Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 264–281. doi:10.2298/AADM0902264N

STABILITY OF HOMOMORPHISMS AND (θ, ϕ)-DERIVATIONS

Abbas Najati, Themistocles M. Rassias

In this paper, we prove the generalized Hyers–Ulam stability of homomorphisms and (θ, ϕ)-derivations on a ring $\mathcal{R}$ into a Banach $\mathcal{R}$-bimodule $\mathcal{M}$.

1. INTRODUCTION

The stability problem of functional equations originated from a question of Ulam [**37**] concerning the stability of group homomorphisms: *Let* $(G_1, *)$ *be a group and let* $(G_2, \diamond, d)$ *be a metric group with the metric* $d(\cdot,\cdot)$. *Given* $\epsilon > 0$, *does there exist* $\delta(\epsilon) > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality*

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, *then there is a homomorphism* $H : G_1 \to G_2$ *with*

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations where homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a homomorphism near it. Hyers [**12**] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces: *Assume that* $f : X \to Y$ *satisfies*

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon$$

for some $\varepsilon \ge 0$ *and all* $x, y \in X$. *Then there exists a unique additive mapping* $T : X \to Y$ *such that*

$$\|f(x) - T(x)\| \le \varepsilon$$

2000 Mathematics Subject Classification. Primary: 39B72; Secondary 47H09.
Keywords and Phrases. Generalized metric space, fixed point, stability, Banach algebra, semiprime ring, Jordan derivation, generalized Jordan derivation.

for all $x \in X$.

Aoki [**2**] and Rassias [**31**] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.1. (Th. M. Rassias). *Let* $f : E \to E'$ *be a mapping from a normed vector space* E *into a Banach space* E' *subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, *where* ε *and* p *are constants with* $\epsilon > 0$ *and* $p < 1$. *Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\,\|x\|^p \tag{1.2}$$

for all $x \in E$. *If* $p < 0$ *then inequality* (1.1) *holds for* $x, y \neq 0$ *and* (1.2) *for* $x \neq 0$. *Also, if for each* $x \in E$ *the mapping* $t \mapsto f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then* L *is linear.*

The inequality (1.1) has provided a lot of influence in the development of what is now known as a *generalized Hyers–Ulam stability* of functional equations. In 1994, a generalization of the Th. M. Rassias' theorem was obtained by Găvruta [**8**], who replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Since then the stability problems of various functional equations and mappings and their Pexiderized versions with more general domains and ranges have been investigated by a number of authors (see [**21**]–[**29**]). We also refer the readers to the books [**7**], [**13**], [**16**] and [**32**].

Let A be a real or complex algebra. A mapping $D : A \to A$ is said to be a (*ring*) *derivation* if

$$D(a+b) = D(a) + D(b), \qquad D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. If, in addition, $D(\lambda a) = \lambda D(a)$ for all $a \in A$ and all $\lambda \in \mathbb{F}$, then D is called a *linear derivation*, where $\mathbb{F}$ denotes the scalar field of A. Singer and Wermer [**35**] proved that if A is a commutative Banach algebra and $D : A \to A$ is a continuous linear derivation, then $D(A) \subseteq \operatorname{rad}(A)$. They also conjectured that the same result holds even D is a discontinuous linear derivation. Thomas [**36**] proved the conjecture. As a direct consequence, we see that there are no non-zero linear derivations on a semi-simple commutative Banach algebra, which had been proved by Johnson [**15**]. On the other hand, it is not the case for ring derivations. Hatori and Wada [**9**] determined a representation of ring derivations on a semi-simple commutative Banach algebra (see also [**33**]) and they proved that only the zero operator is a ring derivation on a semi-simple commutative Banach algebra with the maximal ideal space without isolated points. The stability of derivations between operator algebras was first obtained by Šemrl [**34**]. Badora [**3**] and Miura *et al.* [**22**] proved the generalized Hyers–Ulam stability of ring derivations on Banach algebras.

Let $\mathcal{R}$ be an associative ring, $\mathcal{N}$ be a $\mathcal{R}$-bimodule and let θ, ϕ be automorphisms of $\mathcal{R}$. An additive mapping $D : \mathcal{R} \to \mathcal{N}$ is called a *derivation* if $D(ab) = D(a)b + aD(b)$ holds for all pairs $a, b \in \mathcal{R}$ and is called a *Jordan derivation* in case $D(a^2) = D(a)a + aD(a)$ is fulfilled for all $a \in \mathcal{R}$. Every derivation is a Jordan derivation. The converse is in general not true (see [**6**, **10**]). The concept of generalized derivation has been introduced by Bresar [**4**]. Hvala [**11**] and Lee [**18**] introduced a concept of (θ, ϕ)-derivation (see also [**19**]). An additive mapping $F : \mathcal{R} \to \mathcal{N}$ is called a *(θ, ϕ)-derivation* in case $F(ab) = F(a)\theta(b) + \phi(a)F(b)$ holds for all pairs $a, b \in \mathcal{R}$. An additive mapping $F : \mathcal{R} \to \mathcal{N}$ is called a *(θ, ϕ)-Jordan derivation* in case $F(a^2) = F(a)\theta(a) + \phi(a)F(a)$ holds for all $a \in \mathcal{R}$. An additive mapping $F : \mathcal{R} \to \mathcal{N}$ is called a *generalized (θ, ϕ)-derivation* in case $F(ab) = F(a)\theta(b) + \phi(a)D(b)$ holds for all pairs $a, b \in \mathcal{R}$, where $D : \mathcal{R} \to \mathcal{N}$ is a (θ, ϕ)-derivation. An additive mapping $F : \mathcal{R} \to \mathcal{N}$ is called a *generalized (θ, ϕ)-Jordan derivation* in case $F(a^2) = F(a)\theta(a) + \phi(a)D(a)$ holds for all $a \in \mathcal{R}$, where $D : \mathcal{R} \to \mathcal{N}$ is a (θ, ϕ)-Jordan derivation. It is clear that every generalized (θ, ϕ)-derivation is a generalized (θ, ϕ)-Jordan derivation.

The aim of the present paper is to establish the stability problem of homomorphisms and generalized (θ, ϕ)-derivations by using the fixed point method (see [**1**, **5**, **17**, **21**]).

Let E be a set. A function $d : E \times E \to [0, \infty]$ is called a *generalized metric* on E if d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in E$;

(iii) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz.

Theorem 1.2. [**20**] *Let (E, d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all non-negative integers n or there exists a non-negative integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \ge n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{\, y \in E : d(J^{n_0} x, y) < \infty \,\}$;
(4) $d(y, y^*) \le \dfrac{1}{1-L}\, d(y, Jy)$ *for all* $y \in Y$.

2. STABILITY OF HOMOMORPHISMS

In this section, we assume that $\mathcal{R}$ is an associative ring, $\mathcal{X}$ is a normed algebra, $\mathcal{Y}$ is a Banach algebra, and $n \geq 3$ is a fixed integer.

