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TWO SUBCLASSES OF 2-CONVEX POLYOMINOES: PROPERTIES FOR RECONSTRUCTION

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A polyomino P is called 2-convex if for every two cells there exists a monotone path included in P with at most 2 changes of direction. This paper studies the tomographical aspects of two subclasses of 2-convex polyominoes called η_{2L} and η'_{2L}. In the first part, the uniqueness results of the two subclasses of HV-convex polyominoes η and η' are investigated using the switching components (that is the elements of these subclasses that have the same projections). In the second part, using the uniqueness results and the algorithm by Chrobak and Dürr, two paths connecting the feet and a tomographical condition are given to verify whether P is in η_{2L} or η'_{2L}.

1. INTRODUCTION

There are many notions of discrete convexity of polyominoes (namely HV-convex [**3**], Q-convex [**4**], L-convex polyominoes [**7**]) and each one leads to interesting studies. One natural notion of convexity on the discrete plane is the class of HV-convex polyominoes, that is polyominoes with consecutive cells in rows and columns. Following the works of Del Lungo, Nivat, Barcucci and Pinzani [**3**] we are able to reconstruct polyominoes that are HV-convex according to their horizontal and vertical projections. In addition to that, for an HV-convex polyomino P every pair of cells of P can be reached using a path included in P with only two kinds of unit steps (such a path is called monotone). A polyomino is called k-convex if for every two cells we find a monotone path with at most k changes of direction. Obviously a k-convex polyomino is an HV-convex polyomino. Thus, the families of k-convex polyominoes for $k \in \mathbb{N}$ forms a hierarchy of HV-convex polyominoes. When the value of k is equal to 1 we have the so called L-convex polyominoes, where this terminology is motivated by the L-shape of the path that connects any two

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cells. This notion of L-convex polyominoes has been considered by several points of view. Combinatorial aspects of L-convex polyominoes were analyzed in [**5**], giving the enumeration according to the semi-perimeter and the area. In [**6**] it is given an algorithm that reconstructs an L-convex polyomino from the set of its maximal L-polyominoes. Similarly in [**7**] it is given another way to reconstruct an L-convex polyomino from the size of some special paths, called bordered L-paths. In fact, 2-convex polymoninoes are more geometrically complex and there was no result for their direct reconstruction. We could notice that Duchi, Rinaldi, and Schaeffer were able to enumerate this class in an interesting and technical article [**9**]. But the enumeration technique gives no idea for the tomographical reconstruction.

The first subclass that creates the link with 2-convex polyominoes is the class of HV-centered polyominoes. In [**14**], it is showed that if P is an HV-centered polyomino then P is 2-convex. Note that the tomographical properties of this subclass have been studied in [**8**] and its reconstruction algorithm is well known.

Now in order to study the geometrical and tomographical properties of all 2-convex polyominoes which are not L-convex and HV-centered, we choose to decompose them into different subclasses regarding the position of their non-intersecting feet, that is there does not exist a row (resp. a column) going from one foot to another.

By regarding the position of the non-intersecting feet of convex polyominoes, it is showed in [**14**] that if P is a convex (HV-convex) polyomino and if the N-foot is situated to the left (resp. to the right) of the S-foot and the E-foot is situated to the north (resp. to the south) of the W-foot then P is a 2-convex polyomino. To reconstruct P in this subclass, it is sufficient to fix its feet and then applying the algorithm of Chrobak and Dürr to reconstruct an HV-convex polyomino (see [**15**]).

Another subclass of 2-convex polyominoes has been studied in [**14**] where the N-foot is situated to the left (resp. to the right) of the S-foot and the E-foot is situated to the south (resp. to the north) of the W-foot. This subclass is called the two empty corners subclass, where the upper left corner and the lower right corner of the polyomino are characterized by the property that these two corners are empty i.e. there are no cells in these two corners. The geometrical properties and the uniqueness for this subclass are well studied in [**14**], also they are used to reconstruct directly all convex polyominoes belonging to this subclass (see [**15**]). Directed 2-convex polyominoes have been studied in [**14**], and their direct reconstruction is always possible using tomographical properties and the fact that any directed convex polyomino P is unique.

To complete the study of all subclasses of 2-convex polyominoes, we introduce in this paper another subclass called η_{2L} (resp. η'_{2L}) where the lower right corner (resp. upper left corner) of all polyominoes belonging to it is empty. This subclass has a complicated geometry and we are not yet able to reconstruct it directly. This is why by investigating the switching components made of 1-cycle, 2-cycle and other geometrical properties, we are able to give a necessary and sufficient condition in order to verify whether reconstructed polyominoes are 2-convex in η_{2L} (resp. η'_{2L}).

This paper is divided into 6 sections. After basics on polyominoes, section 3 gives the characterizations of two subclasses of 2-convex polyominoes. In section 4, the possible configurations of polyominoes in the classes η and η' are investigated using switching components made of 1-cycle or 2-cycle. We also focus on the unicity results for these classes of polyominoes. Section 5 gives the algorithm for the reconstruction of these two classes and the necessary and sufficient condition to obtain 2-convex polyominoes in η_{2L} and η'_{2L}. The last section is reserved for the final comments.

2. DEFINITION AND NOTATION

A planar discrete set is a finite subset of the integer lattice $\mathbb{Z}^2$ defined up to translation. A discrete set can be represented either by a set of cells, i.e. unitary squares of the cartesian plane, or by a binary matrix, where the 1's determine the cells of the set (see Fig. 1).

A polyomino P is a finite connected set of adjacent cells, defined up to translation, in the cartesian plane. A polyomino is said to be *column-convex* (resp. *row-convex*) if every column (resp. row) is connected (see [**2**, **13**]).

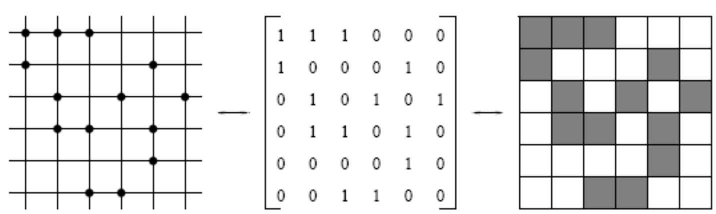


Figure 1: A finite set of $\mathbb{N} \times \mathbb{N}$, and its representation in terms of a binary matrix and a set of cells.

Finally, a polyomino is said to be *convex* (or HV-convex) if it is both column and row-convex (see Fig. 2). To each discrete set S, represented as a $m \times n$ binary matrix, we associate two integer vectors $H = (h_1, \ldots, h_m)$ and $V = (v_1, \ldots, v_n)$ such that, for each $1 \leq i \leq m$, $1 \leq j \leq n$, h_i and v_j are numbers of cells of S (fields of the matrix that contain 1) which lie on row i and column j, respectively. The vectors H and V are called the *horizontal* and *vertical* projections of S, respectively (see Fig. 3). Moreover if S has H and V as horizontal and vertical projections, respectively, then we say that S satisfies (H,V). Using the usual matrix notations, the element (i, j) denotes the entry in row i and column j. For any two cells A and B in a polyomino, a *path* $\prod_{AB}$, from A to B, is a sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)$

of adjacent disjoint cells belonging in P, with $A = (i_1, j_1)$, and $B = (i_r, j_r)$. For each $1 \leq k \leq r-1$, we can say that the two consecutive cells $(i_k, j_k), (i_{k+1}, j_{k+1})$ form:

- an *east* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k + 1$;
- a *north* step if $i_{k+1} = i_k - 1$ and $j_{k+1} = j_k$;
- a *west* step if $i_{k+1} = i_k$ and $j_{k+1} = j_k - 1$;
- a *south* step if $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k$.

