Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 303–309. doi:10.2298/AADM0902303C

LIMIT DISTRIBUTION OF ASCENT, DESCENT OR EXCEDANCE LENGTH SUMS OF PERMUTATIONS

Lane Clark

Let $A_n(\sigma)$ denote the sum of the lengths of ascents of a permutation σ of $\{1,\ldots,n\}$ chosen uniformly at random. We find the exact expectation and variance and prove a central limit theorem for the A_n. Identical results hold for the sum of the lengths of descents or of excedances of a permutation of $\{1,\ldots,n\}$ chosen uniformly at random.

1. INTRODUCTION

The set of permutations of $[n] = \{1,\ldots,n\}$ is $\mathfrak{S}_n = \{(\sigma(1),\ldots,\sigma(n)) : \sigma(1),\ldots,\sigma(n) \in [n] \text{ are distinct}\}$ with equality of tuples. A permutation $\sigma = (\sigma(1),\ldots,\sigma(n)) \in \mathfrak{S}_n$ has

an *ascent* at $i \in [n-1]$ iff $\sigma(i) < \sigma(i+1)$ with *length of ascent at* i of $\sigma(i+1) - \sigma(i)$,
a *descent* at $i \in [n-1]$ iff $\sigma(i) > \sigma(i+1)$ with *length of descent at* i of $\sigma(i) - \sigma(i+1)$,
an *excedance* at $i \in [n]$ iff $\sigma(i) > i$ with *length of excedance at* i of $\sigma(i) - i$.

We consider the uniform probability space $\Omega_n = (\mathfrak{S}_n, \mathcal{P}_n, \Pr = \Pr_n)$ on $\mathfrak{S}_n$, i.e., $\mathcal{P}_n$ is the powerset of $\mathfrak{S}_n$ and $\Pr(\sigma) = 1/n!$ for each $\sigma \in \mathfrak{S}_n$. Any function $X : \mathfrak{S}_n \to \mathbb{N}$ is then a random variable on Ω_n with finite moments $\mathrm{E}(X^r) = \sum\limits_{k=0}^{\infty} k^r \Pr(X = k)$.

Let $A_{n,m}(\sigma)$ denote the number of ascents of $\sigma \in \mathfrak{S}_n$ of length at least $m \in \mathbb{P}$. Then $A_{n,1}(\sigma)$ counts the number of ascents of σ and the Eulerian numbers $A(n,k) = \#\{\sigma \in \mathfrak{S}_n : A_{n,1}(\sigma) = k\}$. Central limit theorems for the random variables $A_{n,m}$ on Ω_n were proved by Carlitz st al. [**2**] for $m = 1$, and, more generally, by the author [**3**] for $m = o(n)$. Recently, Balcza [**1**] found the exact expectation and variance for the sum of the lengths of inversions of σ on Ω_n.

2000 Mathematics Subject Classification. 05A10, 60F05.
Keywords and Phrases. Ascents, descents, excedances, length sums, central limit theorems.

Let $A_n(\sigma)$ (respectively, $D_n(\sigma)$ and $E_n(\sigma)$) denote the sum of the lengths of ascents (respectively, descents and excedances) of $\sigma \in \mathfrak{S}_n$. For example, $\sigma = (4,1,7,10,6,3,8,2,9,5)$ has ascents at $2,3,6,8$ of lengths $6,3,5,7$; descents at $1,4,5,7,9$ of lengths $3,4,3,6,4$; and excedances at $1,3,4,5,7$ of lengths $3,4,6,1,1$. Hence, $A_{10}(\sigma) = 6+3+5+7 = 21$, $D_{10}(\sigma) = 3+4+3+6+4 = 20$ and $E_{10}(\sigma) = 3+4+6+1+1 = 15$. Evidently the statistic $A_n(\sigma)$ refines the statistic $A_{n,1}(\sigma)$. It is easily seen that $\max\{A_n(\sigma) : \sigma \in \mathfrak{S}_n\} = \max\{D_n(\sigma) : \sigma \in \mathfrak{S}_n\} = \max\{E_n(\sigma) : \sigma \in \mathfrak{S}_n\} = M_n$ $(n \in \mathbb{P})$ where $M_n = \lceil n/2 \rceil^2$ (even n) and $M_n = \lceil n/2 \rceil^2 - \lceil n/2 \rceil$ (odd n).