Lemma 2.1. *Let X and Y be linear spaces. A mapping $f : X \to Y$ (with $f(0) = 0$ if $n = 3$) satisfies*

$$\sum_{j=1}^{n} f\Big(-x_j + \sum_{\substack{1\leq i\leq n\\ i\neq j}} x_i\Big) = (n-2)\sum_{i=1}^{n} f(x_i) \tag{2.1}$$

for all $x_1, \ldots, x_n \in X$, if and only if f is additive.

Proof. Let f satisfy (2.1). Letting $x_1 = \cdots = x_n = 0$ in (2.1), we get $f(0) = 0$. Letting $x_2 = \cdots = x_n = 0$ in (2.1), we infer that f is odd. So by letting $x_3 = \cdots = x_n = 0$ in (2.1) and using the oddness of f, we get that the mapping f is additive. The converse is obvious. □

Theorem 2.2. *Let $f : \mathcal{R} \to \mathcal{Y}$ be a mapping for which there exist functions $\varphi : \mathcal{R}^n \to [0, \infty)$ and $\psi : \mathcal{R}^2 \to [0, \infty)$ such that*

$$\lim_{k\to\infty} \frac{1}{r^k}\, \varphi(r^k a_1, \ldots, r^k a_n) = 0, \tag{2.2}$$

$$\lim_{k\to\infty} \frac{1}{r^k}\, \psi(r^k a, b) = \lim_{k\to\infty} \frac{1}{r^k}\, \psi(a, r^k b) = \lim_{k\to\infty} \frac{1}{r^k}\, \psi(r^k a, r^k b) = 0, \tag{2.3}$$

$$\Big\| \sum_{j=1}^{n} f\Big(-a_j + \sum_{\substack{1\leq i\leq n\\ i\neq j}} a_i\Big) - (n-2)\sum_{i=1}^{n} f(a_i)\Big\| \leq \varphi(a_1, \ldots, a_n), \tag{2.4}$$

$$\|f(ab) - f(a)f(b)\| \leq \psi(a, b) \tag{2.5}$$

for all $a, b, a_1, \ldots, a_n \in \mathcal{R}$, where $r = n - 2 > 1$. If there exists a constant $L < 1$ such that

$$\varphi(ra, \ldots, ra) \leq rL\varphi(a, \ldots, a)$$

for all $a \in \mathcal{R}$, then there exists a unique homomorphism $H : \mathcal{R} \to \mathcal{Y}$ satisfying

$$\|f(a) - H(a)\| \leq \frac{1}{n(n-2)(1-L)}\, \varphi(a, \ldots, a), \tag{2.6}$$

$$H(a)[H(b) - f(b)] = [H(a) - f(a)]H(b) = 0 \tag{2.7}$$

for all $a, b \in \mathcal{R}$.

Proof. Letting $a_1 = \cdots = a_n = a$ in (2.4), we get

$$\|f(ra) - rf(a)\| \leq \frac{1}{n}\, \varphi(a, \ldots, a) \tag{2.8}$$

for all $a \in \mathcal{R}$. Let $E := \{ g : \mathcal{R} \to \mathcal{Y} \}$. We introduce a generalized metric on E as follows:

$$d_\varphi(g,h) := \inf\{ C \in [0,\infty] : \|g(a) - h(a)\| \le C\varphi(a,\dots,a) \ \text{ for all } a \in \mathcal{R} \}.$$

It is easy to show that (E, d_φ) is a generalized complete metric space [**5**].

Now we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda g)(a) = \frac{1}{r}\, g(ra), \quad \text{for all } g \in E \text{ and } a \in \mathcal{R}.$$

Let $g, h \in E$ and let $C \in [0,\infty]$ be an arbitrary constant with $d_\varphi(g,h) \le C$. From the definition of d_φ, we have

$$\|g(a) - h(a)\| \le C\varphi(a,\dots,a)$$

for all $a \in \mathcal{R}$. By the assumption and last inequality, we have

$$\|(\Lambda g)(a) - (\Lambda h)(a)\| = \frac{1}{r}\,\|g(ra) - h(ra)\| \le \frac{C}{r}\,\varphi(ra,\dots,ra) \le CL\varphi(a,\dots,a)$$

for all $a \in \mathcal{R}$. So $d_\varphi(\Lambda g, \Lambda h) \le L d_\varphi(g,h)$ for any $g, h \in E$. It follows from (2.8) that $d_\varphi(\Lambda f, f) \le \dfrac{1}{n(n-2)}$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^k f\}$ converges to a fixed point H of Λ, i.e.,

$$H : \mathcal{R} \to \mathcal{Y}, \quad H(a) = \lim_{k\to\infty} (\Lambda^k f)(a) = \lim_{k\to\infty} \frac{1}{r^k}\, f(r^k a)$$

and $H(ra) = rH(a)$ for all $a \in \mathcal{R}$. Also H is the unique fixed point of Λ in the set $E_\varphi = \{g \in E : d_\varphi(f,g) < \infty \}$ and

$$d_\varphi(H,f) \le \frac{1}{1-L}\, d_\varphi(\Lambda f, f) \le \frac{1}{n(n-2)(1-L)}\,,$$

i.e., inequality (2.6) holds true for all $a \in \mathcal{R}$. It follows from the definition of H, (2.2) and (2.4) that

$$\sum_{j=1}^{n} H\Big(- a_j + \sum_{\substack{1\le i\le n\\ i\ne j}} a_i \Big) = (n-2)\sum_{i=1}^{n} H(a_i)$$

for all $a_1, \dots, a_n \in \mathcal{R}$. Since $H(0) = 0$, by Lemma 2.1 the mapping H is additive. So it follows from the definition of H, (2.3) and (2.5) that

$$\begin{aligned} \|H(ab) - H(a)H(b)\| &= \lim_{k\to\infty} \frac{1}{r^{2k}}\,\|f(r^{2k}ab) - f(r^k a) f(r^k b)\| \\ &\le \lim_{k\to\infty} \frac{1}{r^{2k}}\,\psi(r^k a, r^k b) = 0 \end{aligned}$$

for all $a, b \in \mathcal{R}$. So H is homomorphism. Similarly, we have from (2.3) and (2.5) that

$$H(ab) = H(a)f(b), \qquad H(ab) = f(a)H(b) \tag{2.9}$$

for all $a, b \in \mathcal{R}$. Since H is homomorphism, we get (2.7) from (2.9).

Finally it remains to prove the uniqueness of H. Let $H_1 : \mathcal{R} \to \mathcal{Y}$ be another homomorphism satisfying (2.6). Since $d_\varphi(f, H_1) \le \dfrac{1}{n(n-2)(1-L)}$ and H_1 is additive, we get $H_1 \in E_\varphi$ and $(\Lambda H_1)(a) = \frac{1}{r} H_1(ra) = H_1(a)$ for all $a \in \mathcal{R}$, i.e., H_1 is a fixed point of Λ. Since H is the unique fixed point of Λ in E_φ, we get $H_1 = H$. □

We need the following lemma in the proof of the next theorem.