Finally, we define a path to be *monotone* if it is entirely made of only two of the four types of steps defined above.

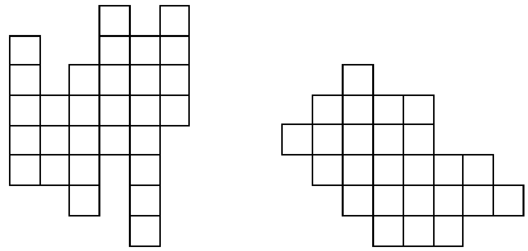


Figure 2: Column convex and convex polyomino.

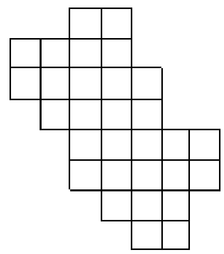


Figure 3: A polyomino P with $H = (2, 4, 5, 4, 5, 5, 3, 2)$ and $V = (2, 3, 6, 7, 6, 4, 2)$.

Proposition 1. [**6**] *A polyomino P is HV-convex if and only if every pair of cells is connected by a monotone path.*

Let us consider a polyomino P. A path in P has a change of direction in the cell (i_k, j_k), for $2 \leq k \leq r-1$, if $i_k \neq i_{k-1} \Longleftrightarrow j_{k+1} \neq j_k$.

Definition 1. *A HV-convex polyomino will be called k-convex if every pair of its cells can be connected by a monotone path with at most k changes of direction.*

In [**6**], it is proposed a hierarchy on convex polyominoes based on the number of changes of direction in the paths connecting any two cells of a polyomino.
For $k = 1$, we have the first level of hierarchy, i.e. the class of 1-convex polyominoes, also denoted L-convex polyominoes for the typical shape of each path having at most one single change of direction. In the present studies we focus our attention to the next level of the hierarchy, i.e. the class of 2-convex polyominoes, whose tomographical properties turn to be more interesting and substantially harder to be investigated than those of L-convex polyominoes (see Fig. 4).

3. GEOMETRICAL PROPERTIES

Let (H, V) be two vectors of projections and let P be a convex polyomino, that satisfies (H, V).

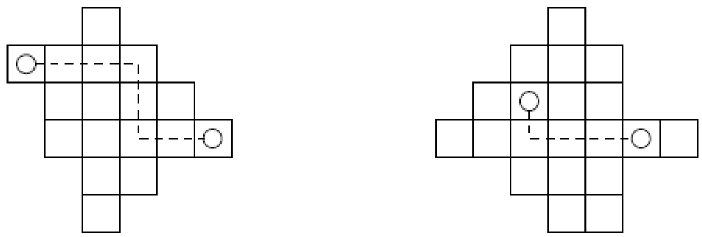


Figure 4: The convex polyomino on the left is 2-convex, while the one on the right is L-convex. For each polyomino, two cells and a monotone path connecting them are shown.

By a classical argument P is contained in a rectangle R of size $m \times n$ (called minimal bounding box). Let $[\min(S), \max(S)]$ ($[\min(E), \max(E)]$, $[\min(N), \max(N)]$, $[\min(W), \max(W)]$) be the intersection of P's boundary on the lower (right, upper, left) side of R (see [**3**]). By abuse of notation, for each $1 \le i \le m$ and $1 \le j \le n$, we call $\min(S)$ [resp. $\min(E)$, $\min(N)$, $\min(W)$] the cell at the position $(m, \min(S))$ [resp. $(\min(E), n)$, $(1, \min(N))$, $(\min(W), 1)$] and $\max(S)$ [resp. $\max(E)$, $\max(N)$, $\max(W)$] the cell at the position $(m, \max(S))$ [resp. $(\max(E), n)$, $(1, \max(N))$, $(\max(W), 1)$] (see Fig. 5).

Definition 2. *The segment* $[\min(S), \max(S)]$ *is called the* S*-foot. Similarly, the segments* $[\min(E), \max(E)]$, $[\min(N), \max(N)]$ *and* $[\min(W), \max(W)]$ *are called* E*-foot,* N*-foot and* W*-foot.*

Definition 3. *Let* P *be an* HV*-convex polyomino, we say that* P *is h-centered* [*resp. v-centered*], *if its* W*-foot and* E*-foot* [*resp.* N*-foot and* S*-foot*] *intersect, that is there at least one row going from one foot to another* (*see Fig.* 6), (*they are defined in* [**8**]).

The following property links h-centered polyominoes or v-centered polyominoes to 2-convex polyominoes:

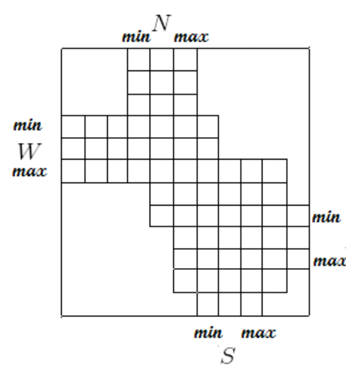


Figure 5: Min and max of the four feet in the rectangle R.

Proposition 2. *If* P *is an h-centered polyomino or a v-centered polyomino, then it is a* 2*-convex polyomino.*

Proof. Let us assume that P is h-*centered.* The W*-foot* and the E*-foot* intersect in a row i. The row i is used to go from any point of P to any other point of P. Thus there are at most two changes of direction. That is P is a 2-convex polyomino. If P is v*-centered* a similar reasoning holds. □

For a bounding rectangle R and for a given polyomino P, let us define the following sets:

- $WN = \{(i,j) \in P \mid i < \min(W) \text{ and } j < \min(N)\}$,
- $SE = \{(i,j) \in P \mid i > \max(E) \text{ and } j > \max(S)\}$,
- $NE = \{(i,j) \in P \mid i < \min(E) \text{ and } j > \max(N)\}$,
- $WS = \{(i,j) \in P \mid i > \max(W) \text{ and } j < \min(S)\}$.

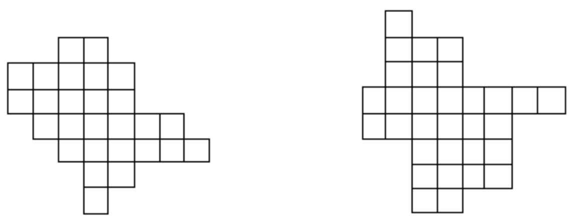


Figure 6: A v-centered polyomino on the left and an h-centered polyomino on the right.

From now on, we suppose that P is not *h-centered*, *v-centered* and L-convex polyomino. Let $\mathcal{C}$ be the class of convex polyominoes, thus we have four classes of polyominoes regarding the position of the non-intersecting feet.