Let $a(n,k) = \#\{\sigma \in \mathfrak{S}_n : A_n(\sigma) = k\}$, $d(n,k) = \#\{\sigma \in \mathfrak{S}_n : D_n(\sigma) = k\}$ and $e(n,k) = \#\{\sigma \in \mathfrak{S}_n : E_n(\sigma) = k\}$ (see Table 1 below). Then A_n, D_n and E_n are random variables on Ω_n with

$$(1) \qquad \Pr(A_n = k) = \frac{a(n,k)}{n!}\,,\ \Pr(D_n = k) = \frac{d(n,k)}{n!} \text{ and } \Pr(E_n = k) = \frac{e(n,k)}{n!}\,. \quad (k \in \mathbb{N})$$

It is clear that $a(n,k) = d(n,k)$ $(n \in \mathbb{P}, k \in \mathbb{N})$ by just reading the permutations in the opposite direction. Lemma 2.1 proves that $a(n,k) = e(n,k)$ $(n \in \mathbb{P}, k \in \mathbb{N})$. Hence, A_n, D_n and E_n are identically distributed (they are not pair-wise independent, however) on Ω_n. Therefore, from here on we let $\{X_n\} = \{A_n\}, \{D_n\}$ or $\{E_n\}$ on Ω_n. In this paper, we derive $\mu_n = \mathrm{E}(X_n) = (n^2-1)/6$ and $\sigma_n^2 = \mathrm{Var}(X_n) = (2n^3+2n^2+7n+7)/180$ and prove the central limit theorem $(X_n - \mu_n)/\sigma_n \xrightarrow{d} N(0,1)$, i.e, we prove that for every $x \in \mathbb{R}$

$$\Pr\left(\frac{X_n - \mu_n}{\sigma_n} \le x\right) \to \Phi(x)\,,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t$$

is the distribution function of a standard normal random variable $N(0,1)$.

n/k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
3	1	2	3	0	0	0	0	0	0	0	0	0	0	0
4	1	3	7	9	4	0	0	0	0	0	0	0	0	0
5	1	4	12	24	35	24	20	0	0	0	0	0	0	0
6	1	5	18	46	93	137	148	136	100	36	0	0	0	0
7	1	6	25	76	187	366	591	744	884	832	716	360	252	0

$a(n,k)$

Table 1.

In passing we note that the entries in Table 1, read by rows, is *The Online Encyclopedia of Integer Sequences* sequence A062869 (date: June 26, 2001) about which very little was known, until this paper, apart from a table of values for $1 \le n \le 9$.

Here $\mathbb{N}$ (respectively, $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}$) denotes the nonnegative integers (respectively, the positive integers, the rational numbers and the real numbers). Also $\lfloor x \rfloor$ is the largest integer at most $x \in \mathbb{R}$. Let $(r)_0 = 1$ and $(r)_k = (r)\cdots(r-k+1)$ $(k \in \mathbb{P}, r \in \mathbb{R})$. See Comtet [**4**] for combinatorics and Durrett [**5**] for probability.

2. MAIN RESULTS

2.1. Equidistribution of A_n, D_n and E_n

Stanley [**8**; Proposition 1.3.12] gave an explicit bijection $f : \mathfrak{S}_n \to \mathfrak{S}_n$ so that the number of ascents of σ equals the number of excedances of $f(\sigma)$ for all $\sigma \in \mathfrak{S}_n$. We next give a different bijection f that also satisfies $A_n(\sigma) = E_n(f(\sigma))$ for all $\sigma \in \mathfrak{S}_n$.

