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ON FUNCTIONS OF ARAKAWA AND KANEKO AND MULTIPLE ZETA VALUES

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We study two functions $\xi_k(s)$ and $\xi_{k_1,\dots,k_r}(s)$ introduced by Arakawa and Kaneko [Nagoya Math. J., **153** (1999), 189–209] and relate them with (finite) multiple zeta values and multiple zeta star values using elementary methods. In particular, we give an alternative proof of a result of Ohno [Y. Ohno: J. Number Theory, **74** (1999), 39–43].

1. INTRODUCTION

Let $\mathrm{Li}_{k_1,\dots,k_r}(z)$ denote the multiple polylogarithm function defined by

$$\mathrm{Li}_{k_1,\dots,k_r}(z) = \sum_{n_1>n_2>\cdots>n_r\ge 1} \frac{z^{n_1}}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}},$$

with $k_1 \in \mathbb{N}\setminus\{1\}$ and $k_i \in \mathbb{N} = \{1,2,\dots\}$, $2\le i\le r$, and $|z|\le 1$. For $z=1$ the multiple polylogarithm function $\mathrm{Li}_{k_1,\dots,k_r}(1) = \zeta(k_1,\dots,k_r)$ simplifies to a multiple zeta value, sometimes also called multiple zeta function, where $\zeta(k_1,\dots,k_r)$ and $\zeta_N(k_1,\dots,k_r)$ denote the (finite) multiple zeta value defined by

$$\zeta(k_1,\dots,k_r) = \sum_{n_1>n_2>\cdots>n_r\ge 1} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}},$$

$$\zeta_N(k_1,\dots,k_r) = \sum_{N\ge n_1>n_2>\cdots>n_\ell\ge 1} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}},$$

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with $k_1 \in \mathbb{N} \setminus \{1\}$, and $k_2, \dots, k_r \in \mathbb{N}$ for the infinite series and $N, k_1, \dots, k_r \in \mathbb{N}$ for the finite counterpart. Arakawa and Kaneko [**3**] introduced and studied the functions $\xi_k(s)$ and $\xi_{k_1,\dots,k_r}(s)$, defined by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \operatorname{Li}_k(1 - e^{-t}) \mathrm{d}t,$$
$$\xi_{k_1,\dots,k_r}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} \operatorname{Li}_{k_1,\dots,k_r}(1 - e^{-t}) \mathrm{d}t,$$

respectively, being absolutely convergent for $\Re(s) > 0$. Arakawa and Kaneko related $\xi_k(s)$ and $\xi_{k_1,\dots,k_r}(s)$ for several choices of $s \in \mathbb{C}$ and $k_1, \dots k_r \in \mathbb{N}$ to multiple zeta values. Ohno [**14**] applied his generalization of the duality and sum formulas for multiple zeta values to the result obtained by Arakawa and Taneko for $\xi_k(n)$ in order to express $\xi_k(n)$ for positive integers n in terms of so-called multiple zeta star values or non-strict multiple zeta values. Arakawa and Taneko [**3**] posed several questions concerning the function $\xi_{k_1,\dots,k_r}(s)$. In particular, they asked for evaluations of $\xi_{k_1,\dots,k_r}(s)$ to multiple zeta values, for $k_1, \dots, k_r \in \mathbb{N}$ and $s \in \mathbb{C}$. We answer this question for arbitrary $k_1, \dots, k_r \in \mathbb{N}$ and positive integers n by providing evaluations of the function $\xi_{k_1,\dots,k_r}(n)$ to multiple zeta (star) values. In particular, we reobtain Ohno's result for $\xi_k(n)$, giving an alternative short and self-contained proof.

