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ZERO AND COEFFICIENT INEQUALITIES FOR STABLE POLYNOMIALS

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In this paper an elementary inequality and Cardan-Viète formulae are used to obtain some inequalities involving the zeros and coefficients of stable polynomials with complex coefficients.

1. INTRODUCTION

No doubt that one of the classical problems of the theory of equations is to find relations between the zeroes and coefficients of a polynomial. Surely, one of the first of these relations are the well known Cardan-Viète formulae [**1**]. Since the beginning, a lot of papers devoted to obtain inequalities between zeros and coefficients were written giving new bounds, implicit or explicit, or improving the classical known ones. These results have been fully documented by Marden [**2**], Milovanović, Mitrinović and Rassias [**3**] (see also [**4**]), and recently in the work of Rahman and Schmeisser [**5**]. Polynomials possess a long history, as it is well-known, but they have recently come under extensive revision because of their importance in several areas of contemporary applied mathematics, including linear control systems, electrical networks, signal processing, and coding theory ([**6**, **7**]), where root location and stability problems arise, among others, in a natural way. In this paper, using an elementary inequality, several inequalities involving zeros, coefficients and the inner and the outer radius of a ring shaped region where lie the zeros of a complex polynomial are obtained. Furthermore, applying the same procedure to stable polynomials, new inequalities similar to the ones obtained by Rubió-Massegú, Díaz-Barrero and Rubió-Díaz in [**8**] are also given.

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2. MAIN RESULTS

Let $A(z) = \sum_{k=0}^{n} a_k z^k$, $(a_n \neq 0)$ be a polynomial with complex coefficients. Recall that $A(z)$ is *Schur stable* if all its zeros lie in the open unit disk $|z| < 1$, and *Hurwitz stable* if all its zeros lie in the open left half-plane $\mathrm{Re}(z) < 0$.

In what follows we state several results involving the zeros and coefficients of stable polynomials. We begin with

Theorem 1. *Let* $A(z) = \sum_{k=0}^{n} a_k z^k$, $(a_n \neq 0)$ *be a Schur stable complex polynomial with zeros* $z_1, z_2, \ldots, z_n$. *If* p *and* q *are strictly positive real numbers, then*

$$\sum_{k=1}^{n} \frac{1}{\left(1 - |z_k|^p\right)^{1/q}} \geq \frac{(p+q)^{\frac{1}{p}+\frac{1}{q}}}{p^{1/q} q^{1/p}} \left| \frac{a_{n-1}}{a_n} \right| .$$

Equality holds if and only if $A(z) = a_n (z - w)^n$, *where* w *is a nonzero complex number such that* $|w| = \left(\frac{q}{p+q} \right)^{1/p}$.

Proof. To prove the preceding result we need the following lemma that follows from elementary considerations.

Lemma 1. *Let* p *and* q *be strictly positive real numbers. Then*

$$\sup_{0 \leq \alpha \leq 1} \alpha^q (1 - \alpha^p) = \frac{p}{q} \left(\frac{q}{p+q} \right)^{1+\frac{q}{p}} .$$

The supremum is attained only when $\alpha = \left(\frac{q}{p+q} \right)^{1/p}$.

Applying the previous Lemma we have

$$|z_k|^q \left(1 - |z_k|^p\right) \leq \frac{p}{q} \left(\frac{q}{p+q} \right)^{1+\frac{q}{p}}$$

or, equivalently,

$$\frac{1}{1 - |z_k|^p} \geq |z_k|^q \, \frac{q}{p} \left(\frac{p+q}{q} \right)^{1+\frac{q}{p}} = \frac{(p+q)^{1+\frac{q}{p}}}{p \, q^{q/p}} \, |z_k|^q.$$

