Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **3** (2009), 14–26. doi:10.2298/AADM0901014C

FUNCTIONAL EQUATIONS IN SCHWARTZ DISTRIBUTIONS AND THEIR STABILITIES

Jae-Young Chung

Dedicated to the Memory of Professor D. S. Mitrinović (1908–1995)

Employing two methods we consider a class of n-dimensional functional equations in the space of Schwartz distributions. As the first approach, employing regularizing functions we reduce the equations in distributions to classical ones of smooth functions and find the solutions. Secondly, using differentiation in distributions, converting the functional equations to differential equations and find the solutions. Also we consider the Hyers-Ulam stability of the equations.

1. INTRODUCTION

The main purpose of this article is to introduce two approaches of solving functional equations in Schwartz distributions. As the first approach, using regularizing sequence of test functions and converting given distributional versions of functional equations to classical ones [**4**, **5**] we obtain the solutions. In particular, this approach is useful to consider the Hyers-Ulam stability problems of functional equations in Schwartz distributions. Secondly, in the theory of Schwartz distributions one can differentiate freely underlining functions, which is one of the most powerful tools of the Schwartz theory and can be applied to solving some class of functional equations by reducing the equations to differential equations [2, 3, 6, 7]. Here we denote by $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j > 0,\, j = 1, \ldots, n\}$ and $I^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_j > -1,\, j = 1, \ldots, n\}$ and $xy = (x_1y_1, \ldots, x_ny_n)$ for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. First we introduce the distributional analogue of the following functional equations.

2000 Mathematics Subject Classification. 39B82, 46F10.
Keywords and Phrases. d'Alembert equation, Pompeiu equation, logarithmic functional equation, distribution, Hyers-Ulam stability.

Quadratic functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

D'ALEMBERT equation:

$$f(x+y)+f(x-y)=2f(x)f(y). \tag{1.2}$$

POMPEIU equations:

$$f(x+y+xy)=g(x)+h(y)+g(x)\,h(y), \quad x,y\in I^n, \tag{1.3}$$
$$f(x+y+xy)=f(x)+f(y)+f(x)\,f(y), \quad x,y\in I^n. \tag{1.4}$$

Logarithmic functional equations:

$$f(xy)=g(x)+h(y), \quad x,y\in\mathbb{R}^n_+, \tag{1.5}$$
$$f(xy)=f(x)+f(y), \quad x,y\in\mathbb{R}^n_+. \tag{1.6}$$

Multiplicative functional equations:

$$f(xy)=g(x)h(y), \quad x,y\in\mathbb{R}^n_+, \tag{1.7}$$
$$f(xy)=f(x)f(y), \quad x,y\in\mathbb{R}^n_+. \tag{1.8}$$

The functional equations (1.1) – (1.8) can be generalized to the space of distributions, respectively, as follows.
Quadratic functional equation in $\mathcal{D}'(\mathbb{R}^n)$:

$$u\circ A+u\circ B=2u\circ P_1+2u\circ P_2, \tag{1.9}$$

D'ALEMBERT equation in $\mathcal{D}'(\mathbb{R}^n)$:

$$u\circ A+u\circ B=2u\otimes u. \tag{1.10}$$

POMPEIU equations in $\mathcal{D}'(I^n)$:

$$u\circ S=v\circ P_1+w\circ P_2+v\otimes w, \tag{1.11}$$

$$u\circ S=u\circ P_1+u\circ P_2+u\otimes u. \tag{1.12}$$

Logarithmic functional equations in $\mathcal{D}'(\mathbb{R}^n_+)$:

$$u\circ T=v\circ P_1+w\circ P_2, \tag{1.13}$$
$$u\circ T=u\circ P_1+u\circ P_2. \tag{1.14}$$

Multiplicative functional equations in $\mathcal{D}'(\mathbb{R}^n_+)$:

$$u\circ T=v\otimes w, \tag{1.15}$$
$$u\circ T=u\otimes u. \tag{1.16}$$

where $A(x,y)=x+y$, $B(x,y)=x-y$, $P_1(x,y)=x$, $P_2(x,y)=y$, $x,y\in\mathbb{R}^n$, $S(x,y)=x+y+xy$, $x,\,y\in I^n$, $T(x,y)=xy$, $x,\,y\in\mathbb{R}^n_+$ and $\circ$ denotes the pullback of distributions.