For the evaluation of the general case $\xi_{k_1,\dots,k_r}(n)$ we use a finite version (see for example [**9**]) of the well known stuffle identity for multiple zeta values [**7**]. Subsequently, we will use a variant of (finite) multiple zeta values, called multiple zeta star values or non-strict multiple zeta values $\zeta_N^*(k_1, \dots, k_r)$, which recently attracted some interest, [**2**, **14**, **16**, **15**, **11**, **13**, **10**, **17**] where the summation indices satisfy $N \ge n_1 \ge n_2 \ge \dots \ge n_r \ge 1$ in contrast to $N \ge n_1 > n_2 > \dots > n_r > 1$, as in the usual definition (1),

$$\zeta_N^*(k_1, \dots, k_r) = \sum_{N \ge n_1 \ge n_2 \ge \dots \ge n_r \ge 1} \frac{1}{n_1^{k_1} n_2^{k_2} \dots n_r^{k_r}},$$

with $N, k_1, \dots, k_r \in \mathbb{N}$. Note that multiple zeta star values are frequently used in particle physics and are called harmonics sums in this context. Various properties and algorithmic aspects of multiple zeta star values have been considered in [**4**, **6**]. The multiple zeta star value can be converted into ordinary finite multiple zeta values by considering all possible deletions of commas, e.g.

$$(1)\quad \zeta_N^*(k_1, \dots, k_r) = \sum_{h=1}^{r} \sum_{1 \le \ell_1 < \ell_2 < \dots < \ell_{h-1} < r} \zeta_N\Bigg(\sum_{i_1=1}^{\ell_1} k_{i_1}, \sum_{i_2=\ell_1+1}^{\ell_2} k_{i_2}, \dots, \sum_{i_h=\ell_{h-1}+1}^{r} k_{i_h} \Bigg);$$

note that the first term $h = 1$ should be interpreted as $\zeta_N\Big(\sum_{i_1=\ell_0+1}^{r} k_{i_1} \Big)$, subject to $\ell_0 = 0$. The notation $\zeta_N^*(k_1, \dots, k_r)$ is chosen in analogy with Aoki and Ohno [**2**] where infinite counterparts of $\zeta_N^*(k_1,\dots,k_r)$ have been treated. First we will study the instructive case of $\xi_k(n)$, reproving the result of Ohno. The main results of this work concerning the evaluation of $\xi_{k_1,\dots,k_r}(n)$ into multiple zeta (star) values will be stated in Theorems 2, 3, 4.

2. AN ALTERNATIVE PROOF OF OHNO'S EVALUATION

Ohno evaluated the sum $\xi_k(n)$ for $k,n\in\mathbb{N}$ by an application of his generalization of the duality and sum formulas for multiple zeta values to a result of Arakawa and Kaneko [**3**]. He obtained the following result.

Theorem 1. (Ohno [**14**]) *The function $\xi_k(n)$ is for arbitrary $k,n\in\mathbb{N}$ given by*

$$\xi_k(n)=\sum_{m_1\ge m_2\ge\cdots\ge m_n\ge 1}\frac{1}{m_1^{k+1}m_2\dots m_n}=\zeta^*(k+1,\{1\}_{n-1}).$$

Note that one can convert the multiple star zeta value above into ordinary multiple zeta values according to (1) (with respect to the corresponding relation for infinite series), or can directly simplify the multiple zeta star value using (cycle) sum formulas, see e.g. Ohno and Wakabayashi [**16**] or Ohno and Okuda [**15**].

In the following we will give a short alternative and self-contained proof of Theorem 1. In order to evaluate $\xi_k(n)$ for $k,n\in\mathbb{N}$ we only use the two basic facts stated below.

$$\begin{aligned}&\frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-t\ell}\mathrm{d}t=\frac{1}{\ell^n},\quad\text{for}\quad \ell,n\in\mathbb{N},\\ &\sum_{\ell=1}^{n}\binom{n}{\ell}\frac{(-1)^{\ell-1}}{\ell^r}=\zeta_n^*(\underbrace{1,\dots,1}_{r})=\zeta_n^*(\{1\}_r)\quad\text{for}\quad r\in\mathbb{N}.\end{aligned}\tag{2}$$

The second identity can be immediately deduced by repeated usage of the formula $\binom{n}{k}=\sum_{\ell=k}^{n}\binom{\ell-1}{k-1}$. We proceed as follows.