Lemma 2.3 [**30**] *Let X and Y be linear spaces and $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1 := \{\, \mu \in \mathbb{C} : |\mu| = 1 \,\}$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.4. *Let X and Y be linear spaces. A mapping $f : X \to Y$ satisfies*

$$\sum_{j=1}^{n} f\Big(-\mu x_j + \sum_{\substack{1 \le i \le n \\ i \ne j}} \mu x_i \Big) = (n-2)\mu \sum_{i=1}^{n} f(x_i) \tag{2.10}$$

for all $x_1, \ldots, x_n \in X$ and all $\mu \in \mathbb{T}^1$, if and only if f is $\mathbb{C}$-linear.

Proof. Let f satisfy (2.10). Letting $x_1 = \cdots = x_n = 0$ in (2.10), we get $f(0) = 0$. By Lemma 2.1, the mapping f is additive. Letting $x_2 = \cdots = x_n = 0$ in (2.10) and using the oddness of f, we get that $f(\mu x_1) = \mu f(x_1)$ for all $x_1 \in X$ and all $\mu \in \mathbb{T}^1$. So by Lemma 2.3, the mapping f is $\mathbb{C}$-linear. The converse is obvious. □

The following theorem is an alternative result of Theorem 2.2.

Theorem 2.5. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exist functions $\varphi : \mathcal{X}^n \to [0, \infty)$ and $\psi : \mathcal{X}^2 \to [0, \infty)$ such that*

$$\lim_{k\to\infty} r^k \varphi\Big(\frac{1}{r^k} a_1, \ldots, \frac{1}{r^k} a_n\Big) = 0,$$

$$\lim_{k\to\infty} r^k \psi\Big(\frac{1}{r^k} a, b\Big) = \lim_{k\to\infty} r^k \psi\Big(a, \frac{1}{r^k} b\Big) = \lim_{k\to\infty} r^{2k} \psi\Big(\frac{1}{r^k} a, \frac{1}{r^k} b\Big) = 0,$$

$$\Big\| \sum_{j=1}^{n} f\Big(-\mu a_j + \textstyle\sum_{\substack{1 \le i \le n \\ i \ne j}} \mu a_i \Big) - (n-2)\mu \displaystyle\sum_{i=1}^{n} f(a_i) \Big\| \le \varphi(a_1, \ldots, a_n),$$

$$\|f(ab) - f(a)f(b)\| \le \psi(a, b)$$

for all $a, b, a_1, \ldots, a_n \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$, where $r = n - 2 > 1$. If there exists a constant $L < 1$ such that

$$r\varphi\Big(\frac{1}{r} a, \ldots, \frac{1}{r} a\Big) \le L\varphi(a, \ldots, a)$$

for all $a \in \mathcal{X}$, then there exists a unique homomorphism $H : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(a) - H(a)\| \le \frac{L}{n(n-2)(1-L)}\,\varphi(a,\dots,a),$$
$$H(a)[H(b) - f(b)] = [H(a) - f(a)]H(b) = 0$$

for all $a, b \in \mathcal{X}$.

Proof. It follows from the assumptions that $\varphi(0,\dots,0) = 0$, and so $f(0) = 0$. Letting $\mu = 1$ and using the same method as in the proof of Theorem 2.2, we have

$$\|f(ra) - rf(a)\| \le \frac{1}{n}\,\varphi(a,\dots,a) \tag{2.11}$$

for all $a \in \mathcal{R}$. Let $E := \{\, g : \mathcal{X} \to \mathcal{Y} \mid g(0) = 0 \,\}$. We introduce the same definition d_φ as in the proof of Theorem 2.2 such that (E, d_φ) becomes a generalized complete metric space. Let $\Lambda : E \to E$ be the mapping defined by

$$(\Lambda g)(a) = rg\Big(\frac{1}{r}\,a\Big), \quad \text{for all } g \in E \text{ and } a \in \mathcal{X}.$$

One can show that

$$d_\varphi(\Lambda g, \Lambda h) \le L d_\varphi(g, h)$$

for any $g, h \in E$. It follows from the assumption and (2.11) that $d_\varphi(\Lambda f, f) \le \frac{L}{n(n-2)}$. Due to Theorem 1.2, the sequence $\{\Lambda^k f\}$ converges to a fixed point H of Λ, i.e., $H : \mathcal{X} \to \mathcal{Y}$,

$$H(a) = \lim_{k\to\infty} (\Lambda^k f)(a) = \lim_{n\to\infty} r^k f\Big(\frac{1}{r^k}\,a\Big), \quad H(ra) = rH(a)$$

for all $a \in \mathcal{X}$. Also

$$d_\varphi(H, f) \le \frac{1}{1-L}\,d_\varphi(\Lambda f, f) \le \frac{L}{n(n-2)(1-L)}$$

i.e., the inequality

$$\|f(a) - H(a)\| \le \frac{L}{n(n-2)(1-L)}\,\varphi(a,\dots,a)$$

holds true for all $a \in \mathcal{X}$.

The rest of the proof is similar to the proof of Theorem 3.1 and we omit the details. □

Corollary 2.6. *Let $p, q, \delta, \varepsilon$ be non-negative real numbers with $0 < p, q < 1$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping such that*

$$\Big\|\sum_{j=1}^{n} f\Big(-\mu a_j + \sum_{\substack{1\le i\le n\\ i\ne j}} \mu a_i\Big) - (n-2)\mu \sum_{i=1}^{n} f(a_i)\Big\| \le \delta + \varepsilon \sum_{i=1}^{n} \|a_i\|^p,$$
$$\|f(ab) - f(a)f(b)\| \le \delta + \varepsilon(\|a\|^q + \|b\|^q)$$

for all $a, b, a_1, \ldots, a_n \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique homomorphism $H : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(a) - H(a)\| \leq \frac{\delta}{(r+2)(r-r^p)} + \frac{\varepsilon}{r-r^p}\|a\|^p,$$
$$H(a)[H(b) - f(b)] = [H(a) - f(a)]H(b) = 0$$

for all $a, b \in \mathcal{X}$, where $r = n - 2 > 1$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(a_1, \ldots, a_n) := \delta + \varepsilon \sum_{i=1}^{n} \|a_i\|^p, \quad \psi(a, b) := \delta + \varepsilon(\|a\|^q + \|b\|^q)$$

for all $a, b, a_1, \ldots, a_n \in \mathcal{X}$. Then we can choose $L = r^{p-1}$ and we get the desired results. □

Corollary 2.7. *Let p, q, ε be non-negative real numbers with $p > 1$ and $q > 2$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping such that*

$$\Big\| \sum_{j=1}^{n} f\Big(-\mu a_j + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} \mu a_i\Big) - (n-2)\mu \sum_{i=1}^{n} f(a_i) \Big\| \leq \varepsilon \sum_{i=1}^{n} \|a_i\|^p,$$
$$\|f(ab) - f(a)f(b)\| \leq \varepsilon(\|a\|^q + \|b\|^q)$$

for all $a, b, a_1, \ldots, a_n \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique homomorphism $H : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(a) - H(a)\| \leq \frac{\varepsilon}{r^p - r}\|a\|^p$$

for all $a \in \mathcal{X}$, where $r = n - 2 > 1$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(a_1, \ldots, a_n) := \varepsilon \sum_{i=1}^{n} \|a_i\|^p, \quad \psi(a, b) := \varepsilon(\|a\|^q + \|b\|^q)$$

for all $a, b, a_1, \ldots, a_n \in \mathcal{X}$. Then we can choose $L = r^{1-p}$ and we get the desired results. □