Raising to $1/q$ both sides of the preceding inequality, yields

$$\frac{1}{\left(1 - |z_k|^p\right)^{1/q}} \geq \frac{(p+q)^{\frac{1}{p}+\frac{1}{q}}}{p^{1/q} q^{1/p}} \, |z_k|$$

valid for $1 \leq k \leq n$. Adding up the above inequalities for $1 \leq k \leq n$, we get

$$\sum_{k=1}^{n} \frac{1}{\left(1 - |z_k|^p\right)^{1/q}} \geq \frac{(p+q)^{\frac{1}{p}+\frac{1}{q}}}{p^{1/q} q^{1/p}} \sum_{k=1}^{n} |z_k|.$$

Finally, taking into account that

$$\sum_{k=1}^{n}|z_k| \geq \left|\sum_{k=1}^{n} z_k\right| = \left|\frac{a_{n-1}}{a_n}\right|$$

we get

$$\sum_{k=1}^{n}\frac{1}{(1-|z_k|^p)^{1/q}} \geq \frac{(p+q)^{\frac{1}{p}+\frac{1}{q}}}{p^{1/q}q^{1/p}}\left|\frac{a_{n-1}}{a_n}\right|$$

as claimed.

Furthermore, the inequality becomes equality if and only if $|z_k| = \left(\frac{q}{p+q}\right)^{1/p}$ for all k, and

$$\left|\sum_{k=1}^{n} z_k\right| = \sum_{k=1}^{n}|z_k|.$$

The last condition holds if and only if there exists a complex number u with $|u| = 1$ and real numbers $\beta_k \geq 0$, $1 \leq k \leq n$, such that $z_k = \beta_k u$ for all k. From $|z_k| = \left(\frac{q}{p+q}\right)^{1/p}$ and $|z_k| = \beta_k$ it follows that β_k does not depend on k. This implies that there exists $w \in \mathbb{C}$ such that $z_k = w$ for all k. Number w verifies $|w| = \left(\frac{q}{p+q}\right)^{1/p}$ and in particular $w \neq 0$. Summarizing, we have obtained that equality holds if and only if there exists a nonzero complex number w such that $A(z) = a_n(z-w)^n$ and $|w| = \left(\frac{q}{p+q}\right)^{1/p}$ as claimed, and this completes the proof.
$\square$

In particular, setting $q = p > 0$ into the preceding result, we obtain

$$\sum_{k=1}^{n}\frac{1}{(1-|z_k|^p)^{1/p}} \geq 4^{1/p}\left|\frac{a_{n-1}}{a_n}\right|.$$

In this case, equality holds when there exists a nonzero complex number w such that $A(z) = a_n(z-w)^n$ and $|w| = 2^{-1/p}$. Assigning values to p new inequalities can be obtained. For instance, when $p = 1/2$, yields

$$\sum_{k=1}^{n}\frac{1}{(1-\sqrt{|z_k|}\,)^2} \geq 16\left|\frac{a_{n-1}}{a_n}\right|$$

with equality when $A(z) = a_n(z-w)^n$ and $|w| = 1/4$. For $p = 1$, we have $\displaystyle\sum_{k=1}^{n}\frac{1}{1-|z_k|} \geq 4\left|\frac{a_{n-1}}{a_n}\right|$ with equality when $A(z) = a_n(z-w)^n$ and $|w| = 1/2$; and for $p = 2$,

$$\sum_{k=1}^{n}\frac{1}{\sqrt{1-|z_k|^2}} \geq 2\left|\frac{a_{n-1}}{a_n}\right|.$$

In this case equality holds when $A(z) = a_n(z-w)^n$ and $|w| = 1/\sqrt{2}$. Finally, if $\dfrac{1}{p}+\dfrac{1}{q}=1$, then $\displaystyle\sum_{k=1}^{n}\frac{1}{(1-|z_k|^p)^{1/q}} \geq p^{1/p}q^{1/q}\left|\frac{a_{n-1}}{a_n}\right|$. Equality holds if and only if $A(z) = a_n(z-w)^n$ with $|w| = p^{-1/p}$.