- $\eta = \{P \in \mathcal{C} \mid \max(N) < \min(S) \text{ and } \max(W) < \min(E), \text{ and } SE = \emptyset\}$ (see Fig. 8).
- $\psi = \{P \in \mathcal{C} \mid \max(S) < \min(N) \text{ and } \max(E) < \min(W), \text{ and } NE = \emptyset\}$.
- $\eta' = \{P \in \mathcal{C} \mid \max(N) < \min(S) \text{ and } \max(W) < \min(E), \text{ and } WN = \emptyset\}$ (see Fig. 8).
- $\psi' = \{P \in \mathcal{C} \mid \max(S) < \min(N) \text{ and } \max(E) < \min(W), \text{ and } WS = \emptyset\}$.

Let us define the horizontal transformation (symmetry) $S_H : (i,j) \longrightarrow (m-i+1,j)$ which transforms the polyomino P from the class η to the class ψ, η' to ψ'. Note that S_H maps η to ψ and η' to ψ' and obviously by involution ψ to η and ψ' to η'. Indeed the transformation acts on the feet of the polyomino as it is shown in the following table (see Fig. 7). Thus we only investigate the characterizations of the classes η and η'. The proofs of the class η' are similar to those of η.

We make the choice of studying two special cases called η where the lower right corner of the polyomino is empty and η' where the upper left corner of the polyomino is empty (see Fig. 8). For each case, we give a characterization in order to describe the 2-convex polyominoes in terms of monotone paths.

The above sets with the classes η, ψ, η', and ψ' allow us to define the following four classes:

- $\eta_{2L} = \{P \in \mathcal{C} \,|\, \max(N) < \min(S)$ and $\max(W) < \min(E)$, and $SE = \emptyset\}$, where P is a 2-convex polyomino.
- $\psi_{2L} = \{P \in \mathcal{C} \,|\, \max(S) < \min(N)$ and $\max(E) < \min(W)$, and $NE = \emptyset\}$, where P is a 2-convex polyomino.
- $\eta'_{2L} = \{P \in \mathcal{C} \,|\, \max(N) < \min(S)$ and $\max(W) < \min(E)$, and $WN = \emptyset\}$, where P is a 2-convex polyomino.
- $\psi'_{2L} = \{P \in \mathcal{C} \,|\, \max(S) < \min(N)$ and $\max(E) < \min(W)$, and $WS = \emptyset\}$, where P is a 2-convex polyomino.

N, S	W, E
$S \longrightarrow N$	$W \longrightarrow W$
$N \longrightarrow S$	$E \longrightarrow E$
	$\leq \Longleftrightarrow \geq$
min $\longrightarrow$ min	min $\longrightarrow$ max
max $\longrightarrow$ max	max $\longrightarrow$ min

Figure 7: The horizontal transformation S_H on the feet of P.

Note that the horizontal symmetry S_H maps η_{2L} to ψ_{2L} and η'_{2L} to ψ'_{2L}.

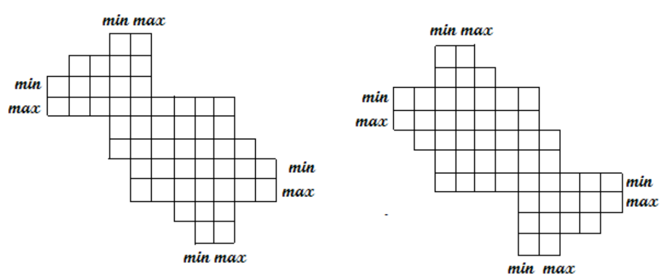


Figure 8: An element of the class η on the left and of the class η' on the right.

The following characterization holds for convex polyominoes in the class η.

Theorem 1. *Let P be a convex polyomino in the class η, P is 2-convex if and only if there exist six paths:*

(1) *from* $\min(N)$ *to* $\max(E)$,
(2) *and from* $\min(N)$ *to* $\max(S)$,
(3) *and from* $\min(W)$ *to* $\max(E)$,
(4) *and from* $\min(W)$ *to* $\max(S)$,
(5) *and from a generic cell* $(i, j) \in WN$ *to* $\max(E)$,
(6) *and from a generic cell* $(i, j) \in WN$ *to* $\max(S)$.

having at most two changes of direction.

Proof. $\Longrightarrow$ It is an immediate consequence of the definition of 2-convex polyomino.

$\Longleftarrow$ Suppose that P is not 2-convex, then there exist two points $(i_0, j_0), (i_1, j_1)$ such that any path between them has more than two changes of direction. Let us suppose that the two points are situated on two distinct feet. We make the proof for the feet N and E (the other cases are similar). Assume that (i_0, j_0) is at the position $(1, \min(N) \leq j_0 \leq \max(N))$ and (i_1, j_1) is at the position $(\min(E) \leq i_1 \leq \max(E), n)$. We consider the following two cases.

1) If the path from $\min(N)$ *to* $\max(E)$ has one change of direction, i.e. there exists an L-path between them, then by convexity there is an L-path between (i_0, j_0) and (i_1, j_1), hence the contradiction.

2) If the path from $\min(N)$ *to* $\max(E)$ has two changes of direction, one can observe the following cases:
2a) either the path goes through $\min(E)$ and then there exists an L-path between $\max(N)$ and $\min(E)$, thus by convexity there exists a $2L$-path from (i_0, j_0) to (i_1, j_1), hence the contradiction.
2b) or the path goes through $\max(N)$ and then there exists an L-path between $\max(N)$ and $\max(E)$, thus there exists a $2L$-path from (i_0, j_0) to (i_1, j_1), hence the contradiction (see Fig. 9).
The proofs for the other possible positions for the points (i_0, j_0) and (i_1, j_1) are analogous and use the paths (2) to (6). □

Corollary 1. *If P satisfies Theorem 1, then P is in the class η_{2L}.*

The following characterization hold for convex polyominoes in the class η'.

Theorem 2. *Let P be a convex polyomino in the class η', P is 2-convex if and only if there exist six paths:*

(1) *from* $\max(E)$ *to* $\min(N)$,
(2) *and from* $\max(S)$ *to* $\min(N)$,
(3) *and from* $\max(E)$ *to* $\min(W)$,
(4) *and from* $\max(S)$ *to* $\min(W)$,
(5) *and from a generic cell* $(i, j) \in SE$ *to* $\min(N)$,
(6) *and from a generic cell* $(i, j) \in SE$ *to* $\min(W)$.

having at most two changes of direction (see Fig. 9).

Proof. Same arguments as in the class η. □

Corollary 2. *If P satisfies Theorem 2, then P is in the class η'_{2L}.*

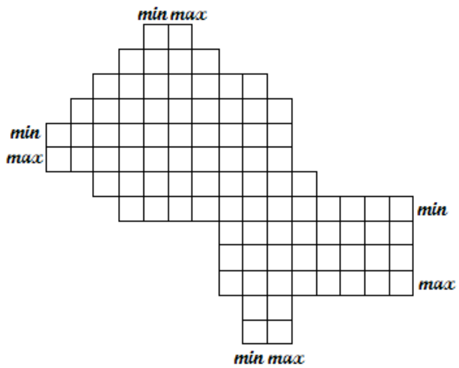


Figure 9: An element of the class η_{2L} on the left and of the class η'_{2L} on the right.