3. STABILITY OF GENERALIZED (θ, ϕ)-DERIVATIONS

In this section, we assume that $\mathcal{R}$ is a 2-divisible associative ring, $\mathcal{M}$ is a Banach $\mathcal{R}$-bimodule, and θ, ϕ are automorphisms of $\mathcal{R}$. For convenience, we use the following abbreviation for given mappings $f, g : \mathcal{R} \to \mathcal{M}$,

$$D_{f,g}^{\theta,\phi}(a, b, c, d) := f(ab + c + d) - f(a)\theta(b) - \phi(a)g(b) - f(c) - f(d),$$
$$J_{f,g}^{\theta,\phi}(a, b, c) := f(a^2 + b + c) - f(a)\theta(a) - \phi(a)g(a) - f(b) - f(c)$$

for all $a, b, c, d \in \mathcal{R}$. Now we prove the generalized Hyers–Ulam stability of generalized (θ, ϕ)-derivations and generalized (θ, ϕ)-Jordan derivations in Banach $\mathcal{R}$-bimodules.

Theorem 3.1. *Let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings for which there exist functions $\varphi, \psi : \mathcal{R}^3 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} 4^n \varphi\Big(\frac{a}{2^n}, 0, 0\Big) = \lim_{n\to\infty} 2^n \varphi\Big(0, \frac{b}{2^n}, \frac{c}{2^n}\Big) = 0, \tag{3.1}$$

$$\|J^{\theta,\phi}_{f,g}(a, b, c)\| \le \varphi(a, b, c), \tag{3.2}$$

$$\lim_{n\to\infty} 4^n \psi\Big(\frac{a}{2^n}, 0, 0\Big) = \lim_{n\to\infty} 2^n \psi\Big(0, \frac{b}{2^n}, \frac{c}{2^n}\Big) = 0, \tag{3.3}$$

$$\|J^{\theta,\phi}_{g,g}(a, b, c)\| \le \psi(a, b, c) \tag{3.4}$$

for all $a, b, c \in \mathcal{R}$. If there exist constants $L, K < 1$ such

$$2\varphi(0, a, a) \le L\varphi(0, 2a, 2a), \qquad 2\psi(0, a, a) \le K\psi(0, 2a, 2a)$$

for all $a \in \mathcal{R}$, then there exist a unique (θ, ϕ)-Jordan derivation $G : \mathcal{R} \to \mathcal{M}$ and a unique generalized (θ, ϕ)-Jordan derivation $F : \mathcal{R} \to \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \le \frac{L}{2 - 2L}\, \varphi(0, a, a), \tag{3.5}$$

$$\|g(a) - G(a)\| \le \frac{K}{2 - 2K}\, \psi(0, a, a) \tag{3.6}$$

for all $a \in \mathcal{R}$.

Proof. It follows from (3.1) and (3.3) that $\varphi(0, 0, 0) = 0 = \psi(0, 0, 0)$ and so we get from (3.2) and (3.4) that $f(0) = g(0) = 0$. Letting $a = 0$ and $b = c$ in (3.2), we get

$$\|f(2c) - 2f(c)\| \le \varphi(0, c, c) \tag{3.7}$$

for all $c \in \mathcal{R}$. Let $E := \{\, h : \mathcal{R} \to \mathcal{M} \mid h(0) = 0 \,\}$. We introduce a generalized metric on E as follows:

$$d_\varphi(h, k) := \inf\{\, C \in [0, \infty] : \|h(a) - k(a)\| \le C\varphi(0, a, a) \ \text{ for all } a \in \mathcal{R} \,\}.$$

It is easy to show that (E, d_φ) is a generalized complete metric space [**5**].

Now we consider the mapping $\Lambda : E \to E$ defined by

$$(\Lambda h)(a) = 2h\Big(\frac{a}{2}\Big), \quad \text{for all } h \in E \text{ and } a \in \mathcal{R}.$$

Let $h, k \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d_\varphi(h, k) \le C$. From the definition of d_φ, we have

$$\|h(a) - k(a)\| \le C\varphi(0, a, a)$$

for all $a \in \mathcal{R}$. By the assumption and last inequality, we have

$$\|(\Lambda h)(a) - (\Lambda k)(a)\| = 2\Big\|h\Big(\frac{a}{2}\Big) - k\Big(\frac{a}{2}\Big)\Big\| \le 2C\varphi\Big(0, \frac{a}{2}, \frac{a}{2}\Big) \le CL\varphi(0,a,a)$$

for all $a \in \mathcal{R}$. So $d_\varphi(\Lambda h, \Lambda k) \le L d_\varphi(h,k)$ for any $h, k \in E$. It follows from the assumption and (3.7) that $d_\varphi(\Lambda f, f) \le L/2$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^n f\}$ converges to a fixed point F of Λ, i.e.,

$$F : \mathcal{R} \to \mathcal{M}, \quad F(a) = \lim_{n\to\infty} (\Lambda^n f)(a) = \lim_{n\to\infty} 2^n f\Big(\frac{a}{2^n}\Big)$$

and $F(2a) = 2F(a)$ for all $a \in \mathcal{R}$. Also F is the unique fixed point of Λ in the set $E_\varphi = \{h \in E : d_\varphi(f,h) < \infty\}$ and

$$d_\varphi(F,f) \le \frac{1}{1-L}\, d_\varphi(\Lambda f, f) \le \frac{L}{2-2L},$$

i.e., inequality (3.5) holds true for all $a \in \mathcal{R}$. Similarly, we obtain that

$$d_\psi(\Lambda h, \Lambda k) \le K d_\psi(h,k), \quad d_\psi(\Lambda g, g) \le K/2.$$

for any $h, k \in E$, where

$$d_\psi(h,k) := \inf\{\, C \in [0,\infty] : \|h(a) - k(a)\| \le C\psi(0,a,a) \ \text{ for all } a \in \mathcal{R}\,\}.$$