$$\xi_k(n)=\frac{1}{\Gamma(n)}\int_0^\infty\frac{t^{n-1}}{e^t-1}\,\mathrm{Li}_k(1-e^{-t})\mathrm{d}t=\frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-t}\sum_{m\ge1}\frac{(1-e^{-t})^{m-1}}{m^k}\mathrm{d}t.$$

We expand $(1-e^{-t})^{m-1}$ by the binomial theorem and interchange summation and integration. According to (2) we obtain

$$\begin{aligned}\xi_k(n)&=\sum_{m\ge1}\frac{1}{m^k}\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}\frac{(-1)^\ell}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-(\ell+1)t}\mathrm{d}t\\&=\sum_{m\ge1}\frac{1}{m^k}\sum_{\ell=0}^{m-1}\binom{m-1}{\ell}\frac{(-1)^\ell}{(\ell+1)^n}.\end{aligned}$$

Since $\binom{m-1}{\ell} = \binom{m}{\ell+1}\frac{\ell+1}{m}$, we get according to (2) after an index shift Theorem 1.

3. THE GENERAL CASE

We obtain the following result for $\xi_{k_1,\dots,k_r}(n)$, generalizing Theorem 1.

Theorem 2. *For arbitrary $k_1,\dots,k_r,n \in \mathbb{N}$ the function $\xi_{k_1,\dots,k_r}(n)$ is given by*

$$\xi_{k_1,\dots,k_r}(n) = \sum_{n_1\geq 1} \frac{\zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\dots,k_r)}{n_1^{k_1+1}}.$$

Proof. By definition we have

$$\begin{aligned}\xi_{k_1,\dots,k_r}(n) &= \frac{1}{\Gamma(n)}\int_0^\infty \frac{t^{n-1}}{e^t-1}\,\mathrm{Li}_{k_1,\dots,k_r}(1-e^{-t})\mathrm{d}t\\ &= \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-t}\sum_{n_1>n_2>\dots>n_r\geq 1}\frac{(1-e^{-t})^{n_1-1}}{n_1^{k_1}n_2^{k_2}\dots n_r^{k_r}}\mathrm{d}t.\end{aligned}$$

Proceeding as before we expand $(1-e^{-t})^{n_1-1}$ by the binomial theorem and interchange summation and integration. We get

$$\xi_{k_1,\dots,k_r}(n) = \sum_{n_1>n_2>\dots>n_r\geq 1}\frac{1}{n_1^{k_1}n_2^{k_2}\dots n_r^{k_r}}\sum_{\ell=0}^{n_1-1}\frac{(-1)^\ell\binom{n_1-1}{\ell}}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-(\ell+1)t}\mathrm{d}t.$$

According to (2) we obtain

$$\xi_{k_1,\dots,k_r}(n) = \sum_{n_1>n_2>\dots>n_r\geq 1}\frac{\zeta^*_{n_1}(\{1\}_{n-1})}{n_1^{k_1+1}n_2^{k_2}\dots n_r^{k_r}} = \sum_{n_1\geq 1}\frac{\zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\dots,k_r)}{n_1^{k_1+1}}. \quad \square$$

Next we will evaluate $\xi_{k_1,\dots,k_r}(n)$ into multiple zeta values. In order to do so, we are going to evaluate the product $S = S_{n_1}(n,k_2,\dots,k_2)$ of finite multiple zeta (star) values, defined by

$$S = S_{n_1}(n,k_2,\dots,k_2) = \zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\dots,k_r)$$

into sums of finite multiple zeta values of the forms $\zeta_{n_1-1}(\mathbf{f})$, and $\frac{1}{n_1^\ell}\zeta_{n_1-1}(\mathbf{f})$, for some $1\leq \ell\leq n-1$ and $\mathbf{f}=(f_1,\dots,f_j)$, with $f_i\in\mathbb{N}$, $1\leq i\leq j$ and $1\leq j\leq n+r-2$. By (1) we can write $\zeta^*_{n_1}(\{1\}_{n-1})$ in terms of ordinary finite multiple zeta values

$$\zeta^*_{n_1}(\{1\}_{n-1}) = \sum_{h=1}^{n-1}\sum_{1\leq\ell_1<\ell_2<\dots<\ell_{h-1}<n-1}\zeta_{n_1}\big(\ell_1,\ell_2-\ell_1,\dots,n-\ell_{h-1}-1\big),$$