On the other hand, under the assumptions of Theorem 1, for $p \leq q$ we have

$$\sum_{k=1}^{n}\frac{1}{(1-|z_k|^p)^{1/q}} \geq \sum_{k=1}^{n}\frac{1}{(1-|z_k|^q)^{1/q}}$$

and applying Theorem 1 with $p = q$, we get

$$\sum_{k=1}^{n}\frac{1}{(1-|z_k|^p)^{1/q}} \geq 4^{1/q}\left|\frac{a_{n-1}}{a_n}\right|.$$

It is easy to see that

$$\frac{(p+q)^{\frac{1}{p}+\frac{1}{q}}}{p^{1/q}q^{1/p}} \geq 4^{1/q}$$

for any p and q with $0 < p \leq q$, so the inequality of Theorem 1 is a refinement of the preceding one. The same conclusion is obtained in the case $p \geq q$ if we use that

$$\sum_{k=1}^{n}\frac{1}{(1-|z_k|^p)^{1/q}} \geq \sum_{k=1}^{n}\frac{1}{(1-|z_k|^p)^{1/p}}$$

and we apply Theorem 1 for $q = p$.

Also observe that for a polynomial $A(z) = \sum\limits_{k=0}^{n} a_k z^k$ whose zeros lie in the disk $|z| < R$, the change of variable $z = Rz'$ transforms $A(z)$ into a SCHUR stable polynomial, and applying Theorem 1 to this polynomial it follows that

$$\sum_{k=1}^{n}\frac{1}{(R^p-|z_k|^p)^{1/q}} \geq \frac{(p+q)^{\frac{1}{p}+\frac{1}{q}}}{p^{1/q}q^{1/p}R^{1+\frac{p}{q}}}\left|\frac{a_{n-1}}{a_n}\right|$$

where $z_1,\dots,z_n$ are the zeros of $A(z)$. Furthermore, equality holds if and only if $A(z) = a_n(z-w)^n$ with $|w| = \left(\dfrac{q}{p+q}\right)^{1/p}R$.

Let $A(z) = \sum\limits_{k=0}^{n} a_k z^k$, $(a_n \neq 0)$ be a complex polynomial such that $a_0 \neq 0$, and let $A^*(z) = a_0 z^n + \cdots + a_{n-1}z + a_n$. The zeros of $A^*(z)$ are the inverses of the zeros of $A(z)$. So $A^*(z)$ is Schur stable if and only if the zeros of $A(z)$ lie in $|z| > 1$. Such polynomials $A(z)$ are usually called *anti–Schur stable* polynomials.

Suppose that $A(z)$ is anti–Schur stable. Then, applying Theorem 1 to $A^*(z)$ we obtain the inequality

$$\sum_{k=1}^{n}\left(\frac{|z_k|^p}{|z_k|^p-1}\right)^{1/q} \geq \frac{(p+q)^{\frac{1}{p}+\frac{1}{q}}}{p^{1/q}q^{1/p}}\left|\frac{a_1}{a_0}\right|.$$

Another inequality for anti–Schur stable polynomials more similar to the one of Theorem 1 is the following

Theorem 2. *Let* $A(z) = \sum_{k=0}^{n} a_k z^k$, $(a_n \neq 0)$ *be an anti–Schur stable polynomial with zeros* $z_1, z_2, \ldots, z_n$. *If* p *and* q *are strictly positive real numbers such that* $q > p$, *then*

$$\sum_{k=1}^{n} \frac{1}{(|z_k|^p - 1)^{1/q}} \geq \frac{q^{1/p}}{p^{1/q}\,(q-p)^{\frac{1}{p}-\frac{1}{q}}} \left|\frac{a_1}{a_0}\right|.$$

Equality holds if and only if $A(z) = a_n\,(z-w)^n$, *where* w *is a nonzero complex number such that* $|w| = \left(\frac{q}{q-p}\right)^{1/p}$.