2. SOME OPERATIONS ON DISTRIBUTIONS

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathcal{D}'(\Omega)$ the space of Schwartz distributions on Ω. Recall that a distribution u is a linear functional on $C_c^\infty(\Omega)$ of infinitely differentiable functions on Ω with compact supports such that for every compact set $K \subset \Omega$ there exist constants C and k satisfying

$$|\langle u, \varphi\rangle| \le C \sum_{|\alpha|\le k} \sup |\partial^\alpha \varphi|$$

for all $\varphi \in C_c^\infty(\Omega)$ with supports contained in K. Here we denote by $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \dfrac{\partial}{\partial x_j}$.

We briefly introduce some basic operations in $\mathcal{D}'(\Omega)$. Let $u \in \mathcal{D}'(\Omega)$. Then the k-th partial derivative $\partial_k u$ of u is defined by

$$\langle \partial_k u,\, \varphi\rangle = -\langle u,\, \partial_k \varphi\rangle$$

for $k = 1, \dots, n$. Let $f \in C^\infty(\Omega)$. Then the multiplication fu is defined by

$$\langle fu,\, \varphi\rangle = \langle u,\, f\varphi\rangle.$$

We denote by Ω_j open subsets of $\mathbb{R}^{n_j}$ for $j = 1, 2$.

Definition 2.1. *Let $u_j \in \mathcal{D}'(\Omega_j)$. Then the tensor product $u_1 \otimes u_2$ of u_1 and u_2 is defined by*

$$\langle u_1 \otimes u_2, \varphi(x_1, x_2)\rangle = \langle u_1\,, \langle u_2\,, \varphi(x_1, x_2)\rangle\,\rangle, \quad \varphi(x_1, x_2) \in C_c^\infty(\Omega_1 \times \Omega_2).$$

The tensor product $u_1 \otimes u_2$ belongs to $\mathcal{D}'(\Omega_1 \times \Omega_2)$.

Definition 2.2. *Let $u_j \in \mathcal{D}'(\Omega_j)$ and $f : \Omega_1 \to \Omega_2$ a smooth function such that for each $x \in \Omega_1$ the derivative $f'(x)$ is surjective. Then there exist a unique continuous linear map $f^* : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ such that $f^*u = u \circ f$ when u is a continuous function. We call f^*u the pullback of u by f and often denoted by $u \circ f$.*

For more details of tensor product and pullback of distributions we refer the reader to [**8**, Chapter IV].

3. CONVOLUTION OF REGULARIZING FUNCTIONS

In this section using regularizing functions we consider the equations (1.9) and (1.10). We denote by $\delta(x)$ the function on $\mathbb{R}^n$ such that

$$\delta(x) = \begin{cases} A \exp(-(1 - |x|^2)^{-1}), & |x| < 1 \\ 0, \quad |x| \ge 1, \end{cases}$$

where

$$A = \left(\int_{|x|<1} \exp(-(1-|x|^2)^{-1}) \, \mathrm{d}x \right)^{-1}.$$

It is easy to see that $\delta(x)$ an infinitely differentiable function with support $\{x : |x| \le 1\}$. Now we employ the function $\delta_t(x) := t^{-n}\delta(x/t),\ t > 0$. Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then for each $t > 0$, $(u * \delta_t)(x) = \langle u_y, \delta_t(x-y) \rangle$ is a smooth function in $\mathbb{R}^n$ and $(u*\delta_t)(x) \to u$ as $t \to 0^+$ in the sense of distributions, that is, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int (u * \delta_t)(x)\varphi(x) \, \mathrm{d}x.$$