4. SWITCHING COMPONENTS IN THE TWO CLASSES η AND $\eta\prime$

In this section, we give the possible configurations of the polyominoes in the classes η and $\eta\prime$. The goal is to study the switching components of 2-convex polyominoes and to give results of existence of switching and uniqueness in the two classes.

Let $U(H,V)$ be the class of discrete sets having H and V as projections.

Definition 4. *We define the* $1-$*switching* (*or* $1-$*cycle*) *as an operator whose successive application allows to move from an element of* $U(H,V)$ *to another element of* $U(H,V)$. *This basic operator, also called an elementary switching operator, transforms each configuration of cells of the kind depicted in Fig.* 10*a into the one in Fig.* 10*b or vice versa.*

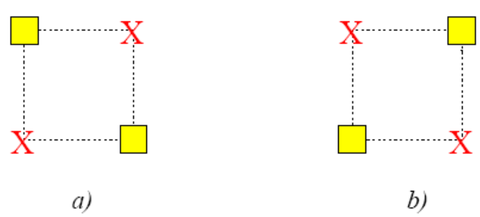


Figure 10: The two kinds of switching components. The switching operator transforms the switching component of a) into the one of b) or vice versa.

In Fig. 10, X represents the position of a point not belonging to the discrete set. The two configurations are called *switching components.*

In order to maintain convexity when the switching operator is applied to the switching component as in Fig. 10a the lower right 1 cannot have a 1 neither to its right nor under it and the upper left 1 cannot have a 1 neither to its left nor on top of it.

Furthermore, always for convexity, all the elements in the rectangle having the four elements of the switching component as vertices must be equal to 1. Analogous properties must hold for the switching on Fig. 10b.

Proposition 3. *A polyomino P of the class η cannot contain any* 1*-cycle.*

Proof. Suppose that P has the switching component

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

Then the lower right 1 is on $\max(S)$ or $\max(E)$ since $\mathrm{card}(ES) = 0$. Suppose that the lower right 1 is on $\max(S)$ then the upper left 1 is on $\min(N)$ and we have $h_1 = h_m = 1$ (vectors of projections), otherwise P is v-centered. Now applying the switching operator on P, in P' which is the image of P by the switching operator, we have that $\min(N) = \max(N) > \min(S) = \max(S)$ and hence P' does not belong to η. In fact the upper left 1 cannot be at a point in WN, otherwise P is v-centered. Suppose that the lower right 1 is on $\max(E)$, then the upper left 1 is on $\min(W)$ and we have $v_1 = v_n = 1$, otherwise P is h-centered. Now applying the switching operator on P, in P' we have that $\min(W) = \max(W) > \min(E) = \max(E)$ and hence P' does not belong to η. In fact the upper left 1 cannot be at a point in WN, otherwise P is h-centered.

Now suppose that P has the switching component

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

Then one can see that this switching component cannot occur on the feet since P is not h-centered or v-centered and so the lower right 0 is in SE. By applying the switching operator to P, we get that $\mathrm{card}(ES) = 1$ and then P' does not belong to η.
To summarize, P has no $1-cycle$ in η (see Fig. 11). □

Proposition 4. *A polyomino P of the class η' cannot contain any* 1*-cycle.*

Proof. Same arguments as in the class η. □

Definition 5. *The switching structures which are obtained by composing* 2 *elementary switchings such that the lower-rightmost point of the first one coincides with the upper-leftmost point of the second one, will be called a* 2*-switching chain* ($2-cycle$)*.*

Note that the n-switching chain is defined in analogous way to 2-switching. The switching in the Fig. 12 is a 2-switching chain represented by the sequence of the six points $(1,2), (1,4), (6,4), (6,6), (4,6), (4,2)$. Notice that two consecutive points share a row or a column.

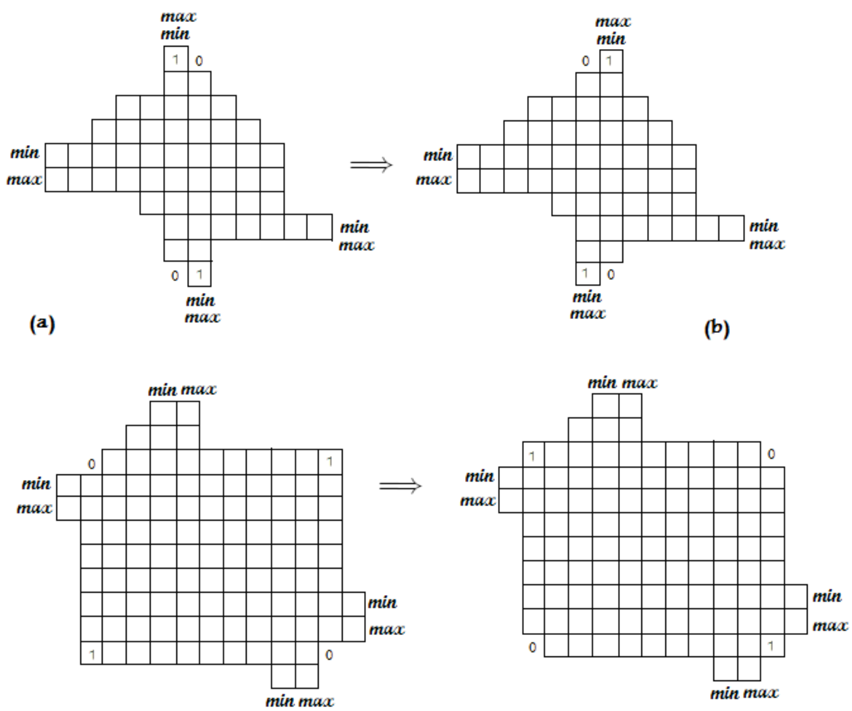


Figure 11: (a): Two polyominoes in η, (b): The image by the switching operator which is not in η.

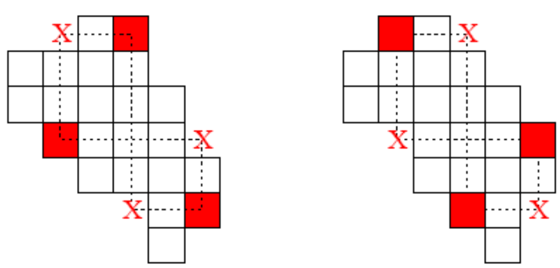


Figure 12: The two kinds of 2-switching components and two polyominoes belonging to the class $U(H,V)$, with $H = (2,4,5,4,4,2,1)$ and $V = (2,3,5,5,5,2)$.