for example $\zeta^*_{n_1}(\{1\}_3) = \zeta_{n_1}(3)+\zeta_{n_1}(1,2)+\zeta_{n_1}(2,1)+\zeta_{n_1}(1,1,1)$. We can convert finite multiple zeta values $\zeta_N(a_1,\dots,a_r)$ into finite multiple zeta values $\zeta_{N-1}(b_1,\dots,b_s)$ by

$$\zeta_N(a_1,\dots,a_r) = \zeta_{N-1}(a_1,\dots,a_r) + \frac{1}{N^{a_1}}\zeta_{N-1}(a_2,\dots,a_r).$$

Consequently, we can express the product $S = \zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\dots,k_r)$ of finite multiple zeta (star) values in the following way.

$$\begin{aligned} S = &\sum_{h=1}^{n-1} \sum_{1\le \ell_1<\ell_2<\dots<\ell_{h-1}<n-1} \zeta_{n_1-1}\big(\ell_1,\ell_2-\ell_1,\dots,n-\ell_{h-1}-1\big)\zeta_{n_1-1}(k_2,\dots,k_r) \\ &+\sum_{h=1}^{n-1} \sum_{1\le \ell_1<\ell_2<\dots<\ell_{h-1}<n-1} \frac{\zeta_{n_1-1}\big(\ell_2,\dots,n-\ell_{h-1}-1\big)\zeta_{n_1-1}(k_2,\dots,k_r)}{n_1^{\ell_1}}. \end{aligned} \tag{3}$$

In order to further simplify S we will use finite versions of the stuffle identities, see e.g. Borwein et al. [**7**] and Costermans et al. [**9**]. In general, stuffle identities provide evaluations of products of multiple zeta values $\zeta(\mathbf{k})\zeta(\mathbf{h})$ into sums of multiple zeta values $\zeta(\mathbf{k})\zeta(\mathbf{h}) = \sum_{\mathbf{f}\in\text{stuffle}(\mathbf{k},\mathbf{h})} \zeta(\mathbf{f})$; here $\zeta(\mathbf{k}) = \zeta(k_1,\dots,k_r)$ and $\zeta(\mathbf{h}) = \zeta(h_1,\dots,h_s)$. For the simplification of S we use finite versions of the stuffle identities, providing evaluations of products of finite multiple zeta values $\zeta_N(\mathbf{k})\zeta_N(\mathbf{h})$ into sums of finite multiple zeta values $\zeta_N(\mathbf{k})\zeta_N(\mathbf{h}) = \sum_{\mathbf{f}\in\text{stuffle}(\mathbf{k},\mathbf{h})} \zeta_N(\mathbf{f})$.

Following [**7**] we define for two given strings $\mathbf{k} = (k_1,\dots,k_r)$ and $\mathbf{h} = (h_1,\dots,h_s)$ the set stuffle$(\mathbf{k},\mathbf{h})$ as the smallest set of strings over the alphabet $\mathcal{A}$, defined by

$$\mathcal{A} = \{k_1,\dots,k_r,h_1,\dots,h_s, \text{“}+\text{''}, \text{“},\text{''}, \text{“}(\text{''}, \text{“})\text{''}\}$$

satisfying $(\mathbf{k},\mathbf{h}) = (k_1,\dots,k_r,h_1,\dots,h_s) \in$ stuffle$(\mathbf{k},\mathbf{h})$ and further if a string of the form $(U,k_n,h_m,V) \in$ stuffle$(\mathbf{k},\mathbf{h})$, then so are the strings $(U,h_m,k_n,V) \in$ stuffle$(\mathbf{k},\mathbf{h})$ and $(U,k_n+h_m,V) \in$ stuffle$(\mathbf{k},\mathbf{h})$. Stuffle identities arise from the definition of (finite) multiple zeta values in terms of sums; the term *stuffle* derives from the manner in which the two upper strings are combined. Other closely related identities are due to different representations of multiple zeta values (see for example [**7**]). We will use the following result of Costermans et al. [**9**].