So according to Theorem 1.2, the sequence $\{\Lambda^n g\}$ converges to a fixed point G of Λ, i.e.,

$$G : \mathcal{R} \to \mathcal{M}, \quad G(a) = \lim_{n\to\infty} (\Lambda^n g)(a) = \lim_{n\to\infty} 2^n g\Big(\frac{a}{2^n}\Big)$$

and $G(2a) = 2G(a)$ for all $a \in \mathcal{R}$. Also G is the unique fixed point of Λ in the set $E_\psi = \{h \in E : d_\psi(g,h) < \infty\}$ and

$$d_\psi(G,g) \le \frac{1}{1-K}\, d_\psi(\Lambda g, g) \le \frac{K}{2-2K},$$

i.e., inequality (3.6) holds true for all $a \in \mathcal{R}$. It follows from the definitions of F, G, (3.1) and (3.2) that

$$\|J^{\theta,\phi}_{F,G}(a,0,0)\| = \lim_{n\to\infty} 4^n \Big\|J^{\theta,\phi}_{f,g}\Big(\frac{a}{2^n}, 0, 0\Big)\Big\| \le \lim_{n\to\infty} 4^n \varphi\Big(\frac{a}{2^n}, 0, 0\Big) = 0,$$

$$\|J^{\theta,\phi}_{F,G}(0,b,c)\| = \lim_{n\to\infty} 2^n \Big\|J^{\theta,\phi}_{f,g}\Big(0, \frac{b}{2^n}, \frac{c}{2^n}\Big)\Big\| \le \lim_{n\to\infty} 2^n \varphi\Big(0, \frac{b}{2^n}, \frac{c}{2^n}\Big) = 0$$

for all $a, b, c \in \mathcal{R}$. Hence

$$F(a^2) = F(a)\theta(a) + \phi(a)G(a), \quad F(b+c) = F(b) + F(c) \tag{3.8}$$

for all $a, b, c \in \mathcal{R}$. Similarly, it follows from the definition of G, (3.3) and (3.4) that

$$G(a^2) = G(a)\theta(a) + \phi(a)G(a), \quad G(b+c) = G(b) + G(c) \tag{3.9}$$

for all $a, b, c \in \mathcal{R}$. Hence G is a (θ, ϕ)-Jordan derivation. So we infer from (3.8) and (3.9) that F is a generalized (θ, ϕ)-Jordan derivation.

Finally it remains to prove the uniqueness of F and G. Let $F_1, G_1 : \mathcal{R} \to \mathcal{M}$ be another additive mappings satisfying (3.5) and (3.6), respectively. Since $d_\varphi(f, F_1) \leq \dfrac{L}{2-2L}$, $d_\psi(g, G_1) \leq \dfrac{K}{2-2K}$ and F_1, G_1 are additive, we get $F_1 \in E_\varphi$, $G_1 \in E_\psi$ and $(\Lambda F_1)(a) = 2F_1(a/2) = F_1(a)$, $(\Lambda G_1)(a) = 2G_1(a/2) = G_1(a)$ for all $a \in \mathcal{R}$, i.e., F_1, G_1 are fixed points of Λ. Since F and G are the unique fixed points of Λ in E_φ and E_ψ, respectively, we get $F_1 = F$ and $G_1 = G$. $\square$

Theorem 3.2 *Let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings with $f(0) = g(0) = 0$ for which there exist functions $\Phi, \Psi : \mathcal{R}^3 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{4^n} \Phi(2^n a, 0, 0) = \lim_{n\to\infty} \frac{1}{2^n} \Phi(0, 2^n b, 2^n c) = 0, \tag{3.10}$$

$$\|J_{f,g}^{\theta,\phi}(a, b, c)\| \leq \Phi(a, b, c), \tag{3.11}$$

$$\lim_{n\to\infty} \frac{1}{4^n} \Psi(2^n a, 0, 0) = \lim_{n\to\infty} \frac{1}{2^n} \Psi(0, 2^n b, 2^n c) = 0, \tag{3.12}$$

$$\|J_{g,g}^{\theta,\phi}(a, b, c)\| \leq \Psi(a, b, c) \tag{3.13}$$

for all $a, b, c \in \mathcal{R}$. If there exist constants $L, K < 1$ such

$$\Phi(0, 2a, 2a) \leq 2L\Phi(0, a, a), \quad \Psi(0, 2a, 2a) \leq 2K\Psi(0, a, a)$$

for all $a \in \mathcal{R}$, then there exist a unique (θ, ϕ)-Jordan derivation $G : \mathcal{R} \to \mathcal{M}$ and a unique generalized (θ, ϕ)-Jordan derivation $F : \mathcal{R} \to \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \leq \frac{1}{2-2L} \Phi(0, a, a), \tag{3.14}$$

$$\|g(a) - G(a)\| \leq \frac{1}{2-2K} \Psi(0, a, a) \tag{3.15}$$

for all $a \in \mathcal{R}$.

Proof. Using the same method as in the proof of Theorem 3.1, we have

$$\left\|\frac{1}{2} f(2c) - f(c)\right\| \leq \frac{1}{2} \Phi(0, c, c), \quad \left\|\frac{1}{2} g(2c) - g(c)\right\| \leq \frac{1}{2} \Psi(0, c, c) \tag{3.16}$$

for all $c \in \mathcal{R}$. We introduce the same definitions for E, d_Φ and d_Ψ as in the proof of Theorem 3.1 such that (E, d_Φ) and (E, d_Ψ) become generalized complete metric spaces. Let $\Lambda : E \to E$ be the mapping defined by

$$(\Lambda h)(a) = \frac{1}{2} h(2a), \quad \text{for all } h \in E \text{ and } a \in \mathcal{R}.$$

One can show that

$$d_\Phi(\Lambda h, \Lambda k) \leq L d_\Phi(h, k), \quad d_\Psi(\Lambda h, \Lambda k) \leq K d_\Psi(h, k)$$

for any $h,k \in E$. It follows from (3.16) that $d_\Phi(\Lambda f, f) \le \frac{1}{2}$ and $d_\Psi(\Lambda g, g) \le \frac{1}{2}$. Due to Theorem 1.2, the sequences $\{\Lambda^n f\}$ and $\{\Lambda^n g\}$ converge to fixed points F and G of Λ, i.e., $F, G : \mathcal{R} \to \mathcal{M}$,

$$F(a) = \lim_{n\to\infty} (\Lambda^n f)(a) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n a), \quad G(a) = \lim_{n\to\infty} (\Lambda^n g)(a) = \lim_{n\to\infty} \frac{1}{2^n} g(2^n a),$$

$F(2a) = 2F(a)$ and $G(2a) = 2G(a)$ for all $a \in \mathcal{R}$. Also

$$d_\Phi(F,f) \le \frac{1}{1-L}\, d_\Phi(\Lambda f, f) \le \frac{1}{2-2L}\,,$$
$$d_\Psi(G,g) \le \frac{1}{1-K}\, d_\Psi(\Lambda g, g) \le \frac{1}{2-2K}\,,$$

i.e., the inequalities (3.14) and (3.15) hold true for all $a \in \mathcal{R}$.