Proposition 5. *A 2-convex polyomino cannot contain any n-switching chain, with $n \geq 3$.*

Proof. Let us proceed by contradiction assuming that there exists a 2-convex polyominoes P containing a 3-switching chain, say (i_1, j_1), (i_1, j_2), $\ldots$, (i_4, j_4), (i_4, j_1). Let us further suppose that the cell (i_1, j_1) belongs to P, and so it is for the cell (i_3, j_3). An easy check reveals that there does not exist in P a monotone path connecting (i_1, j_1) and (i_3, j_3) and having two changes of direction at most, against the assumption. The same conclusion is obtained if we try to connect the cells (i_1, j_2) and (i_2, j_3) supposing that (i_1, j_1) does not belong to P. Obviously, the same result holds for any n-switching chain, with $n \geq 3$. □

Proposition 6. *Let P be an HV-convex polyomino in the class η and assume that* $\max(E)$ *is not at the position* $(m-1, n)$ *and that* $\max(S)$ *is not at the position* $(m, n-1)$. *If P has one of the two 2-switching components, then P' which is the image of P by the 2-switching operator does not belong to η, and so P has no 2-cycle in this class.*

Proof. Let P be an HV-convex in the class η with $\max(E) \neq (m-1, n)$ and $\max(S) \neq (m, n-1)$. Suppose that P has the 2-switching component

$$\begin{matrix} 0 & 1 & \\ 1 & 1 & 0 \\ & 0 & 1 \end{matrix}$$

Then the lower rightmost 1 of the 2-switching component cannot be in SE since $\mathrm{card}(SE) = 0$. Let the lower rightmost 1 be on $\max(S)$. By applying the 2-switching operator on P, this yealds a polyomino P' with $\mathrm{card}(SE) \neq 0$, hence P' does not belong to η. Similar reasoning holds if the lower rightmost 1 is on $\max(E)$.
Now suppose that P has the 2-switching component

$$\begin{matrix} 1 & 0 & \\ 0 & 1 & 1 \\ & 1 & 0 \end{matrix}$$

Then the lower rightmost 0 is at the point $(\max(E) + 1, \max(S) + 1)$ and so by applying the 2-switching operator on P we get a polyomino P' with $\mathrm{card}(SE) = 1$, hence P' does not belong to the class η (see Fig. 13). □

Proposition 7. *Let P be an HV-convex polyomino in the class η. Suppose that P contains a 2-cycle, then at least one of the two statements is true:*

- $\max(E)$ *is at the position* $(m-1, n)$,
- $\max(S)$ *is at the position* $(m, n-1)$.

Moreover, the 2-cycle occur on the E or (and) S-feet (see Fig. 14).

Proof. It is an immediate result from Proposition 6 and the fact that

$$\mathrm{card}(SE) = 0. \qquad \square$$

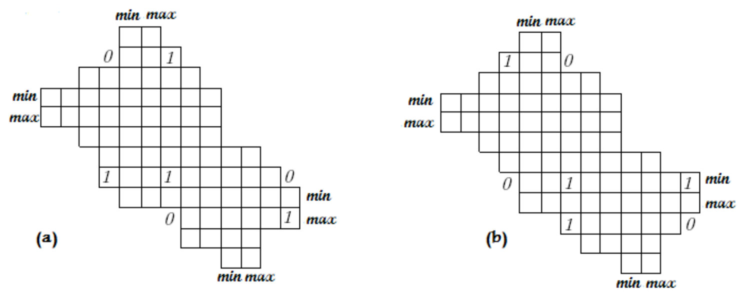
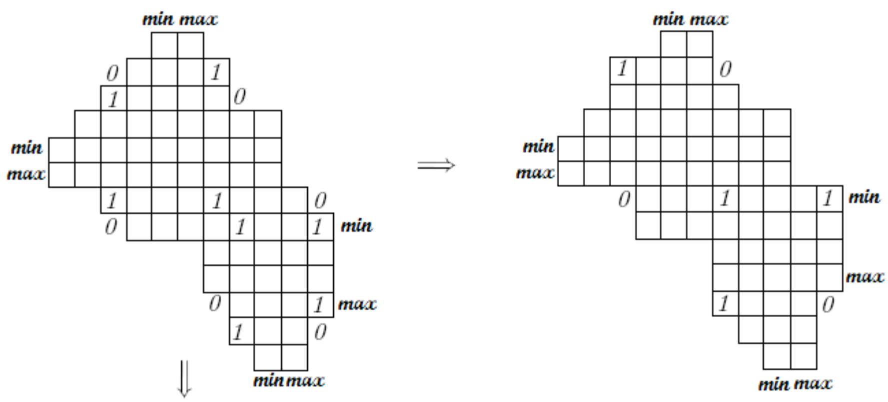


Figure 13: (a): A convex polyomino in η, (b): the image by the 2-switching operator is not in η.



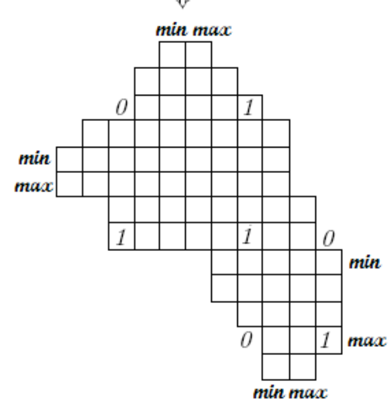


Figure 14: (a): 2-cycle in the class η.

Proposition 8. *Let P be an HV-convex polyomino in the class η' and following suppose that* $\min(W)$ *is not at the position* $(2,1)$ *and* $\min(N)$ *is not at the position* $(1,2)$. *If P has one of the two 2-switching components, then P' which is the image of P by the 2-switching operator does not belong to the class η', and so P has no 2-cycle in this class.*

Proof. Same arguments as in the class η. □

Proposition 9. *Let P be an HV-convex polyomino in the class η'. Suppose that P contains 2-cycle, then at least one of the two statements is true:*

- $\min(W)$ *is at the position* $(2,1)$,
- $\min(N)$ *is at the position* $(1,2)$.

Moreover, the 2*-cycle occurs on the* W *or* (*and*) N*-feet.*

Now we will give a condition on an HV-convex belonging to the classes η and η' to avoid the k-cycle with $k \geq 3$. This condition is important for the reconstruction of 2-convex polyominoes since there is no k-cycle with $k \geq 3$. So as we will see in the next section, all HV-convex polyominoes reconstructed by the algorithm of Chrobak and Dürr do not have any k-cycle, with $k \geq 3$ in the class η and η'.

Theorem 3. *Let P be an HV-convex polyomino. If P contains an L-path from* $\max(N)$ *to* $\min(E)$ *and an L-path from* $\max(W)$ *to* $\min(S)$ *then P does not contain any k-cycle, with* $k \geq 3$.

Proof. To prove that there is no k-cycle in P, it is sufficient to take $k = 3$ and to show that the two 3-switching components below:

$$\begin{matrix} 0 & 1 & & \\ 1 & 1 & 0 & \\ & 0 & 1 & 1 \\ & & 1 & 0 \end{matrix} \qquad \text{and} \qquad \begin{matrix} 1 & 0 & & \\ 0 & 1 & 1 & \\ & 1 & 1 & 0 \\ & & 0 & 1 \end{matrix}$$

do not exist. It is an immediate check on P with L-paths from $\max(N)$ to $\min(E)$ and from $\max(W)$ to $\min(S)$. □

5. CONDITIONS FOR RECONSTRUCTION

In this section we give some properties to verify whether P is 2-convex polyomino in the class η_{2L} (resp. η'_{2L}). By using the algorithm of Chrobak and Dürr, we impose the condition mentioned in Theorem 3 to avoid the presence of a k-cycle, with $k \geq 3$, then we reconstruct a convex polyomino in the class η (resp. η'). The uniqueness results of the two classes η and η' with the condition of the 2-cycle on the feet allow us to give a necessary and sufficient condition in order to establish whether P is a 2-convex polyomino.