Lemma 2.1. *The function $f : \mathfrak{S}_n \to \mathfrak{S}_n$ defined in the proof is a bijection where the number of ascents of σ equals the number of excedances of $f(\sigma)$ and $A_n(\sigma) = E_n(f(\sigma))$ for all $\sigma \in \mathfrak{S}_n$. Hence, $a(n,k) = e(n,k)$ for all $n \in \mathbb{P}$ and $k \in \mathbb{N}$.*

Proof. Suppose $\sigma = (\sigma(1), \dots, \sigma(n)) \in \mathfrak{S}_n$ has ascents at $1 \le i_1 < \cdots < i_\ell \le n-1$. Order $\sigma(i_1), \dots, \sigma(i_\ell)$ as $1 \le \sigma(j_1) < \cdots < \sigma(j_\ell) \le n$ and $[n] - \{\sigma(j_k+1) : 1 \le k \le \ell\}$ as $1 \le t_1 < \cdots < t_{n-\ell} \le n$. Now construct $\tau \in \mathfrak{S}_n$ as follows. Place $\sigma(j_k+1)$ at coordinate $\sigma(j_k)$ of τ $(1 \le k \le \ell)$. Necessarily, $t_1 = 1$ (as all $\sigma(j_k+1) \ge \sigma(j_k)+1 \ge 2$) which we place in the left-most unused coordinate s_1 of τ. Having placed $t_1, \dots, t_q$ in (the left-most unused) coordinates $1 \le s_1 < \cdots < s_q$, we place t_{q+1} in the left-most unused coordinate s_{q+1} $(> s_q$, necessarily) of τ $(1 \le q \le n-\ell-1)$. Clearly, $\tau \in \mathfrak{S}_n$.

Assume that all of $1, \dots, t_q$ have appeared in coordinates $1, \dots, s_q$ of τ where $1 \le q \le n-\ell-1$. Let $s_{q+1} = s_q + a$, $t_{q+1} = t_q + b$ and $s_q = t_q + c$ with $a, b \in \mathbb{P}$ and $c \in \mathbb{N}$. Suppose that $t_{q+1} \ge s_{q+1} + 1$. For $1 \le x \le a + c$, $t_q + 1 \le t_q + x \le t_q + a + c = s_{q+1} \le t_{q+1} - 1$. Then, each $t_q + x = \sigma(j_k+1)$ is at coordinate $\sigma(j_k)$ with $\sigma(j_k) \le t_q + x - 1 \le t_q + a + c - 1 = s_{q+1} - 1$. Hence, all of $t_q + 1, \dots, t_q + a + c = s_{q+1}$ appear in coordinates $1, \dots, s_{q+1} - 1$. Consequently, all of $1, \dots, s_{q+1}$ appear in coordinates $1, \dots, s_{q+1} - 1$, which is a contradiction. Then $t_{q+1} \le s_{q+1}$ and, as above, all of $t_q + 1, \dots, t_{q+1}$ appear in coordinates $1, \dots, s_{q+1}$. Hence, all of $1, \dots, t_{q+1}$ appear in coordinates $1, \dots, s_{q+1}$.

Consequently, $s_q \ge t_q$ $(1 \le q \le n-\ell)$, so that the number of ascents of σ equals the number of excedances of τ and $A_n(\sigma) = E_n(\tau)$. It is immediately seen that $f : \mathfrak{S}_n \to \mathfrak{S}_n$ by $f : \sigma \mapsto \tau$ is a bijection proving our result. $\square$

2.2. Expectation and Variance of X_n

Theorem 2.2. *The random variables X_n on Ω_n have*

$$\mu_n = E(X_n) = \frac{n^2-1}{6} \quad \text{and} \quad \sigma_n^2 = Var(X_n) = \frac{2n^3+2n^2+7n+7}{180}.$$

Proof. We prove the theorem for the statistic A_n. Let $I = \{(i,r,s) : 1 \le i \le n-1, 1 \le r < s \le n\}$. For $(i,r,s) \in I$ and $\sigma \in \mathfrak{S}_n$, let

$$A_{(i,r,s)}(\sigma) = \begin{cases} s-r\,, & \sigma(i)=r,\ \sigma(i+1)=s; \\ 0\,, & \text{otherwise;} \end{cases}$$

and $A_n = \sum_{(i,r,s)\in I} A_{(i,r,s)}$. Then $A_n(\sigma)$ is the sum of the lengths of ascents of σ.