Theorem 3.1. *Every solution u in $\mathcal{D}'(\mathbb{R}^n)$ of the equation* (1.9) *has the form*

$$u = \sum_{1 \le j \le k \le n} a_{jk} \, x_j x_k.$$

Proof. Convolving $\delta_t(x)\delta_s(y)$ in each side of (1.9) we have

$$\begin{aligned}
[(u \circ A) * (\delta_t(x)\delta_s(y))](\xi\,,\eta) &= \langle u \circ A\,,\ \delta_t(\xi - x)\delta_s(\eta - y) \rangle \\
&= \langle u_x\,, \int \delta_t(\xi - x + y)\delta_s(\eta - y)\, \mathrm{d}y \rangle \\
&= \langle u_x\,, \int \delta_t(\xi + \eta - x - y)\delta_s(y)\, \mathrm{d}y \rangle \\
&= \langle u_x\,, (\delta_t * \delta_s)(\xi + \eta - x) \rangle \\
&= (u * \delta_t * \delta_s)(\xi + \eta).
\end{aligned}$$

Similarly we have

$$[(u \circ B) * \delta_t(x)\delta_s(y)](\xi\,,\eta) = (u * \delta_t * \delta_s)(\xi + \eta), \tag{3.1}$$

$$[(u \circ P_1) * \delta_t(x)\delta_s(y)](\xi\,,\eta) = (u * \delta_t)(\xi), \tag{3.2}$$

$$[(u \circ P_2) * \delta_t(x)\delta_s(y)](\xi\,,\eta) = (u * \delta_t)(\eta), \tag{3.3}$$

Thus the equation (1.9) is converted to the following equation

$$(u * \delta_t * \delta_s)(x + y) + (u * \delta_t * \delta_s)(x - y) - 2(u * \delta_t)(x) - 2(u * \delta_s)(y) = 0 \tag{3.4}$$

for all $x,\ y \in \mathbb{R}^n$, $t, s > 0$. In view of (3.4) it is easy to see that

$$f(x) := \limsup_{t \to 0^+} (u * \delta_t)(x)$$

exists. Letting $y = 0$ in (3.4) we have

$$(u * \delta_t * \delta_s)(x) - (u * \delta_t)(x) - (u * \delta_s)(0) = 0. \tag{3.5}$$

Fix $x \in \mathbb{R}^n$ and let $t = t_n \to 0^+$ so that $(u * \delta_{t_n})(x) \to f(x)$ in (3.5) to get

$$(u * \delta_s)(x) - f(x) - (u * \delta_s)(0) = 0. \tag{3.6}$$

From the inequality (3.4), (3.5), (3.6) and the triangle inequality we have

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0 \tag{3.7}$$

for all $x, y \in \mathbb{R}^n$. In view of (3.6), f is a smooth function. Thus the solution f of the equation (3.7) has the form

$$f(x) = \sum_{1 \le j \le k \le n} a_{jk}\, x_j x_k. \tag{3.8}$$

It follows from (3.6) that

$$\begin{aligned} \langle u, \varphi \rangle &= \lim_{t_n \to 0^+} \int (u * \delta_{t_n})(x)\varphi(x)\, \mathrm{d}x \\ &= \int f(x)\varphi(x)\, \mathrm{d}x + \lim_{t_n \to 0^+} \int (u * \delta_{t_n})(0)\varphi(x)\, \mathrm{d}x \\ &= \int f(x)\varphi(x)\, \mathrm{d}x. \end{aligned}$$

This completes the proof. □

Now we consider the equation (1.10).