Proof. Applying Lemma 1 to the numbers $p' = p$ and $q' = q - p$, and taking into account that $0 < \frac{1}{|z_k|} < 1$, we obtain

$$\frac{1}{|z_k|^{q-p}}\left(1 - \frac{1}{|z_k|^p}\right) \leq \frac{p}{q-p}\left(\frac{q-p}{q}\right)^{q/p}$$

or, equivalently,

$$\frac{1}{|z_k|^p - 1} \geq \frac{q-p}{p}\left(\frac{q}{q-p}\right)^{q/p} \frac{1}{|z_k|^q} = \frac{q^{q/p}}{p(q-p)^{\frac{q}{p}-1}}\,\frac{1}{|z_k|^q}.$$

Raising to $1/q$, yields

$$\frac{1}{(|z_k|^p - 1)^{1/q}} \geq \frac{q^{1/p}}{p^{1/q}\,(q-p)^{\frac{1}{p}-\frac{1}{q}}}\,\frac{1}{|z_k|}$$

Adding up the above inequalities for $1 \leq k \leq n$, we get

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{(|z_k|^p - 1)^{1/q}} &\geq \frac{q^{1/p}}{p^{1/q}\,(q-p)^{\frac{1}{p}-\frac{1}{q}}} \sum_{k=1}^{n} \frac{1}{|z_k|} \\
&\geq \frac{q^{1/p}}{p^{1/q}\,(q-p)^{\frac{1}{p}-\frac{1}{q}}} \left|\sum_{k=1}^{n} \frac{1}{z_k}\right| \\
&= \frac{q^{1/p}}{p^{1/q}\,(q-p)^{\frac{1}{p}-\frac{1}{q}}} \left|\frac{a_1}{a_0}\right|.
\end{aligned}$$

Notice that equality holds if and only if $|z_k| = \left(\frac{q}{q-p}\right)^{1/p}$ for all k, and $1/z_k = \beta_k u$, $k = 1, 2, \ldots, n$, where $\beta_k \geq 0$ and $|u| = 1$. That is, when $A(z) = a_n\,(z-w)^n$ where w is a complex number such that $|w| = \left(\frac{q}{q-p}\right)^{1/p}$. This completes the proof. $\square$

We close the paper establishing two results similar to the ones presented in [**8**] for polynomials which are both Schur stable and Hurwitz stable.

Theorem 3. *Let* $A(z) = \sum_{k=0}^{n} a_k z^k$, $(a_n \neq 0)$ *be a complex polynomial with zeros* $z_1, z_2, \ldots, z_n$. *Suppose that* $A(z)$ *is both Schur and Hurwitz stable, and let* $u = \max_{1\leq k\leq n} \left\{ \left| \frac{\mathrm{Im}\,(z_k)}{\mathrm{Re}\,(z_k)} \right| \right\}$. *If* p *and* q *are strictly positive real numbers, then*

$$\sum_{k=1}^{n} (1 - |z_k|^p)^{1/q} \leq \sqrt{1+u^2}\, \frac{p^{1/q} q^{1/p}}{(p+q)^{\frac{1}{p}+\frac{1}{q}}}\, \mathrm{Re}\Big(\frac{a_1}{a_0}\Big)$$

Equality holds if and only if there exist nonnegative integer numbers s *and* t *with* $s + t = n$, *such that* $A(z) = a_n(z-w)^s(z-\bar{w})^t$, *where* w *is the number*

$$w = \frac{1}{\sqrt{1+u^2}} \Big(\frac{q}{p+q}\Big)^{1/p} (-1 + ui)$$

Proof. As in the proof of Theorem 1, we have

$$|z_k|^q\,(1 - |z_k|^p) \leq \frac{p}{q} \left(\frac{q}{p+q}\right)^{1+\frac{q}{p}}$$

from which immediately follows

$$1 - |z_k|^p \leq \frac{p}{q}\Big(\frac{q}{p+q}\Big)^{1+\frac{q}{p}} \frac{1}{|z_k|^q}$$