For $(i,r,s) \in I$, $\mathrm{E}(A_{(i,r,s)}) = \dfrac{s-r}{(n)_2}$, hence,

$$\mathrm{E}(A_n) = \sum_{(i,r,s)\in I} \mathrm{E}(A_{(i,r,s)}) = \sum_{i=1}^{n-1}\sum_{s=2}^{n}\sum_{r=1}^{s-1} \frac{s-r}{(n)_2} = \frac{n^2-1}{6}\,.$$

Let $J = I^2 - \{((i,r,s),(i,r,s)) : (i,r,s) \in I\}$. Then,

$$A_n^{\,2} = \sum_{((i,r,s),(j,t,u))\in I^2} A_{(i,r,s)}A_{(j,t,u)} = \sum_{(i,r,s)\in I} A^2_{(i,r,s)} + \sum_{((i,r,s),(j,t,u))\in J} A_{(i,r,s)}A_{(j,t,u)}\,.$$

Obviously $\mathrm{E}(A_{(i,r,s)}A_{(j,t,u)}) = \mathrm{E}(A_{(j,t,u)}A_{(i,r,s)})$ for $((i,r,s),(j,t,u)) \in I^2$. Further, $\mathrm{E}(A_{(i,r,s)}A_{(j,t,u)}) = 0$ if (1) $j = i+1$ and $s \ne t$, (2) $i = j+1$ and $r \ne u$, or, (3) $|i-j| \ge 2$ and r,s,t,u are not distinct. Hence, we need only consider the sets

$J_1 = \{((i,r,s),(i+1,s,t)) \in J : 1 \le i \le n-2\}$, J_1^*,
$J_2 = \{((i,r,s),(j,t,u)) \in J : 1 \le i \le j-2 \le n-3$ and r,s,t,u are distinct$\}$, J_2^*

where $K^* = \{(b,a) : (a,b) \in K\}$ for $K \subseteq J$. First, for $(i,r,s) \in I$, $\mathrm{E}(A^2_{(i,r,s)}) = \dfrac{(s-r)^2}{(n)_2}$, hence,

$$\Sigma_1 := \sum_{(i,r,s)\in I} \mathrm{E}(A^2_{(i,r,s)}) = \sum_{i=1}^{n-1}\sum_{s=2}^{n}\sum_{r=1}^{s-1} \frac{(s-r)^2}{(n)_2} = \frac{n^3-n}{12}\,.$$

Second, for $((i,r,s),(i+1,s,t)) \in J_1$, $\mathrm{E}(A_{(i,r,s)}A_{(i+1,s,t)}) = \dfrac{(s-r)(t-s)}{(n)_3}$, hence,

$$\begin{aligned}
\Sigma_2 &:= \sum_{((i,r,s),(i+1,s,t))\in J_1} \mathrm{E}(A_{(i,r,s)}A_{(i+1,s,t)}) = \sum_{i=1}^{n-2}\sum_{t=3}^{n}\sum_{s=2}^{t-1}\sum_{r=1}^{s-1} \frac{(s-r)(t-s)}{(n)_3} \\
&= \frac{1}{2(n)_3}\sum_{i=1}^{n-2}\sum_{t=3}^{n}\sum_{s=1}^{t-1} \left\{-s^3 + (t+1)s^2 - ts\right\} \\
&= \frac{1}{24(n)_3}\sum_{i=1}^{n-2}\sum_{t=1}^{n} \left\{t^4 - 2t^3 - t^2 + 2t\right\} \\
&= \frac{1}{120(n)_3}\sum_{i=1}^{n-2} \left\{n^5 - 5n^3 + 4n\right\} = \frac{n^3+n^2-4n-4}{120}
\end{aligned}$$