5.1 HV-convex polyominoes

Assume that H, V denote strictly positive row and column sum vectors. We also assume that $\sum_i h_i = \sum_j v_j$, since otherwise (H, V) are not consistent.

The idea of Chrobak and Dürr [**8**] for the control of the HV-convexity is in fact to impose convexity on the four corner regions outside of the polyomino. An object A is called an *upper-left corner region* if $(i+1, j) \in A$ or $(i, j+1) \in A$ implies $(i, j) \in A$. In an analogous fashion they can define other corner regions. Let $\bar{P}$ be the complement of P. The definition of HV-convex polyominoes directly implies the following lemma.

Lemma 1. *P is an HV-convex polyomino if and only if $\bar{P} = A \cup B \cup C \cup D$, where A, B, C, D are disjoint corner regions* (*upper-left, upper-right, lower-left and lower-right, respectively*) *such that*

(i) $(i-1, j-1) \in A$ *implies* $(i,j) \notin D$, *and*
(ii) $(i-1, j+1) \in B$ *implies* $(i,j) \notin C$.

Given an HV-convex polyomino P and two row indices $1 \leq k, l \leq m$. P is *anchored* at (k,l) if $(k,1),(l,n) \in P$. The idea of Chrobak and Dürr is, given (H,V), to construct a 2SAT expression (a boolean expression in conjunctive normal form with at most two literals in each clause) $F_{k,l}(H,V)$ with the property that $F_{k,l}(H,V)$ is satisfiable if and only if there is an HV-convex polyomino realization P of (H,V) that is anchored at (k,l). $F_{k,l}(H,V)$ consists of several sets of clauses, each set expressing a certain property: "Corners" (Cor), "Disjointness" (Dis), "Connectivity" (Con), "Anchors" (Anc), "Lower bound on column sums" (LBC) and "Upper bound on row sums" (UBR).

$$\text{Cor} = \bigwedge_{i,j} \left\{ \begin{array}{llll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

$$\text{Dis} = \bigwedge_{i,j} \left\{ X_{i,j} \Rightarrow \bar{Y}_{i,j} : \text{ for symbols } X, Y \in \{A,B,C,D\}, X \neq Y \right\}$$

$$\text{Con} = \bigwedge_{i,j} \left\{ A_{i,j} \Rightarrow \bar{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \bar{C}_{i+1,j-1} \right\}$$

$$\text{Anc} = \left\{ \bar{A}_{k,1} \wedge \bar{B}_{k,1} \wedge \bar{C}_{k,1} \wedge \bar{D}_{k,1} \wedge \bar{A}_{l,n} \wedge \bar{B}_{l,n} \wedge \bar{C}_{l,n} \wedge \bar{D}_{l,n} \right\}$$

$$\text{LBC} = \bigwedge_{i,j} \left\{ \begin{array}{ll} A_{i,j} \Rightarrow \bar{C}_{i+v_j,j} & A_{i,j} \Rightarrow \bar{D}_{i+v_j,j} \\ B_{i,j} \Rightarrow \bar{C}_{i+v_j,j} & B_{i,j} \Rightarrow \bar{D}_{i+v_j,j} \end{array} \right\} \wedge \bigwedge_j \left\{ \bar{C}_{v_j,j} \quad \bar{D}_{v_j,j} \right\}$$

$$\text{UBR} = \bigwedge_j \left\{ \begin{array}{ll} \wedge_{i \leq min\{k,l\}} \bar{A}_{i,j} \Rightarrow B_{i,j+h_i} & \wedge_{k \leq i \leq l} \bar{C}_{i,j} \Rightarrow B_{i,j+h_i} \\ \wedge_{l \leq i \leq k} \bar{A}_{i,j} \Rightarrow D_{i,j+h_i} & \wedge_{max\{k,l\} \leq i} \bar{C}_{i,j} \Rightarrow D_{i,j+h_i} \end{array} \right\}$$

Define $F_{k,l}(H,V) = \text{Cor} \wedge \text{Dis} \wedge \text{Con} \wedge \text{Anc} \wedge \text{LBC} \wedge \text{UBR}$. All literals with indices outside the set $\{1,\ldots,m\} \times \{1,\ldots,n\}$ are assumed to have value 1.

Algorithm1
Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$
W.l.o.g assume:$\forall i : h_i \in [1,n], \forall j : v_j \in [1,m], \sum_i h_i = \sum_j v_j$ and $m \leq n$.
For $k,l = 1,\ldots,m$ **do begin**
If $F_{k,l}(H,V)$ is satisfiable,
then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**.
end
output "failure".

Theorem 4. [**8**] *$F_{k,l}(H,V)$ is satisfiable if and only if (H,V) have a realization P that is an HV-convex polyomino anchored at (k,l).*

Each formula $F_{k,l}(H,V)$ has size $O(mn)$ and can be implemented in time $O(mn)$. Since 2SAT can be solved in linear time [**1**, **10**], Chrobak and Dürr give the following result.

Theorem 5. [**8**] *Algorithm 1 solves the reconstruction problem for HV-convex polyominoes in time $O\big(mn \min(m^2,n^2)\big)$.*

5.2 Clauses for the classes η and η'

Here we give the set of clauses that reconstruct a polyomino P in the class η or η'.

$$\text{Pos} = \left\{ \; C_{(\min(E),1)} \wedge C_{(m,\max(N))\wedge A_{1,1}\wedge D_{m,n}} \; \right\}$$

$$\text{Cor} = \bigwedge_{i,j} \left\{ \begin{array}{llll} A_{i,j} \Rightarrow A_{i-1,j} & B_{i,j} \Rightarrow B_{i-1,j} & C_{i,j} \Rightarrow C_{i+1,j} & D_{i,j} \Rightarrow D_{i+1,j} \\ A_{i,j} \Rightarrow A_{i,j-1} & B_{i,j} \Rightarrow B_{i,j+1} & C_{i,j} \Rightarrow C_{i,j-1} & D_{i,j} \Rightarrow D_{i,j+1} \end{array} \right\}$$

$$\text{Dis} = \bigwedge_{i,j} \left\{ \; X_{i,j} \Rightarrow \bar{Y}_{i,j} : \text{ for symbols } X, Y \in \{A, B, C, D\}, X \neq Y \; \right\}$$

$$\text{Con} = \bigwedge_{i,j} \left\{ \; A_{i,j} \Rightarrow \bar{D}_{i+1,j+1} \quad B_{i,j} \Rightarrow \bar{C}_{i+1,j-1} \; \right\}$$