The rest of the proof is similar to the proof of Theorem 3.1 and we omit the details. □

Corollary 3.3. *Let $\varepsilon, \delta, p, q$ be non-negative real numbers with $0 < p, q < 1$ or $p, q > 2$. If $\mathcal{R}$ is a normed ring and $f, g : \mathcal{R} \to \mathcal{M}$ are mappings satisfy the inequalities*

$$\|J^{\theta,\phi}_{f,g}(a,b,c)\| \le \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p), \quad \|J^{\theta,\phi}_{g,g}(a,b,c)\| \le \delta(\|a\|^q + \|b\|^q + \|c\|^q)$$

for all $a, b, c \in \mathcal{R}$, then there exist a unique (θ,ϕ)-Jordan derivation $G : \mathcal{R} \to \mathcal{M}$ and a unique generalized (θ,ϕ)-Jordan derivation $F : \mathcal{R} \to \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \le \frac{2\varepsilon}{|2-2^p|}\, \|a\|^p, \quad \|g(a) - G(a)\| \le \frac{2\delta}{|2-2^q|}\, \|a\|^q$$

for all $a \in \mathcal{R}$.

Proof. Let

$$L := \begin{cases} 2^{p-1}, & 0 < p < 1; \\ 2^{1-p}, & p > 2. \end{cases} \qquad K := \begin{cases} 2^{q-1}, & 0 < q < 1; \\ 2^{1-q}, & q > 2. \end{cases}$$

So the result follows from Theorems 3.1 and 3.2. □

Corollary 3.4. *Let ε and δ be non-negative real numbers and let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings satisfying $f(0) = g(0) = 0$ and the inequalities*

$$\|J^{\theta,\phi}_{f,g}(a,b,c)\| \le \varepsilon, \quad \|J^{\theta,\phi}_{g,g}(a,b,c)\| \le \delta$$

for all $a, b, c \in \mathcal{R}$. Then there exist a unique (θ,ϕ)-Jordan derivation $G : \mathcal{R} \to \mathcal{M}$ and a unique generalized (θ,ϕ)-Jordan derivation $F : \mathcal{R} \to \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \le \varepsilon, \quad \|g(a) - G(a)\| \le \delta$$

for all $a \in \mathcal{R}$.

Proof. The proof follows from Theorem 3.2 by taking

$$\Phi(a,b,c) := \varepsilon, \quad \Psi(a,b,c) := \delta$$

for all $a, b, c \in \mathcal{R}$. Then we can choose $L = K = 1/2$ and we get the desired results. □

Theorem 3.5. *Let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings with $f(0) = g(0) = 0$ for which there exists a function $\Phi : \mathcal{R}^4 \to [0, \infty)$ satisfying*

$$\lim_{n\to\infty} \frac{1}{2^n}\,\Phi(2^n a, 2^n b, 2^n c, 2^n d) = \lim_{n\to\infty} \frac{1}{2^n}\,\Phi(2^n a, b, 0, 0) = \lim_{n\to\infty} \frac{1}{2^n}\,\Phi(a, 2^n b, 0, 0) = 0, \tag{3.17}$$

$$\|D^{\theta,\phi}_{f,g}(a,b,c,d)\| \le \Phi(a,b,c,d) \tag{3.18}$$

for all $a, b, c, d \in \mathcal{R}$. If $\mathcal{R}$ has the identity e, $\mathcal{M}$ is unit linked and there exists a constant $L < 1$ such

$$\Phi(0,0,2a,2a) \le 2L\Phi(0,0,a,a)$$

for all $a \in \mathcal{R}$, then g is a (θ,ϕ)-derivation and f is a generalized (θ,ϕ)-derivation. Moreover, $f = a\theta + g$, where $a = \lim_{n\to\infty} \frac{1}{2^n} f(2^n e)$.

Proof. Letting $a = b = 0$ and $c = d$ in (3.18), we get

$$\|f(2c) - 2f(c)\| \le \Phi(0,0,c,c)$$

for all $c \in \mathcal{R}$. Using the same method as in the proof of Theorem 3.2, we infer that the limit

$$F(a) := \lim_{n\to\infty} \frac{1}{2^n}\, f(2^n a) \tag{3.19}$$

exists for all $a \in \mathcal{R}$ and the mapping $F : \mathcal{R} \to \mathcal{M}$ is additive. Letting $c = d = 0$ and replacing a and b by $2^n e$ and $2^n b$, respectively, in (3.18), we get

$$\|f(4^n b) - f(2^n e)\theta(2^n b) - \phi(2^n e) g(2^n b)\| \le \Phi(2^n e, 2^n b, 0, 0)$$

for all $b \in \mathcal{R}$ and all $n \in \mathbb{N}$. Since $\phi(e) = e$, we have

$$\Big\|\frac{1}{4^n}\, f(4^n b) - \frac{1}{2^n}\, f(2^n e)\theta(b) - \frac{1}{2^n}\, g(2^n b)\Big\| \le \frac{1}{4^n}\,\Phi(2^n e, 2^n b, 0, 0) \tag{3.20}$$

for all $b \in \mathcal{R}$ and all $n \in \mathbb{N}$. It follows from (3.17), (3.19) and (3.20) that the limit

$$G(b) := \lim_{n\to\infty} \frac{1}{2^n}\, g(2^n b)$$

exists and $G(b) = F(b) - F(e)\theta(b)$ for all $b \in \mathcal{R}$. Hence G is additive. It follows from the definitions of F, G, (3.17) and (3.18) that

$$\begin{aligned}\|F(ab) - F(a)\theta(b) - \phi(a)G(b)\| \\ &= \lim_{n\to\infty} \frac{1}{4^n} \|f(4^n ab) - f(2^n a)\theta(2^n b) - \phi(2^n a)g(2^n b)\| \\ &\le \lim_{n\to\infty} \frac{1}{4^n} \Phi(2^n a, 2^n b, 0, 0) = 0\end{aligned}$$

for all $a, b \in \mathcal{R}$. Therefore

$$F(ab) = F(a)\theta(b) + \phi(a)G(b) \tag{3.21}$$

for all $a, b \in \mathcal{R}$. Further, by (3.21) we have

$$\begin{aligned}G(ab) &= F(ab) - F(e)\theta(ab) = F(a)\theta(b) + \phi(a)G(b) - F(e)\theta(a)\theta(b) \\ &= [F(a) - F(e)\theta(a)]\theta(b) + \phi(a)G(b) = G(a)\theta(b) + \phi(a)G(b)\end{aligned}$$

for all $a, b \in \mathcal{R}$. Thus G is a (θ,ϕ)-derivation and (3.21) shows that F is a generalized (θ,ϕ)-derivation.

By (3.17), (3.18) and the definitions of F, G, we have

$$F(ab) - F(a)\theta(b) = \phi(a)g(b), \tag{3.22}$$

$$F(ab) - \phi(a)G(b) = f(a)\theta(b) \tag{3.23}$$

for all $a, b \in \mathcal{R}$. Since $G(e) = 0$ and $\theta(e) = \phi(e) = e$, letting $a = e$ in (3.22) and $b = e$ in (3.23), we get $g(b) = F(b) - F(e)\theta(b) = G(b)$ and $F(a) = f(a)$, respectively, for all $a, b \in \mathcal{R}$. So g is a (θ,ϕ)-derivation and f is a generalized (θ,ϕ)-derivation. Moreover, $f = F(e)\theta + g$. □

Corollary 3.6. *Let ε, δ, p be non-negative real numbers with $0 < p < 1$. If $\mathcal{R}$ is a normed ring with the identity e, $\mathcal{M}$ is unit linked and $f, g : \mathcal{R} \to \mathcal{M}$ are mappings with $f(0) = g(0) = 0$ and satisfy the inequality*

$$\|D^{\theta,\phi}_{f,g}(a, b, c, d)\| \le \delta + \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p),$$

for all $a, b, c, d \in \mathcal{R}$, then g is a (θ,ϕ)-derivation and f is a generalized (θ,ϕ)-derivation. Moreover, $f = a\theta + g$, where $a = \lim_{n\to\infty} \frac{1}{2^n} f(2^n e)$.