Lemma 1. (Stuffle identity; finite version [**9**]) *Let* $\zeta_N(\mathbf{k}) = \zeta_N(k_1,\dots,k_r)$ *and* $\zeta_N(\mathbf{h}) = \zeta_N(h_1,\dots,h_s)$, *with* $N,r,s\in\mathbb{N}$ *and* $k_i,h_j\in\mathbb{N}$, $1\le i\le r$, $1\le j\le s$. *Then,*

$$\zeta_N(\mathbf{k})\zeta_N(\mathbf{h}) = \sum_{\mathbf{f}\in\text{stuffle}(\mathbf{k},\mathbf{h})} \zeta_N(\mathbf{f}). \tag{4}$$

Remark 1. A few examples of finite stuffle identities are given below, assuming that

$r, s, t, N \in \mathbb{N}$.

$$\zeta_N(r)\zeta_N(t) = \zeta_N(r,t) + \zeta_N(t+r) + \zeta_N(t,r),$$
$$\zeta_N(r,s)\zeta_N(t) = \zeta_N(r,s,t) + \zeta_N(r,s+t) + \zeta_N(r,t,s) + \zeta_N(r+t,s) + \zeta_N(t,r,s).$$

Remark 2. The finite stuffle identity can be shown using algebraic methods, see Costermans et al. [**9**]; we also refer to the work of Hoffman [**12**]. Alternatively, an elementary proof can be carried out using induction with respect to the total length $|\mathbf{k}| + |\mathbf{h}|$ of $\zeta_N(\mathbf{k})$ and $\zeta_N(\mathbf{h})$. The stuffle identity and the stuffle algebra was studied algorithmically in [**6**].

Next we provide the evaluation of S into sums of finite multiple zeta values.

Lemma 2. *The product $S = S_{n_1}(n, k_2, \dots, k_2) = \zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2, \dots, k_r)$ of multiple zeta (star) values can be evaluated into sums of finite multiple zeta values:*

$$S = \sum_{h=1}^{n-1} \sum_{1\le \ell_1<\ell_2<\dots<\ell_{h-1}<n-1} \sum_{\mathbf{f}\in\text{stuffle}(\ell_n^{[1]},\mathbf{k})} \zeta_{n_1-1}(\mathbf{f})$$
$$+ \sum_{h=1}^{n-1} \sum_{1\le \ell_1<\ell_2<\dots<\ell_{h-1}<n-1} \frac{1}{n_1^{\ell_1}} \sum_{\mathbf{f}\in\text{stuffle}(\ell_n^{[2]},\mathbf{k})} \zeta_{n_1-1}(\mathbf{f}),$$

with $\ell_n^{[1]} = (\ell_1, \ell_2 - \ell_1, \dots, n - \ell_{h-1} - 1)$, $\ell_n^{[2]} = (\ell_2, \ell_3 - \ell_2, \dots, n - \ell_{h-1} - 1)$, and $\mathbf{k} = (k_2, \dots, k_r)$.

Proof. For the simplification of S we apply the stuffle identity of Lemma 1 to all values of the form $\zeta_{n_1-1}(\ell_n^{[1]})\zeta_{n_1-1}(\mathbf{k})$ and $\zeta_{n_1-1}(\ell_n^{[2]})\zeta_{n_1-1}(\mathbf{k})$, as occurring in the representation of S in (3), and obtain the stated result. □

Now we can state the explicit evaluation of $\xi_{k_1,\dots,k_r}(n)$.