(appropriate summands for $s=1$ and $t=1,2$ are 0). Third, for $((i,r,s),(j,t,u)) \in J_2$, $\mathrm{E}(A_{(i,r,s)}A_{(j,t,u)}) = \dfrac{(s-r)(u-t)}{(n)_4}$. For fixed $1 \le i \le j-2 \le n-3$,

$$\begin{aligned}
\Sigma_3 := \sum_{\text{all such } ((i,r,s),(j,t,u))} \mathrm{E}(A_{(i,r,s)}A_{(j,t,u)}) &= \underset{r,s,t,u \text{ distinct}}{\sum_{s=2}^{n}\sum_{r=1}^{s-1}\sum_{u=2}^{n}\sum_{t=1}^{u-1}} \frac{(s-r)(u-t)}{(n)_4} \\
= \frac{1}{(n)_4}\sum_{s=2}^{n}\sum_{r=1}^{s-1}(s-r)\bigg\{\sum_{u=2}^{n}\sum_{t=1}^{u-1}(u-t) &- \sum_{t=1}^{r-1}(r-t) - \sum_{t=1}^{s-1}(s-t) \\
&- \sum_{u=r+1}^{n}(u-r) - \sum_{u=s+1}^{n}(u-s) + (s-r)\bigg\} \\
= \frac{1}{(n)_4}\sum_{s=2}^{n}\sum_{r=1}^{s-1}(s-r)\bigg\{\binom{n+1}{3} + (s-r) & \\
&- \big\{r^2 - r + s^2 - s + n^2 - (r+s-1)n\big\}\bigg\}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{s=2}^{n}\sum_{r=1}^{s-1}(s-r)\{r^2 - r + s^2 - s + n^2 - (r+s-1)n\} \\
&\qquad = \sum_{s=2}^{n}\sum_{r=1}^{s-1}\big\{-r^3 + (n+1+s)r^2 - (n^2+n+s^2)r \\
&\qquad\qquad + \big[s^3 - (n+1)s^2 + (n^2+n)s\big]\big\} \\
&\qquad = \frac{1}{12}\sum_{s=1}^{n}\big\{7s^4 - (8n+14)s^3 + (6n^2+12n+5)s^2 \\
&\qquad\qquad - (6n^2+4n-2)s\big\} = \frac{7n^5 - 15n^3 + 8n}{60}.
\end{aligned}$$

Then,

$$\begin{aligned}
\Sigma_3 &= \frac{1}{(n)_4}\left\{\binom{n+1}{3}^2 + \frac{n^4-n^2}{12} - \frac{7n^5-15n^3+8n}{60}\right\} \\
&= \frac{10n^5 - 42n^4 + 10n^3 + 90n^2 - 20n - 48}{360(n-1)_3},
\end{aligned}$$

hence, summing over all such i and j,

$$\begin{aligned}
\Sigma_4 &:= \sum_{((i,r,s),(j,t,u)) \in J_2} \mathrm{E}(A_{(i,r,s)}, A_{(j,t,u)}) \\
&= \binom{n-2}{2}\Sigma_3 = \frac{5n^4 - 16n^3 - 11n^2 + 34n + 24}{360}.
\end{aligned}$$

Consequently, $\mathrm{E}(A_n^2) = \Sigma_1 + 2\Sigma_2 + 2\Sigma_4 = \dfrac{5n^4 + 2n^3 - 8n^2 + 7n + 12}{180}$, hence, $\mathrm{Var}(A_n) = \dfrac{2n^3 + 2n^2 + 7n + 7}{180}$. □

2.3. Central Limit Theorem for $\{X_n\}$

We now prove a central limit theorem for $\{X_n\}$. Our proof is based on the following result of HOEFFDING [**7**; Theorem 3]. Given $c_n : [n]^2 \to \mathbb{R}$, let $d_n : [n]^2 \to \mathbb{R}$ be defined by $d_n(i,j) = c_n(i,j) - \frac{1}{n}\sum_{g=1}^{n} c_n(g,j) - \frac{1}{n}\sum_{h=1}^{n} c_n(i,h) + \frac{1}{n^2}\sum_{g=1}^{n}\sum_{h=1}^{n} c_n(g,h)$.