Raising to $1/q$ both sides of the preceding inequality, yields

$$(1 - |z_k|^p)^{1/q} \leq \frac{p^{1/q} q^{1/p}}{\Big(p+q\Big)^{\frac{1}{p}+\frac{1}{q}}} \frac{1}{|z_k|}$$

valid for $1 \leq k \leq n$. Adding up the above inequalities for $1 \leq k \leq n$, we get

$$\sum_{k=1}^{n} (1 - |z_k|^p)^{1/q} \leq \frac{p^{1/q} q^{1/p}}{(p+q)^{\frac{1}{p}+\frac{1}{q}}} \sum_{k=1}^{n} \frac{1}{|z_k|}\,.$$

Now we will find an upper bound for the sum $S = \sum_{k=1}^{n} \frac{1}{|z_k|}$. In fact, taking into account Cardan-Viète formulae, we have

$$\sum_{k=1}^{n} \frac{1}{z_k} = \sum_{k=1}^{n} \frac{\bar{z}_k}{|z_k|^2} = -\frac{a_1}{a_0}\,.$$

On the other hand, $|z_k| = \sqrt{(\mathrm{Re}\,(z_k))^2 + (\mathrm{Im}\,(z_k))^2} \leq \sqrt{1+u^2}\; |\mathrm{Re}\,(z_k)|$. Therefore,

$$\begin{aligned} S &= \sum_{k=1}^{n} \frac{1}{|z_k|} = \sum_{k=1}^{n} \frac{|z_k|}{|z_k|^2} \leq \sqrt{1+u^2} \sum_{k=1}^{n} \frac{|\mathrm{Re}\,(z_k)|}{|z_k|^2} = -\sqrt{1+u^2} \sum_{k=1}^{n} \frac{\mathrm{Re}\,(z_k)}{|z_k|^2} \\ &= -\sqrt{1+u^2} \sum_{k=1}^{n} \frac{\mathrm{Re}\,(\bar{z}_k)}{|z_k|^2} = \sqrt{1+u^2}\, \mathrm{Re}\left(-\sum_{k=1}^{n} \frac{\bar{z}_k}{|z_k|^2}\right) = \sqrt{1+u^2}\, \mathrm{Re}\Big(\frac{a_1}{a_0}\Big) \end{aligned}$$

and

$$\sum_{k=1}^{n}(1-|z_k|^p)^{1/q} \le \frac{p^{1/q}q^{1/p}}{(p+q)^{\frac{1}{p}+\frac{1}{q}}}\sum_{k=1}^{n}\frac{1}{|z_k|} \le \sqrt{1+u^2}\,\frac{p^{1/q}q^{1/p}}{(p+q)^{\frac{1}{p}+\frac{1}{q}}}\,\mathrm{Re}\Big(\frac{a_1}{a_0}\Big)$$

as claimed. Equality holds when (a) $|z_k| = \Big(\frac{q}{p+q}\Big)^{1/p}$ for all k, and (b) $|z_k| = \sqrt{1+u^2}\,|\mathrm{Re}\,(z_k)|$, $k = 1,2,\ldots,n$. Condition (b) is equivalent to $|\mathrm{Im}\,(z_k)| = u\,|\mathrm{Re}\,(z_k)|$. In other words, $z_k = |\mathrm{Re}\,(z_k)|\,(-1 \pm ui)$. In particular it must be verified that $|z_k| = \Big(\frac{q}{p+q}\Big)^{1/p} = \sqrt{1+u^2}\,|\mathrm{Re}\,(z_k)|$, from which we have $|\mathrm{Re}\,(z_k)| = \frac{1}{\sqrt{1+u^2}}\Big(\frac{q}{p+q}\Big)^{1/p}$. So, $z_k = w$ or $z_k = \bar{w}$, where

$$w = \frac{1}{\sqrt{1+u^2}}\Big(\frac{q}{p+q}\Big)^{1/p}(-1+ui).$$

If there are s zeros equal to w and t zeros equal to $\bar{w}$, $s+t=n$, then $A(z) = a_n(z-w)^s(z-\bar{w})^t$ as desired, and the proof is complete. □