Theorem 3. *For $k_1, \dots, k_r, n \in \mathbb{N}$ the function $\xi_{k_1,\dots,k_r}(n)$ can be evaluated into sums of multiple zeta values,*

$$\xi_{k_1,\dots,k_r}(n) = \sum_{h=1}^{n-1} \sum_{1\le \ell_1<\ell_2<\dots<\ell_{h-1}<n-1} \sum_{\mathbf{f}\in\text{stuffle}(\ell_n^{[1]},\mathbf{k})} \zeta(k_1+1,\mathbf{f})$$
$$+ \sum_{h=1}^{n-1} \sum_{1\le \ell_1<\ell_2<\dots<\ell_{h-1}<n-1} \sum_{\mathbf{f}\in\text{stuffle}(\ell_n^{[2]},\mathbf{k})} \zeta(k_1+1+\ell_1,\mathbf{f}),$$

with $\ell_n^{[1]} = (\ell_1, \ell_2 - \ell_1, \dots, n - \ell_{h-1} - 1)$, $\ell_n^{[2]} = (\ell_2, \ell_3 - \ell_2, \dots, n - \ell_{h-1} - 1)$, and $\mathbf{k} = (k_2, \dots, k_r)$.

Proof. By Theorem 2 we have

$$\xi_{k_1,\dots,k_r}(n) = \sum_{n_1\ge 1} \frac{\zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\dots,k_r)}{n_1^{k_1+1}} = \sum_{n_1\ge 1} \frac{S_{n_1}(n,k_2,\dots,k_r)}{n_1^{k_1+1}}.$$

By Lemma 2 we obtain further

$$\xi_{k_1,\ldots,k_r}(n) = \sum_{n_1\geq 1} \frac{1}{n_1^{k_1+1}} \left(\sum_{h=1}^{n-1} \sum_{1\leq \ell_1<\ell_2<\cdots<\ell_{h-1}<n-1} \sum_{\mathbf{f}\in \mathrm{stuffle}(\ell_n^{[1]},\mathbf{k})} \zeta_{n_1-1}(\mathbf{f}) \right. \\ \left. + \sum_{h=1}^{n-1} \sum_{1\leq \ell_1<\ell_2<\cdots<\ell_{h-1}<n-1} \frac{1}{n_1^{\ell_1}} \sum_{\mathbf{f}\in \mathrm{stuffle}(\ell_n^{[2]},\mathbf{k})} \zeta_{n_1-1}(\mathbf{f}) \right).$$

Interchanging summations leads to the stated result. □

Remark 3. Although the result of Theorem 3 is very involved, in can be used to obtain simple expressions for $\xi_{k_1,\ldots,k_r}(n)$ for small n and r:

$$\xi_{k_1,k_2}(2) = \zeta(k_1+2,k_2) + \zeta(k_1+1,1,k_2) + \zeta(k_1+1,k_2+1) + \zeta(k_1+1,k_2,1),$$
$$\xi_{k_1,k_2,k_3}(2) = \zeta(k_1+1,2,k_2,k_3) + \zeta(k_1+1,1,k_2,k_3) + \zeta(k_1+1,k_2+1,k_3) \\ + \zeta(k_1+1,k_2,1,k_3) + \zeta(k_1+1,k_2,k_3+1) + \zeta(k_1+1,k_2,k_3,1).$$

The results above can be used for the evaluation of $\xi_{k_1,k_2}(2), \xi_{k_1,k_2,k_3}(2)$ for particular choices of $k_1, k_2, k_3 \in \mathbb{N}$ using previous evaluations of multiple zeta values with two, three or four parameters in the literature [**8**]. Note that there exists evaluations of multiple zeta values up to weight 22, see [**5**], which can be used for further simplification. Moreover, many calculations can be carried out algorithmically by using computer algebra systems, see [**1**] and the references therein.

4. AN EVALUATION INTO MULTIPLE ZETA STAR VALUES

In the case of $k_1, \ldots, k_r \in \mathbb{N} \setminus \{1\}$ and $n \in \mathbb{N}$ the function $\xi_{k_1,\ldots,k_r}(n)$ can be evaluated into products of multiple zeta values and multiple zeta star values.

Theorem 4. *For $k_1, \ldots, k_r \in \mathbb{N} \setminus \{1\}$ and $n \in \mathbb{N}$ we have*

$$\xi_{k_1,\ldots,k_r}(n) = \sum_{j=1}^{r} (-1)^{j+1} \zeta(k_{j+1},\ldots,k_r) \zeta^*(k_j, k_{j-1}\ldots,k_2,k_1+1,\{1\}_{n-1}),$$

according to the conventions $\zeta(k_{r+1},\ldots,k_r) = 1$ in the case of $j = r$ and $\zeta^*(k_1,k_0,\ldots,k_2,k_1+1,\{1\}_{n-1}) = \zeta^*(k_1+1,\{1\}_{n-1})$ in the case of $j = 1$.

