, let $x_{u_1} \leq x_{u_2}$ and let $G = \hat{G} - v_1u_1 + v_1u_2$. Then $G \in \mathcal{C}(2m, k)$, and by Lemma 2.1, $\rho(G) > \rho(\hat{G})$, a contradiction. This completes the proof. $\quad\square$

Let $T_i = v_0v_iw_iv_0$ ($i = 1, 2, \ldots, k$) be a triangle in $\hat{G}$. Recall first that for each i at most one edge is attached at v_i, or w_i. We classify the T_i's as follows:

type-1: v_i and w_i have no pendant edges attached;

type-2: both, v_i and w_i, have just one pendant edge attached;

type-3: either v_i, or w_i, has just one pendant edge attached.

Lemma 3.5. *Any hanging path of length 2 (in $\hat{G}$) is attached to v_0. Moreover, any triangle T_i is of type-1 or type-2.*

Proof. Assume, on the contrary, that there exists a path $P = v_1vw$ attached at vertex v_1 of T_1. Then $vw \in M$, and $v_1v \notin M$. Consider next a triangle T_2 (recall $k \geq 2$). If $x_{v_0} \geq x_{v_1}$, let $G = \hat{G} - vv_1 + vv_0$. Then $G \in \mathcal{C}(2m, k)$, and we are done (by Lemma 2.1). So let $x_{v_0} < x_{v_1}$. If both edges v_0v_2 and v_0w_2 do not belong to M, consider $G = \hat{G} - \{v_2v_0, w_2v_0\} + \{v_2v_1, w_2v_1\}$. But then $G \in \mathcal{C}(2m, k)$, and by Lemma 2.1, $\rho(G) > \rho(\hat{G})$, a contradiction. So let, say $v_0w_2 \in M$. But then T_2 must be of type-3 (w_2 has no pendant edges attached, while v_2 must have a pendant edge attached). Having in mind Lemma 3.4, we easily deduce that $\hat{G} = H \cdot L_a$ (in the notation used in Lemma 2.5). But this is a contradiction (due to Lemma 2.5). So the first claim is proved. To prove the second claim, we only need to reproduce the last arguments from the proof of the first claim. This completes the proof. $\quad\square$

By Lemmas 3.1–3.6 it follows that $\hat{G}$ must be one of the graphs $G(m, k, i)$, for some i. To resolve this situation, we prove the following lemma:

Lemma 3.6. *Let $m \geq 3$ and $k \geq 2$. Then*

$$\rho(G(m, k, i+1)) < \rho(G(m, k, i)),$$

where $i = 0, 1, \ldots, k-1$.

Proof. Let $H = G(m, k, i+1)$ and $H' = G(m, k, i)$. Note that H' can be obtained from H as follows: two hanging edges attached at the triangle T_{i+1} of type-2 are deleted (say sp and tq, where for short, $s = v_{i+1}$ and $t = w_{i+1}$ are the vertices of T_{i+1}), and a hanging path of length 2 is attached at v_0 (by inserting edges $v_0 p$ and pq).

Recall next that

$$R(\mathbf{z}; K) = \frac{\mathbf{z}^T A(K) \mathbf{z}}{\mathbf{z}^{\mathbf{T}} \mathbf{z}} \quad (\mathbf{z} \in I\!\!R^n \setminus \{\mathbf{0}\})$$

is the Rayleigh's quotient for a graph K. Then $\rho(K) \leq R(\mathbf{z}; K)$, where $\mathbf{z}$ runs over the sphere $||\mathbf{z}|| = 1$; the equality holds if and only if $\mathbf{z}$ is an eigenvector of K (see, for example, [**9**, pp. 11–12]).

Next, assume that $K = H$, and that $\mathbf{x} = (x_1, \ldots, x_n)^T$ is the Perron vector of H. Bearing in mind the above facts we have:

$$\rho(H') - \rho(H) > R(\mathbf{x}; H') - R(\mathbf{x}; H).$$

By symmetry, and for simplicity, let $x_s = x_t = a$, $x_p = x_q = b$ and $x_{v_0} = c$. Therefore we have:

$$R(\mathbf{x}; H') - R(\mathbf{x}; H) = 2[(b^2 + bc) - 2ab].$$

The latter expression is non-negative if $b + c \geq 2a$. Since $\rho(H)a = a + b + c$ (the eigenvalue equation at s or t in H), we are done provided $\rho(H) \geq 3$.

Consider now any graph $G \in \mathcal{C}(2m, k)$ with $m \geq 3$ and $k \geq 2$. It is easy to see that $G(6, 2, 2)$ is its induced subgraph. By direct calculations, we obtain $\rho(G(6, 2, 2)) \geq 3.03840$. So, by Lemma 2.3, $\rho(G(m, k, i)) > 3$, as required. Therefore, the proof follows.   □

REMARK 3.1. When $m \leq 3$, we can use newGRAPH[1] to show that $G(m, k, 0)$ is the unique graph with the largest spectral radius in $\mathcal{C}(2m, k)$, except for $m = 3$ and $k = 1$.

Proof of Theorem 3.1. So far, by Lemmas 3.1-3.5, we have proved that $\hat{G} = G(m, k, i)$ for some i. Next, by Lemma 3.6, $G = G(2m, k, 0)$, if $k \geq 2$. This completes the proof.   □

For $k < 2$ the the corresponding classes $\mathcal{C}(2m, k)$ are the sets of all trees ($k = 0$) and unicyclic graphs ($k = 1$). This cases are covered by the following results from the literature:

Theorem 3.2 ([**4**, **18**]). *Let T be a tree in $\mathcal{C}(2m, 0)$. Then $\rho(T) \leq \rho(G(m, 0, 0))$ and the equality holds if and only if $T \cong G(m, 0, 0)$.*

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[1]newGRAPH is available at the following address: http://www.mi.sanu.ac.rs/newgraph/

Theorem 3.3 ([**5**]). *Let U be a unicyclic graph in $\mathcal{C}(2m, 1)$, with $m \geq 4$. Then $\rho(U) \leq \rho(G(m, 1, 0))$ and the equality holds if and only if $U \cong G(m, 1, 0)$.*

Finally, observe that $G(m, k-1, 0)$ is a proper subgraph of $G(m, k, 0)$. By Lemma 2.3, we have $\rho(G(m, 0, 0)) < \rho(G(m, 1, 0)) < \cdots < \rho(G(m, m-1, 0))$. So, we also have:

Theorem 3.4. *The graph $G(2m, m-1, 0)$ has maximal spectral radius in the set of all cactuses on $2m$ vertices having perfect matchings.*

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