Note that when $u = 0$, that is, when A has only real zeroes (A is hyperbolic), the inequality becomes

$$\sum_{k=1}^{n}(1-|z_k|^p)^{1/q} \le \frac{p^{1/q}q^{1/p}}{(p+q)^{\frac{1}{p}+\frac{1}{q}}}\,\frac{a_1}{a_0}$$

Equality holds if and only if

$$A(z) = a_n\bigg(z + \Big(\frac{q}{p+q}\Big)^{1/p}\bigg)^n$$

since in this case $w = \bar{w} = -\Big(\frac{q}{p+q}\Big)^{1/p}$.

On the other hand, setting $q = 1$ in Theorem 3, we get

$$\sum_{k=1}^{n}(1-|z_k|^p) \le \frac{p\sqrt{1+u^2}}{(p+1)^{1+\frac{1}{p}}}\,\mathrm{Re}\Big(\frac{a_1}{a_0}\Big)$$

and rearranging terms it follows that

$$\sum_{k=1}^{n}|z_k|^p \ge n - \frac{p\sqrt{1+u^2}}{(p+1)^{1+\frac{1}{p}}}\,\mathrm{Re}\Big(\frac{a_1}{a_0}\Big)$$

thus obtaining a lower bound for the sum $\sum\limits_{k=1}^{n}|z_k|^p$ in terms of the coefficients of the polynomial. This bound becomes equality when $A(z) = a_n(z-w)^s(z-\bar{w})^t$

with $s+t=n$ and $w=\dfrac{1}{\sqrt{1+u^2}}\Big(\dfrac{1}{1+p}\Big)^{1/p}(-1+ui)$. For hyperbolic polynomials we have

$$\sum_{k=1}^{n}|z_k|^p \geq n-\frac{p}{(p+1)^{1+\frac{1}{p}}}\,\frac{a_1}{a_0}\,.$$

Finally, we state and prove the following

Theorem 4. *Let* $A(z)=\sum\limits_{k=0}^{n}a_k z^k$, $(a_n\neq 0)$ *be a complex polynomial with zeros* $z_1,z_2,\ldots,z_n$. *Suppose that* $A(z)$ *is both anti–Schur and Hurwitz stable, and let* $u=\max\limits_{1\leq k\leq n}\left\{\left|\dfrac{\mathrm{Im}(z_k)}{\mathrm{Re}(z_k)}\right|\right\}$. *If* p *and* q *are strictly positive real numbers such that* $q>p$, *then*

$$\sum_{k=1}^{n}\left(|z_k|^p-1\right)^{1/q}\leq\sqrt{1+u^2}\,(q-p)^{\frac{1}{p}-\frac{1}{q}}\,\frac{p^{1/q}}{q^{1/p}}\,\mathrm{Re}\Big(\frac{a_{n-1}}{a_n}\Big)$$

Equality holds if and only if there exist nonnegative integer numbers s *and* t *with* $s+t=n$, *such that* $A(z)=a_n(z-w)^s(z-\bar{w})^t$, *where* w *is the number*

$$w=\frac{1}{\sqrt{1+u^2}}\left(\frac{q}{q-p}\right)^{1/p}(-1+ui)$$

Proof. From the proof of Theorem 2, we have

$$\frac{1}{|z_k|^{q-p}}\left(1-\frac{1}{|z_k|^p}\right)\leq\frac{p}{q-p}\Big(\frac{q-p}{q}\Big)^{q/p}$$

or, equivalently,

$$|z_k|^p-1\leq\frac{p}{q-p}\Big(\frac{q-p}{q}\Big)^{q/p}|z_k|^q$$

Raising to $1/q$ both sides of the preceding inequality, and adding up the resulting inequalities for $1\leq k\leq n$, we get

$$\sum_{k=1}^{n}\left(|z_k|^p-1\right)^{1/q}\leq(q-p)^{\frac{1}{p}-\frac{1}{q}}\,\frac{p^{1/q}}{q^{1/p}}\,\sum_{k=1}^{n}|z_k|$$