Proof. By Theorem 2 we get

$$\xi_{k_1,\ldots,k_r}(n) = \sum_{n_1\geq 1} \frac{\zeta^*_{n_1}(\{1\}_{n-1})\zeta_{n_1-1}(k_2,\ldots,k_r)}{n_1^{k_1+1}} \\ = \sum_{n_1\geq 2} \frac{\zeta^*_{n_1}(\{1\}_{n-1})}{n_1^{k_1+1}} \sum_{j=1}^{n_1-1} \frac{\zeta_{j-1}(k_3,\ldots,k_r)}{j^{k_2}}.$$

Interchanging summation gives

$$\begin{aligned}\xi_{k_1,\ldots,k_r}(n) &= \sum_{j\ge 1}\frac{\zeta_{j-1}(k_3,\ldots,k_r)}{j^{k_2}}\sum_{n_1\ge j+1}\frac{\zeta^*_{n_1}(\{1\}_{n-1})}{n_1^{k_1+1}}\\ &= \zeta(k_2,\ldots,k_r)\zeta^*(k_1+1,\{1\}_{n-1}) - \sum_{j\ge 1}\frac{\zeta_{j-1}(k_3,\ldots,k_r)}{j^{k_2}}\sum_{n_1=1}^{j}\frac{\zeta^*_{n_1}(\{1\}_{n-1})}{n_1^{k_1+1}}\\ &= \zeta(k_2,\ldots,k_r)\zeta^*(k_1+1,\{1\}_{n-1}) - \sum_{j\ge 1}\frac{\zeta_{j-1}(k_3,\ldots,k_r)\zeta^*_j(k_1+1,\{1\}_{n-1})}{j^{k_2}}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\xi_{k_1,\ldots,k_r}(n) = \zeta(k_2,\ldots,k_r)\zeta^*(k_1+1,\{1\}_{n-1}) - \zeta(k_3,\ldots,k_r)\zeta^*(k_2,k_1+1,\{1\}_{n-1})\\ + \sum_{j\ge 1}\frac{\zeta_{j-1}(k_4,\ldots,k_r)\zeta^*_j(k_2,k_1+1,\{1\}_{n-1})}{j^{k_3}};\end{aligned}$$

This reasoning (repeated interchangement of summations) gives the stated result. □

One can convert the multiple zeta star values appearing in Theorem 4 to ordinary multiple zeta values using (1). This gives an alternative evaluation of $\xi_{k_1,\ldots,k_r}(n)$ into multiple zeta values. We obtain the following result.

Corollary 1. *For $k_1,\ldots,k_r \in \mathbb{N}\setminus\{1\}$ and $n\in\mathbb{N}$ we have*

$$\begin{aligned}&\xi_{k_1,\ldots,k_r}(n) = \sum_{j=1}^{r}(-1)^{j+1}\zeta(k_{j+1},\ldots,k_r)\\ &\times\sum_{h=1}^{n+j-1}\ \sum_{1\le \ell_1<\ell_2<\cdots<\ell_{h-1}<n+j-1}\zeta\Big(\sum_{i_1=1}^{\ell_1}m_{i_1},\sum_{i_2=\ell_1+1}^{\ell_2}m_{i_2},\ldots,\sum_{i_h=\ell_{h-1}+1}^{n+j-1}m_{i_h}\Big)\end{aligned}$$

with $m_1 = k_j,\ldots,m_{j-1}=k_2, m_j = k_1+1$, and $m_g = 1$ for $j<g\le n+j-1$ in the sum $\zeta\big(\sum_{i_1=1}^{\ell_1}m_{i_1},\sum_{i_2=\ell_1+1}^{\ell_2}m_{i_2},\ldots,\sum_{i_h=\ell_{h-1}+1}^{n+j-1}m_{i_h}\big)$.

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