Theorem 2.3 (HOEFFDING [**7**]). *Suppose random variables $S_n : \mathfrak{S}_n \to \mathbb{R}$ on Ω_n are defined by $S_n(\sigma) = \sum_{i=1}^{n} c_n(i,\sigma(i))$. If $\lim_{n\to\infty} \dfrac{\max_{1\le i,j\le n} d_n^2(i,j)}{\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} d_n^2(i,j)} = 0$, then $\dfrac{S_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0,1)$ where $\mu_n = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} c_n(i,j)$ and $\sigma_n = \frac{1}{n-1}\sum_{i=1}^{n}\sum_{j=1}^{n} d_n^2(i,j)$.*

Theorem 2.4. *The random variables X_n on Ω_n satisfy the central limit theorem*

$$\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0,1)$$

where $\mu_n = E(X_n) = (n^2-1)/6$ and $\sigma_n^2 = \mathrm{Var}\,(X_n) = (2n^3 + 2n^2 + 7n + 7)/180$. Equivalently (see DURRETT [**5**; Ex. 2.1, p.70]),

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \Pr(X_n \le \lfloor \mu_n + x\sigma_n \rfloor) - \Phi(x) \right| = 0 \,.$$

Proof. We prove the theorem for the statistic E_n. Let $c_n : [n]^2 \to \mathbb{N}$ be defined by $c_n(i,j) = \max\{0, j-i\}$. From Theorem 2.3, $S_n(\sigma) = \sum_{i=1}^{n} c_n(i,\sigma(i)) = E_n(\sigma)$ and $d_n : [n]^2 \to \mathbb{Q}$ is given by

$$d_n(i,j) = \begin{cases} j - i - \frac{1}{n}\binom{j}{2} - \frac{1}{n}\binom{n-i+1}{2} + \frac{1}{n^2}\binom{n+1}{3}, & 1 \le i < j \le n; \\ -\frac{1}{n}\binom{j}{2} - \frac{1}{n}\binom{n-i+1}{2} + \frac{1}{n^2}\binom{n+1}{3}, & 1 \le j \le i \le n. \end{cases}$$

All $d_n(i,j) \le d_n(1,n) = (n^2-1)/6n < n/6$. Set $a = \lceil 2n/3 \rceil$ and $b = \lfloor n/3 \rfloor$. For $a+1 \le i \le n$, $1 \le j \le b$ and $n \ge 88$, $d_n(i,j) \ge n/20$. Then $\sum_{i=1}^{n}\sum_{j=1}^{n} d_n^2(i,j) \ge$

$\sum\limits_{i=a+1}^{n}\sum\limits_{j=1}^{b} d_n^2(i,j) \geq n^4/4000$, hence, $\dfrac{\max\limits_{1\leq i,j\leq n} d_n^2(i,j)}{\frac{1}{n}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} d_n^2(i,j)} = O(n^{-1})$. Lemma 2.1 implies the μ_n and σ_n of Theorems 2.2 and 2.3 are identical (as can be verified). Our result follows from Theorem 2.3. □

Theorem 2.4 and (1) imply the following asymptotic result.

Corollary 2.5. *With* $\mu_n = (n^2-1)/6$ *and* $\sigma_n^2 = (2n^3+2n^2+7n+7)/180$*,*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\frac{1}{n!}\sum_{k=0}^{\lfloor\mu_n+x\sigma_n\rfloor} a(n,k) - \Phi(x)\right| = 0\,.$$

In particular, $\sum\limits_{k=\lfloor\mu_n+\alpha\sigma_n\rfloor+1}^{\lfloor\mu_n+\beta\sigma_n\rfloor} a(n,k) \sim \big\{\Phi(\beta)-\Phi(\alpha)\big\}\, n!$ *uniformly for real* $\alpha<\beta$*.*

Acknowledgement. I wish to thank one anonymous referee for his/her careful reading of, and thoughtful comments and suggestions about, the first version of this paper. This has led to the improved final version of the paper.

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Department of Mathematics,
Southern Illinois University Carbondale,
Carbondale, IL 62901-4408
U.S.A.
E–mail: lclark@math.siu.edu

(Received April 27, 2009)
(Revised May 25, 2009)