Now, as in the proof of Theorem 3, we will find an upper bound for the sum $S=\sum\limits_{k=1}^{n}|z_k|$. Taking into account Cardan-Viète formulae, we have

$$\sum_{k=1}^{n}z_k=-\frac{a_{n-1}}{a_n}$$

From $|z_k|=\sqrt{(\mathrm{Re}\,(z_k))^2+(\mathrm{Im}(z_k))^2}\leq\sqrt{1+u^2}\;|\mathrm{Re}(z_k)|=-\sqrt{1+u^2}\,\mathrm{Re}(z_k)$, we get

$$S=\sum_{k=1}^{n}|z_k|\leq-\sqrt{1+u^2}\,\sum_{k=1}^{n}\mathrm{Re}(z_k)=\sqrt{1+u^2}\,\mathrm{Re}\Big(-\sum_{k=1}^{n}z_k\Big)=\sqrt{1+u^2}\,\mathrm{Re}\Big(\frac{a_{n-1}}{a_n}\Big).$$

Therefore,

$$\sum_{k=1}^{n} \left(|z_k|^p - 1\right)^{1/q} \le (q-p)^{\frac{1}{p}-\frac{1}{q}} \, \frac{p^{1/q}}{q^{1/p}} \sqrt{1+u^2} \, \mathrm{Re}\Big(\frac{a_{n-1}}{a_n}\Big)$$

and the inequality of the statement is proved. Equality holds when (a) $|z_k| = \Big(\dfrac{q}{q-p}\Big)^{1/p}$ for all k, and (b) $|z_k| = \sqrt{1+u^2}\,|\mathrm{Re}(z_k)|$, $k = 1, 2, \ldots, n$. Now, by a similar reasoning to that of the proof of Theorem 3, it is easy to see that equality holds if and only if $A(z) = a_n(z-w)^s(z-\bar{w})^t$ where $s+t = n$ and w is the complex number

$$w = \frac{1}{\sqrt{1+u^2}} \Big(\frac{q}{q-p}\Big)^{1/p} (-1+ui)$$

and the proof is complete. □

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REFERENCES

1. F. Viète: *Opera Mathematica.* Leiden, Germany, 1646.
2. M. Marden: *Geometry of Polynomials.* American Mathematical Society, Providence, Rhode Island, 1989.
3. G. V. Milovanović, D. S. Mitrinović, T. M. Rassias: *Topics in Polynomials: Extremal Problems, Inequalities and Zeros.* World Scientific, Singapore, 1994.
4. T. M. Rassias: *Inequalities for polynomial zeros*, 165 – 202, in T. M. Rassias (ed.): *Topics in Polynomials.* Kluwer Academic Publisher, Dordrecht, 2000.
5. Q. I. Rahman, G. Schmeisser: *Analytic Theory of Polynomials.* Clarendon Press, Oxford 2002.
6. E. I. Jury: *Theory and Application of the z-Transform Method.* Wiley, New York 1964.
7. S. Barnett: *Introduction to Mathematical Control Theory.* Clarendon Press, Oxford 1975.
8. J. Rubió-Massegú, J. L. Díaz-Barrero, P. Rubió-Díaz: *Zero and Coefficient Inequalities for Hyperbolic Polynomials.* JIPAM: Journal of Inequalities in Applied Mathematics., Vol. **7**, No. 1, Article 25 (2006), 1–6.

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