Theorem 3.2 *Every solution u in $\mathcal{D}'(\mathbb{R}^n)$ of the equation* (1.10) *has the form*

$$u = 0 \quad \text{or} \quad u = \cos(c \cdot x), \quad c \in \mathbb{C}^n.$$

Proof. Similarly as in the proof of Theorem 3.1, convolving $\delta_t(x)\delta_s(y)$ in each side of (1.10) we can convert the equation (1.10) to the following equation

$$(u * \delta_t * \delta_s)(x+y) + (u * \delta_t * \delta_s)(x-y) - 2(u * \delta_t)(x)(u * \delta_s)(y) = 0 \tag{3.9}$$

for all x, $y \in \mathbb{R}^n$, $t, s > 0$. In view of (3.9) it is easy to see that

$$f(y) := \limsup_{s \to 0^+} (u * \delta_s)(y)$$

exists. In (3.9), fix x and let $t = t_k \to 0^+$ so that $(u * \delta_{t_k})(x) \to f(x)$ as $k \to \infty$. Then we have

$$(u * \delta_s)(x+y) + (u * \delta_s)(x-y) - 2f(x)\,(u * \delta_s)(y) = 0. \tag{3.10}$$

Putting $y = 0$ in (3.10) we have

$$(u * \delta_s)(x) = (u * \delta_s)(0) f(x). \tag{3.11}$$

If $(u * \delta_s)(0) = 0$ for all $s > 0$ we have $(u * \delta_s)(x) = 0$ for all $x \in \mathbb{R}^n$, $s > 0$, which implies $u = 0$. If $(u * \delta_s)(0) \neq 0$ for some $s > 0$ it follows from (3.10) and (3.11) that

$$f(x+y) + f(x-y) - 2f(x)\,f(y) = 0.$$

Since f is a smooth function in view of (3.11) it follows that $f(x) = \cos(c \cdot x)$ for some $c \in \mathbb{C}^n$. Thus we have

$$(u * \delta_s)(x) = (u * \delta_s)(0)\cos(c \cdot x).$$

Letting $s = s_k \to 0$ so that $(u * \delta_{s_k})(0) \to f(0)$ as $k \to \infty$ in (3.11) we have

$$u = \cos(c \cdot x).$$

This completes the proof. □

Corollary 3.3. *Every locally integrable solution $f(x)$ of the equation* (1.2) *has the forms $f(x) = 0$ or $f(x) = \cos(c \cdot x)$ for some $c \in \mathbb{C}^n$.*

Proof. Since every locally integrable function f can be regarded as a distribution via the equation

$$\langle f, \varphi \rangle = \int f(x)\varphi(x)\,\mathrm{d}x, \quad \varphi \in C_c^\infty(\mathbb{R}^n),$$

we have

$$f = 0 \quad \text{or} \quad f = \cos(c \cdot x)$$

in the sense of distribution, which means that

$$f(x) = 0 \quad \text{or} \quad g(x) := f(x) - \cos(c \cdot x) = 0$$

for all x in a set E with the LEBESGUE measure $m(E^c) = 0$.

Assume that $f(x) = 0$ for all $x \in E$. Let $z \in \mathbb{R}^n$ be given and set $E_z = E \cap (E - z)$. Then E_z is LEBESGUE measurable with $m(E_z^c) = 0$. Let $w \in E_z$ and replace x by $w + z$, y by w in (1.2) to get

$$f(2w+z) + f(z) - 2f(w+z)f(w) = 0.$$

Thus we have

$$f(2w+z) + f(z) = 0$$

for all $w \in E_z$. Since $m(E_z^c) = 0$ we must have $2w + z \in E$ for some $w \in E_z$, which implies $f(z) = 0$. Now assume that $g(x) = 0$ for all $x \in E$. Note that g satisfies the equation

$$g(x+y) + g(x-y) - 2g(x)g(y) - 2f(x)\cos(c \cdot y) + 2f(y)\cos(c \cdot x) = 0. \tag{3.12}$$

Similarly, for given $z \in \mathbb{R}^n$ let $w \in E_z$ and replace x by $w + z$, y by w in (3.12). Then we have

$$g(2w+z) + g(z) = 0$$

for all $w \in E_z$, which implies $g(z) = 0$. This completes the proof. □

4. USING DIFFERENTIATION

In this section, using differentiation in SCHWARTZ distributions we consider the equations (1.11)−(1.16).