Theorem 3.7. *Let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings for which there exist functions $\varphi, \psi : \mathcal{R}^2 \to [0,\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \varphi(na, b) = \lim_{n\to\infty} \frac{1}{n} \varphi(a, nb) = 0, \tag{3.24}$$

$$\lim_{n\to\infty} \frac{1}{n} \psi(na, b) = \lim_{n\to\infty} \frac{1}{n} \psi(a, nb) = 0, \tag{3.25}$$

$$\|f(ab) - f(a)\theta(b) - \phi(a)g(b)\| \leq \varphi(a,b), \tag{3.26}$$

$$\|g(ab) - g(a)\theta(b) - \phi(a)g(b)\| \leq \psi(a,b) \tag{3.27}$$

for all $a, b \in \mathcal{R}$. *If* $\mathcal{R}$ *is normed with the identity* e *and* $\mathcal{M}$ *is unit linked, then*

$$g(ab) = g(a)\theta(b) + \phi(a)g(b), \quad f(ab) = f(a)\theta(b) + \phi(a)g(b) \tag{3.28}$$

for all $a, b \in \mathcal{R}$.

Proof. By (3.24) and (3.27), we get

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n}\,[g(nab) - g(na)\theta(b)] = \phi(a)g(b), \\ &\lim_{n\to\infty} \frac{1}{n}\,[g(nab) - \phi(a)g(nb)] = g(a)\theta(b) \end{aligned} \tag{3.29}$$

for all $a, b \in \mathcal{R}$. Using the Badora's method [**3**] on the inequality (3.27), we have

$$\begin{aligned} &\|g(ab) - g(a)\theta(b) - \phi(a)g(b)\| \\ &\leq \left\|g(ab) - \frac{1}{n}\,g(nabe) + \frac{1}{n}\,\phi(ab)g(ne)\right\| \\ &\quad + \left\|\frac{1}{n}\,g(nab) - \frac{1}{n}\,\phi(a)g(nb) - g(a)\theta(b)\right\| \\ &\quad + \left\|\frac{1}{n}\,g(nab) - \frac{1}{n}\,g(na)\theta(b) - \phi(a)g(b)\right\| \\ &\quad + \frac{1}{n}\,\|\phi(a)g(nb) - \phi(ab)g(ne) + g(na)\theta(b) - g(nab)\| \\ &\leq \left\|g(ab) - \frac{1}{n}\,g(nabe) + \frac{1}{n}\phi(ab)g(ne)\right\| \\ &\quad + \left\|\frac{1}{n}\,g(nab) - \frac{1}{n}\,\phi(a)g(nb) - g(a)\theta(b)\right\| \\ &\quad + \left\|\frac{1}{n}\,g(nab) - \frac{1}{n}\,g(na)\theta(b) - \phi(a)g(b)\right\| \\ &\quad + \frac{1}{n}\,\|\phi(a)\|\,\|g(nb) - g(b)\theta(ne) - \phi(b)g(ne)\| \\ &\quad + \frac{1}{n}\,\|g(nab) - g(na)\theta(b) - \phi(na)g(b)\| \\ &\leq \left\|g(ab) - \frac{1}{n}\,g(nabe) + \frac{1}{n}\,\phi(ab)g(ne)\right\| \\ &\quad + \left\|\frac{1}{n}\,g(nab) - \frac{1}{n}\,\phi(a)g(nb) - g(a)\theta(b)\right\| \\ &\quad + \left\|\frac{1}{n}\,g(nab) - \frac{1}{n}\,g(na)\theta(b) - \phi(a)g(b)\right\| \\ &\quad + \frac{1}{n}\,\|\phi(a)\|\,\psi(b,ne) + \frac{1}{n}\,\psi(na,b) \end{aligned}$$

for all $a, b \in \mathcal{R}$. Applying (3.25) and (3.29), we observe that the right side of the last inequality tends to 0 when n tends to infinity. Therefore

$$g(ab) = g(a)\theta(b) + \phi(a)g(b) \tag{3.30}$$

for all $a, b \in \mathcal{R}$.

Similarly, by (3.24) and (3.26), we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\left[f(nab) - f(na)\theta(b)\right] &= \phi(a)g(b),\\ \lim_{n\to\infty} \frac{1}{n}\left[f(nab) - \phi(a)g(nb)\right] &= f(a)\theta(b)\end{aligned} \tag{3.31}$$

for all $a, b \in \mathcal{R}$. Let $a, b \in \mathcal{R}$ and $n \in \mathbb{N}$ be fixed. Since g satisfies (3.30), we have $g(nb) = g(bne) = ng(b) + \phi(b)g(ne)$. Using (3.26), we have

$$\begin{aligned}
&\|f(ab) - f(a)\theta(b) - \phi(a)g(b)\|\\
&\quad\le \left\|f(ab) - \frac{1}{n} f(nabe) + \frac{1}{n} \phi(ab)g(ne)\right\|\\
&\qquad + \left\|\frac{1}{n} f(nab) - \frac{1}{n} \phi(a)g(nb) - f(a)\theta(b)\right\|\\
&\qquad + \left\|\frac{1}{n} f(nab) - \frac{1}{n} f(na)\theta(b) - \phi(a)g(b)\right\|\\
&\qquad + \frac{1}{n} \|\phi(a)g(nb) - \phi(ab)g(ne) + f(na)\theta(b) - f(nab)\|\\
&\quad= \left\|f(ab) - \frac{1}{n} f(nabe) + \frac{1}{n} \phi(ab)g(ne)\right\|\\
&\qquad + \left\|\frac{1}{n} f(nab) - \frac{1}{n} \phi(a)g(nb) - f(a)\theta(b)\right\|\\
&\qquad + \left\|\frac{1}{n} f(nab) - \frac{1}{n} f(na)\theta(b) - \phi(a)g(b)\right\|\\
&\qquad + \frac{1}{n} \|n\phi(a)g(b) + f(na)\theta(b) - f(nab)\|\\
&\quad\le \left\|f(ab) - \frac{1}{n} f(nabe) + \frac{1}{n} \phi(ab)g(ne)\right\|\\
&\qquad + \left\|\frac{1}{n} f(nab) - \frac{1}{n} \phi(a)g(nb) - f(a)\theta(b)\right\|\\
&\qquad + \left\|\frac{1}{n} f(nab) - \frac{1}{n} f(na)\theta(b) - \phi(a)g(b)\right\| + \frac{1}{n}\varphi(na, b).
\end{aligned}$$