$$\text{Anc} = \left\{ \begin{array}{c} \bar{A}_{\min(W),1} \wedge \bar{A}_{\min(E),n} \wedge \bar{B}_{\min(W),1} \wedge \bar{B}_{\min(E),n} \wedge \\ \bar{C}_{\min(W),1} \wedge \bar{C}_{\min(E),n} \wedge \bar{D}_{\min(W),1} \wedge \bar{D}_{\min(E),n} \wedge \\ \bar{A}_{1,\min(N)} \wedge \bar{A}_{m,\min(S)} \wedge \bar{B}_{1,\min(N)} \wedge \bar{B}_{m,\min(S)} \wedge \\ \bar{C}_{1,\min(N)} \wedge \bar{C}_{m,\min(S)} \wedge \bar{D}_{1,\min(N)} \wedge \bar{D}_{m,\min(S)} \wedge \\ \bar{A}_{\max(W),1} \wedge \bar{A}_{\max(E),n} \wedge \bar{B}_{\max(W),1} \wedge \bar{B}_{\max(E),n} \wedge \\ \bar{C}_{\max(W),1} \wedge \bar{C}_{\max(E),n} \wedge \bar{D}_{\max(W),1} \wedge \bar{D}_{\max(E),n} \wedge \\ \bar{A}_{1,\max(N)} \wedge \bar{A}_{m,\max(S)} \wedge \bar{B}_{1,\max(N)} \wedge \bar{B}_{m,\max(S)} \wedge \\ \bar{C}_{1,\max(N)} \wedge \bar{C}_{m,\max(S)} \wedge \bar{D}_{1,\max(N)} \wedge \bar{D}_{m,\max(S)} \end{array} \right\}$$

$$\text{LBC} = \bigwedge_{i} \left\{ \begin{array}{cc} \wedge_{j<\min(N)} A_{i,j} \Rightarrow \bar{C}_{i+v_j,j} & \wedge_{j>\max(S)} B_{i,j} \Rightarrow \bar{D}_{i+v_j,j} \\ \wedge_{\max(N)<j<\min(S)} B_{i,j} \Rightarrow \bar{C}_{i+v_j,j} & \wedge_{\min(S)\leq j\leq \max(S)} B_{i,j} \Rightarrow \bar{C}_{i+v_j,j} \\ \wedge_{\min(N)\leq j\leq \max(N)} C_{i+v_j,j} \Rightarrow \bar{A}_{i,j} & \end{array} \right\} \wedge$$

$$\bigwedge_j \left\{ \; \bar{C}_{v_j,j} \quad \bar{D}_{v_j,j} \; \right\}$$

$$\text{UBR} = \bigwedge_{j} \left\{ \begin{array}{cc} \wedge_{i<\min(W)} \bar{A}_{i,j} \Rightarrow B_{i,j+h_i} & \wedge_{i>\max(E)} \bar{C}_{i,j} \Rightarrow D_{i,j+h_i} \\ \wedge_{\max(W)<i<\min(E)} \bar{C}_{i,j} \Rightarrow B_{i,j+h_i} & \wedge_{\min(E)\leq i\leq \max(E)} \bar{B}_{i,j+h_i} \Rightarrow C_{i,j} \\ \wedge_{\min(W)\leq i\leq \max(W)} \bar{A}_{i,j} \Rightarrow B_{i,j+h_i} & \end{array} \right\}$$

$$\text{GEO1} = \left\{ \begin{array}{c} \bar{A}_{\max(W),\min(S)} \wedge \bar{B}_{\max(W),\min(S)} \wedge \bar{C}_{\max(W),\min(S)} \wedge \bar{D}_{\max(W),\min(S)} \wedge \\ \bar{A}_{\min(E),\max(N)} \wedge \bar{B}_{\min(E),\max(N)} \wedge \bar{C}_{\min(E),\max(N)} \wedge \bar{D}_{\min(E),\max(N)} \end{array} \right\}$$

$$\text{REC} = \left\{ \; \bar{A}_{\min(W)-1,\min(N)-1} \wedge D_{\max(E)+1,\max(S)+1} \; \right\}$$

$$\text{REC1} = \left\{ \; A_{\min(W)-1,\min(N)-1} \wedge \bar{D}_{\max(E)+1,\max(S)+1} \; \right\}$$

In order to reconstruct an HV-convex polyomino in η with the above conditions, we use the set of clauses:

$$\eta(H, V) = \text{Pos} \wedge \text{Cor} \wedge \text{Dis} \wedge \text{Con} \wedge \text{Anc} \wedge \text{LBC} \wedge \text{UBR} \wedge \text{GEO1} \wedge \text{REC}.$$

In order to reconstruct an HV-convex polyomino P in the class η', we use the set of clauses:

$$\eta'(H, V) = \text{Pos} \wedge \text{Cor} \wedge \text{Dis} \wedge \text{Con} \wedge \text{Anc} \wedge \text{LBC} \wedge \text{UBR} \wedge \text{GEO1} \wedge \text{REC1}.$$

Algorithm2
Input: $H \in \mathbb{N}^m, V \in \mathbb{N}^n$
W.l.o.g assume:$\forall i : h_i \in [1, n], \forall j : v_j \in [1, m], \sum_i h_i = \sum_j v_j$.
For $\min(W), \min(E) = 1, \ldots, m$ and
$\min(N), \min(S) = 1, \ldots, n$ **do begin**
If $\eta(H, V)$ or $\eta'(H, V)$ is satisfiable,
then output $P = \overline{A \cup B \cup C \cup D}$ and **halt**.
end
output "failure".

Proof. We give the proof for the class η which is a very simple modification of the algorithm of Chrobak and Dürr (see Theorem 4). The set Anc gives the feet of suitable size by fixing 8 cells outside the corners A, B, C, D. Thus these cells of the extremities of the feet are in the interior of the polyomino. The set Pos imposes the constraint of the relative positions of feet in the class η (see Fig. 15). The set GEO1 implies that the cells at the position $(\max(W), \min(S))$ and $(\min(E), \max(N))$ belong to P and thus, by convexity there exist L-paths from $\max(W)$ to $\min(S)$ and from $\min(E)$ to $\max(N)$. GEO1 is imposed in P to avoid the k-cycle with $k \geq 3$. The set REC implies that the cell $(\min(W)-1, \min(N)-1)$ belongs to P while the cell $(\max(E)+1, \max(S)+1)$ does not belong to P (see Fig. 16). Hence if the set of clauses $\eta(H, V)$ is satisfiable, then we are able to reconstruct a polyomino P in the class η. □

In order to compute the complexity of this algorithm, one can see that the possible positions of the four feet is $(n-h_m+1)(n-h_1+1)(m-v_1+1)(m-v_n+1) \leq n^2m^2$ (see [**3**]). And so by imposing the four feet in the interior of the polyomino by using the algorithm of Chrobak and Dürr, we obtain the following result.