Lemma 4.1. *Let Y be an open subset of $\mathbb{R}^{n-1}$, I an open interval of $\mathbb{R}$ and let $u \in \mathcal{D}'(Y \times I)$. If $\partial_n u = 0$ then u can be written as*

$$u = u_{n-1} \otimes 1_{x_n},$$

where $u_{n-1} \in \mathcal{D}'(Y)$, which means that

$$\langle u,\, \phi\rangle = \int \langle u_{n-1},\, \phi(\cdot, x_n)\rangle \mathrm{d}x_n, \quad \phi \in C_c^\infty(Y \times I).$$

We refer the reader to [**8**, Theorem 3.1.4′] for a proof of the lemma.

Theorem 4.2. *The solutions $u, v, w \in \mathcal{D}'(I^n)$ of the equation* (1.11) *are either of the forms*

$$\begin{aligned} u &= \alpha\beta(1+x_1)^{a_1} \cdots (1+x_n)^{a_n} - 1, \\ v &= \alpha(1+x_1)^{a_1} \cdots (1+x_n)^{a_n} - 1, \\ w &= \beta(1+x_1)^{a_1} \cdots (1+x_n)^{a_n} - 1, \end{aligned}$$

where α, β, $a_j \in \mathbb{C}$, $j = 1, \ldots, n$ with $\alpha\beta \neq 0$, or else

$$\begin{aligned} &u = v = -1,\ w : \textit{arbitrary}, \\ &u = w = -1,\ v : \textit{arbitrary}. \end{aligned}$$

Proof. Differentiate (1.11) with respect to x_k and y_k respectively, to get

$$(1 + y_k)\,(\partial_k u) \circ S = (\partial_k v) \circ P_1 + (\partial_k v) \otimes w, \tag{4.1}$$

$$(1 + x_k)\,(\partial_k u) \circ S = (\partial_k w) \circ P_2 + v \otimes (\partial_k w), \tag{4.2}$$

for $k = 1, \cdots, n$. It follows from (4.1) and (4.2) that

$$(1 + x_k)\,(\partial_k v) \circ P_1 + (\partial_k v) \otimes w) = (1 + y_k)\,(\partial_k w) \circ P_2 + v \otimes (\partial_k w),$$

which is equivalent to

$$[(1 + x_k)\partial_k v] \otimes (1 + w) = (1 + v) \otimes [(1 + y_k)\partial_k w]. \tag{4.3}$$

We first consider the nontrivial case where $v \neq -1$ and $w \neq -1$. Choose $\psi_0 \in C_c^\infty(I^n)$ so that $\langle 1 + w, \psi_0\rangle \neq 0$. Then for every $\phi \in C_c^\infty(I^n)$ we have

$$\langle (1 + x_k)\partial_k v, \phi\rangle \langle 1 + w, \psi_0\rangle = \langle 1 + v, \phi\rangle \langle (1 + y_k)\partial_k w, \psi_0\rangle, \tag{4.4}$$

which implies

$$(1 + x_k)\partial_k v = a_k(1 + v) \tag{4.5}$$

for some $a_k \in \mathbb{C}$. Put (4.5) in (4.3) to get

$$(1 + y_k)\partial_k w = a_k(1 + w). \tag{4.6}$$

It follows from (4.5) that

$$\partial_k[(1 + x_k)^{-a_k}(v + 1)] = 0. \tag{4.7}$$

Let $k = n$. Then by Lemma 4.1 we have

$$(1 + x_n)^{-a_n}(v + 1) = v_{n-1} \otimes 1_{x_n},$$

which implies

$$v + 1 = v_{n-1} \otimes (1 + x_n)^{a_n}, \tag{4.8}$$

where $v_{n-1} \in \mathcal{D}'(I^{n-1})$. Put (4.8) in (4.5) and let $k = n - 1$ to get