Applying (3.24) and (3.31), we observe that the right side of the last inequality tends to 0 when n tends to infinity. Therefore

$$f(ab) = f(a)\theta(b) + \phi(a)g(b). \qquad \square$$

Corollary 3.8. *Let $\varepsilon, \delta, p, q$ be non-negative real numbers with $0 < p, q < 1$. If $\mathcal{R}$ is a normed ring with the identity e, $\mathcal{M}$ is unit linked and $f, g : \mathcal{R} \to \mathcal{M}$ are mappings satisfy the inequalities*

$$\begin{aligned}\|f(ab) - f(a)\theta(b) - \phi(a)g(b)\| &\le \delta + \varepsilon(\|a\|^p + \|b\|^q),\\ \|g(ab) - g(a)\theta(b) - \phi(a)g(b)\| &\le \delta + \varepsilon(\|a\|^p + \|b\|^q)\end{aligned}$$

for all $a, b \in \mathcal{R}$, then f and g satisfy (3.28) for all $a, b \in \mathcal{R}$.

Acknowledgment. The authors thank the referee(s) for a number of valuable suggestions on a previous version of this paper.

REFERENCES

1. M. Amyari, M. S. Moslehian: *Hyers-Ulam-Rassias stability of derivations on Hilbert C^*-modules.* Contemporary Math., **427** (2007), 31–39.
2. T. Aoki: *On the stability of the linear transformation in Banach spaces.* J. Math. Soc. Japan, **2** (1950), 64–66.
3. R. Badora: *On approximate derivations.* Math. Inequal. Appl., **9** (2006), 167–173.
4. M. Brešar: *Jordan derivations on semiprime rings.* Proc. Amer. Math. Soc., **104** (1988), 1003–1006.
5. L. Cădariu, V. Radu: *On the stability of the Cauchy functional equation: A fixed point approach.* Grazer Math. Ber., **346** (2004), 43–52.
6. J. Cusak: *Jordan derivations on rings.* Proc. Amer. Math. Soc., **53** (1975), 321–324.
7. P. Czerwik: *Functional Equations and Inequalities in Several Variables.* World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.
8. P. Găvruta: *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings.* J. Math. Anal. Appl., **184** (1994), 431–436.
9. O. Hatori, J. Wada: *Ring derivations on semi-simple commutative Banach algebras.* Tokyo J. Math., **15** (1992), 223–229.
10. I. N. Herstein: *Jordan derivations of prime rings.* Proc. Amer. Math. Soc., **8** (1957), 1104–1119.
11. B. Hvala: *Generalized derivations in rings.* Comm. Algebra, **26** (1998), 1147–1166.
12. D. H. Hyers: *On the stability of the linear functional equation.* Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224.
13. D. H. Hyers, G. Isac, Th. M. Rassias: *Stability of Functional Equations in Several Variables.* Birkhäuser, Basel, 1998.
14. W. Jing, S. Lu: *Generalized Jordan derivations on prime rings and standard operator algebras.* Taiwanese J. Math., **7** (2003), 605–613.
15. B. E. Johnson: *Continuity of derivations on commutative algebras.* Amer. J. Math., **91** (1969), 1–10.
16. S.-M. Jung: *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis.* Hadronic Press lnc., Palm Harbor, Florida, 2001.
17. S.-M. Jung, T.-S. Kim: *A fixed point approach to stability of cubic functional equation.* Bol. Soc. Mat. Mexicana, **12** (2006), 51–57.
18. T.-K. Lee: *Generalized derivations of left faithful rings.* Comm. Algebra, **27** (1999), 4057–4073.
19. C.-K. Liu, W.-K. Shiue: *Generalized Jordan triple (θ, ϕ)-derivations on semiprime rings.* Taiwanese J. Math, **11** (2007), 1397–1406.
20. B. Margolis, J. B. Diaz: *A fixed point theorem of the alternative for contractions on a generalized complete metric space.* Bull. Amer. Math. Soc., **74** (1968), 305–309.
21. M. Mirzavaziri, M. S. Moslehian: *A fixed point approach to stability of a quadratic equation.* Bull. Braz. Math. Soc., **37** (2006), 361–376.
22. T. Miura, G. Hirasawa, S.-E. Takahasi: *A perturbation of ring derivations on Banach algebras.* J. Math. Anal. Appl., **319** (2006), 522–530.

23. M. S. Moslehian: *Hyers–Ulam–Rassias stability of generalized derivations.* Int. J. Math. Math. Sci., (2006), Article ID 93942, pages 1–8.
24. A. Najati: *Hyers–Ulam stability of an n-Apollonius type quadratic mapping.* Bull. Belgian Math. Soc. Simon-Stevin, **14** (2007), 755–774.
25. A. Najati: *On the stability of a quartic functional equation.* J. Math. Anal. Appl., **340** (2008), 569–574.
26. A. Najati, M. B. Moghimi: *Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces.* J. Math. Anal. Appl., **337** (2008), 399–415.
27. A. Najati, C. Park: *Hyers–Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation.* J. Math. Anal. Appl., **335** (2007), 763–778.
28. A. Najati, C. Park: *On the stability of an n-dimensional functional equation originating from quadratic forms.* Taiwanese J. Math., **12** (2008), 1609–1624.
29. C. Park: *On the stability of the linear mapping in Banach modules.* J. Math. Anal. Appl., **275** (2002), 711–720.
30. C. Park: *Homomorphisms between Poisson JC^*-algebras.* Bull. Braz. Math. Soc., **36** (2005), 79–97.
31. Th. M. Rassias: *On the stability of the linear mapping in Banach spaces.* Proc. Amer. Math. Soc., **72** (1978), 297–300.
32. Th. M. Rassias: *Functional Equations, Inequalities and Applications.* Kluwer Academic Publishers Co., Dordrecht, Boston, London, 2003.
33. P. Šemrl: *On ring derivations and quadratic functionals.* Aequationes Math., **42** (1991), 80–84.
34. P. Šemrl: *The functional equation of multiplicative derivation is superstable on standard operator algebras.* Integr. Equat. Oper. Theory, **18** (1994), 118–122.
35. I. M. Singer, J. Wermer: *Derivations on commutative normed algebras.* Math. Ann., **129** (1955), 260–264.
36. M. P. Thomas: *The image of a derivation is contained in the radical.* Ann. of Math., **128** (1988), 435–460.
37. S. M. Ulam: *A Collection of the Mathematical Problems.* Interscience Publ. New York, 1960.

(Received May 30, 2009)
(Revised August 1, 2009)

Department of Mathematics,
Faculty of Sciences,
University of Mohaghegh Ardabili,
Ardabil,
Iran
E–mail: a.nejati@yahoo.com

Department of Mathematics,
National Technical University of Athens,
Zografou Campus,
15780 Athens,
Greece
E–mail: trassias@math.ntua.gr