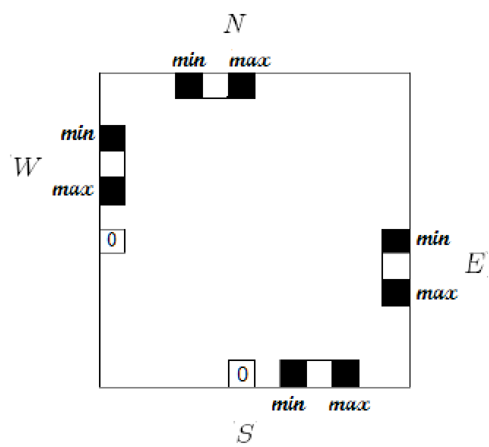


Figure 15: Position and anchors of the feet in the class η. Black cells represent the fixed feet.

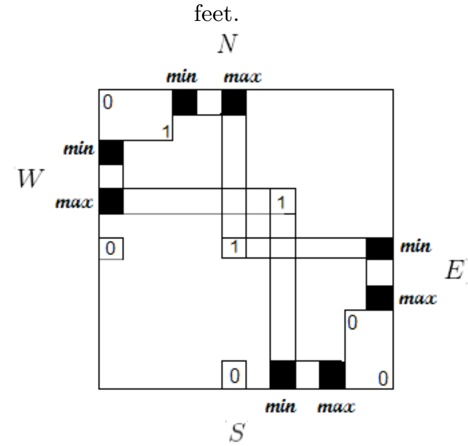


Figure 16: GEO1 and REC in the class η

Theorem 6. *Algorithm 2 solves the reconstruction problem for HV-convex polyominoes in η or η' in time $O(n^3m^3)$.*

Definition 6. *If P and $P^{'}$ are two polyominoes, we say that $P^{'}$ occurs in P if there exists a subset of cells of P that represents $P^{'}$ (see Fig. 17).*

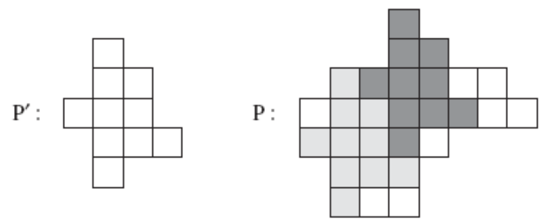


Figure 17: Two occurrences of $P^{'}$ in P.

Now we are able to give a sufficient and necessary condition to obtain a 2-convex polyomino in the class η_{2L}, using the fact that even if we have 2-cycle in the class η, it acts on the feet and hence the unicity of the HV-convex for a fixed feet is guaranteed. Let R_1 be the maximal rectangle in P such that the lowest rightmost cell is $\max(E)$ and the upper rightmost cell is $\min(E)$. Similarly, we define the maximal rectangle R_2 in P such that the lowest rightmost cell is $\max(S)$ and the lowest leftmost cell is $\min(S)$.
Let P_1 be the polyomino which is included in P and having the upper leftmost cell of R_1 as its lowest rightmost cell. Similarly, we define the polyomino P_2 which is included in P and having the upper leftmost cell of R_2 as its lowest rightmost cell (see Fig. 18, 19).

Theorem 7. *Let P be an HV-convex in the class η. P is 2-convex polyomino if and only if the polyominoes P_1 and P_2 are L-convex polyominoes (see Fig. 18, 19).*

Proof. $\Longrightarrow$ Let us proceed by contradiction by assuming that at least P_1 is not an L-convex polyomino, so it is at least a 2-convex. In R_1, we have an L-path from $\max(E)$ to its upper leftmost cell which is the lower rightmost cell of the polyomino P_1. Now if P_1 is not an L-convex then there exists a cell $(i, j) \in P_1$ such that there is no path from (i, j) to the lower right most cell in P_1 having at most one change of direction, hence no $2L$-path from $(i, j) \in P_1$ to $\max(E)$ and P is not 2-convex, hence the contradiction.

$\Longleftarrow$ We have that P_1 and P_2 are L-convex polyominoes. One can see that the two rectangles R_1 and R_2 are also L-convex polyominoes and then we have a monotone path from $\max(E)$ to all other cells in P_1 with at most two changes of direction. Similarly, we have a monotone path from $\max(S)$ to all other cells in P_2 with at most two changes of direction. Note that the cells which belong to P_1 and not to P_2 (resp. $\in P_2$ and not $\in P_1$) can be reached by an L-path from $\max(S)$ (resp. $\max(E)$). Hence P is a 2-convex polyomino. $\square$

Corollary 3. *Let P be an HV-convex polyomino in the class η. If P satisfies Theorem 7 then P is in the class η_{2L}.*

Similarly, in the class η' we call R_1 the maximal rectangle in P with the upper leftmost cell is $\min(W)$ and the lower leftmost cell is $\max(W)$. We define the maximal rectangle R_2 in P with the upper leftmost cell is $\min(N)$ and the upper rightmost cell is $\max(N)$. Let P_1 be the polyomino which is included in P and having the lower rightmost cell of R_1 as its upper leftmost cell. Similarly, we define the polyomino P_2 which is included in P and having the lower rightmost cell of R_2 as its upper leftmost cell.

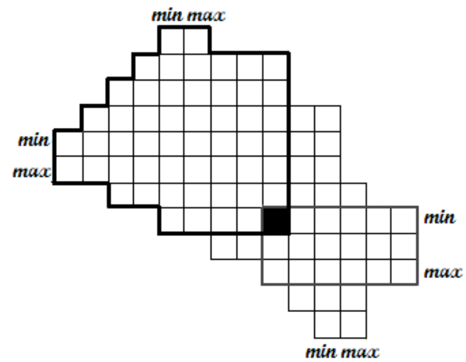


Figure 18: The rectangle R_1 and the polyomino P_1.

Theorem 8. *Let P be an HV-convex in the class η'. P is 2-convex polyomino if and only if the polyominoes P_1 and P_2 are L-convex polyominoes.*

Proof. Same arguments as in the class η. □

Corollary 4. *Let P be an HV-convex polyomino in the class η'. If P satisfies Theorem 8 then P is in the class η'_{2L}.*

To avoid repetitions, once can see that the same characterizations and properties hold for the classes ψ, ψ', ψ_{2L}, and ψ'_{2L} on the basis of the properties of the transformation S_H.

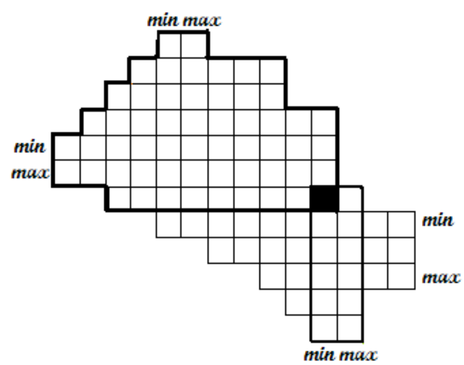


Figure 19: The rectangle R_2 and the polyomino P_2.

6. FINAL COMMENTS

The algorithm presented in this article is a verification method for the 2-convexity in two subclasses of 2-convex polyominoes. It is based on some geometrical and tomographical properties with some modifications in the algorithm presented by Chrobak and Dürr. Note that we are studying the last subclass of 2-convex polyominoes called the non-empty corners subclass where there is at least one cell in SE and WN. The presented method can be extended to verify whether we have a 2-convex polyomino in the non-empty corners subclass, but it becomes heavy especially when the number of cells in one of the two sets SE and WN increases.

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