$$(1 + x_{n-1})\partial_{n-1}v_{n-1} = a_{n-1}v_{n-1}. \tag{4.9}$$

Applying Lemma 4.1 again in (4.9) we have

$$v_{n-1} = v_{n-2} \otimes (1 + x_{n-1})^{a_{n-1}}. \tag{4.10}$$

From (4.8) and (4.10) we have

$$v + 1 = v_{n-2} \otimes (1 + x_{n-1})^{a_{n-1}}(1 + x_n)^{a_n}.$$

By this process we finally have

$$v + 1 = \alpha(1 + x_1)^{a_1} \cdots (1 + x_{n-1})^{a_{n-1}}(1 + x_n)^{a_n}, \tag{4.11}$$

for some $\alpha \in \mathbb{C}$. Also, from (4.6) we have

$$w + 1 = \beta(1 + y_1)^{a_1} \cdots (1 + y_{n-1})^{a_{n-1}}(1 + y_n)^{a_n}, \tag{12}$$

for some $\beta \in \mathbb{C}$. Now putting (4.11) and (4.12) in (1.11) we have

$$u \circ S = \alpha\beta(1 + x_1)^{a_1}(1 + y_1)^{a_1} \cdots (1 + x_n)^{a_n}(1 + y_n)^{a_n} - 1. \tag{4.13}$$

Choose $u_j \in C_c^\infty(I^n)$ such that $u_j \to u$ as $j \to \infty$. Let ϕ, $\psi \in C_c^\infty(I^n)$ with $\int \psi(y)dy = 1$ and set

$$\varphi(x, y) = (1 + y_1) \cdots (1 + y_n)\phi(x + y + x \diamond y)\psi(y). \tag{4.14}$$

Then by simple change of variables we have

$$\langle u_j \circ S,\, \varphi\rangle = \langle u_j,\, \phi\rangle. \tag{4.15}$$

Let $j \to \infty$ in (4.15). Then by the continuity of pullback we have

$$\langle u \circ S,\, \varphi\rangle = \langle u,\, \phi\rangle. \tag{4.16}$$

Applying $\varphi(x, y)$ in (4.16) we have

$$\langle u,\, \phi\rangle = \int [\alpha\beta(1+x_1)^{a_1}\cdots(1+x_n)^{a_n} - 1]\phi(x)\,\mathrm{d}x, \tag{4.17}$$

which implies

$$u = \alpha\beta(1+x_1)^{a_1}\cdots(1+x_n)^{a_n} - 1.$$

Now if $v = -1$ or $w = -1$ it is easy to see that

$$u \circ S = -1. \tag{4.18}$$

for arbitrary w or v, respectively. Applying $\varphi(x, y)$ in (4.18) we have $u = -1$. This completes the proof. □

As a direct consequence of the above result we have the following.

Theorem 4.3. *The solutions $u \in \mathcal{D}'(I^n)$ of the equation* (1.12) *are of the form*

$$u = (1+x_1)^{a_1}\cdots(1+x_n)^{a_n} - 1, \quad \text{or} \quad u = -1$$

for some $a_j \in \mathbb{C}$, $j = 1, \ldots, n$.

Now we consider the logarithmic functional equations.

Theorem 4.4. *The solutions $u, v, w \in \mathcal{D}'(I^n)$ of the equation* (1.13) *are of the forms*

$$u = \ln(x_1^{a_1}\cdots x_n^{a_n}) + \alpha + \beta \tag{4.19}$$

$$v = \ln(x_1^{a_1}\cdots x_n^{a_n}) + \alpha \tag{4.20}$$

$$w = \ln(x_1^{a_1}\cdots x_n^{a_n}) + \beta, \tag{4.21}$$

where α, β, $a_j \in \mathbb{C}$, $j = 1, \ldots, n$.

Proof. Differentiate (1.13) with respect to x_k and y_k respectively, to get

$$y_k\,(\partial_k u \circ T) = (\partial_k v) \circ P_1, \tag{4.22}$$

$$x_k\,(\partial_k u \circ T) = (\partial_k w) \circ P_2, \tag{2.23}$$

for $k = 1, \cdots, n$. It follows from (4.22) and (4.23) that

$$x_k\,(\partial_k v \circ P_1) = y_k\,(\partial_k w \circ P_2), \tag{2.24}$$

which implies

$$x_k \partial_k v = y_k \partial_k w := a_k. \tag{4.25}$$

As in the proof of Theorem 4.2, applying the Lemma 4.1 in (4.25) successively we get the equation (4.20) and (4.21). Putting (4.20) and (4.21) in (1.13) we get (4.19). This completes the proof. □

As a direct consequence of the above result we have the following.

Theorem 4.5. *The solutions $u \in \mathcal{D}'(I^n)$ of the equation* (1.14) *are of the form*

$$u = \ln(x_1^{a_1} \cdots x_n^{a_n}),$$

where $a_j \in \mathbb{C}$, $j = 1, \ldots, n$.

Following the same approach as in the proof of Theorem 4.2 we get the followings.

Theorem 4.6. *The solutions $u, v, w \in \mathcal{D}'(I^n)$ of the equation* (1.15) *are either of the forms*

$$\begin{aligned} u &= \alpha\beta x_1^{a_1} \cdots x_n^{a_n}, \\ v &= \alpha\, x_1^{a_1} \cdots x_n^{a_n}, \\ w &= \beta x_1^{a_1} \cdots x_n^{a_n}, \end{aligned}$$

where α, β, $a_j \in \mathbb{C}$, $j = 1, \ldots, n$ with $\alpha\beta \neq 0$, or else

$$\begin{aligned} &u = v = 0,\ w : \textit{arbitrary}, \\ &u = w = 0,\ v : \textit{arbitrary}. \end{aligned}$$

Theorem 4.7. *The solutions $u \in \mathcal{D}'(I^n)$ of the equation* (1.16) *are of the form*

$$u = x_1^{a_1} \cdots x_n^{a_n},$$

where $a_j \in \mathbb{C}$, $j = 1, \ldots, n$.

5. HYERS-ULAM STABILITIES OF THE EQUATIONS

In this section we consider stability theorems of the quadratic functional inequality

$$\| u \circ A + u \circ B - 2u \circ P_1 - 2u \circ P_2 \| \leq \epsilon, \tag{5.1}$$

and the D'ALEMBERT inequality

$$\| u \circ A + u \circ B - u \otimes u \| \leq \epsilon, \tag{5.2}$$

in the space $\mathcal{D}'(\mathbb{R}^n)$. Also we consider the POMPEIU inequalities

$$\| u \circ S - v \circ P_1 - w \circ P_2 - v \otimes w \| \leq \epsilon, \tag{5.3}$$

(5.4) $$\|u \circ S - u \circ P_1 - u \circ P_2 - u \otimes u\| \le \epsilon,$$

in the space $\mathcal{D}'(I^n)$, logarithmic functional inequalities

(5.5) $$\|u \circ T - v \circ P_1 - w \circ P_2\| \le \epsilon,$$

(5.6) $$\|u \circ T - u \circ P_1 - u \circ P_2\| \le \epsilon,$$

and multiplicative functional inequalities

(5.3) $$\|u \circ T - v \otimes w\| \le \epsilon,$$

(5.4) $$\|u \circ T - u \otimes u\| \le \epsilon,$$

in the space $\mathcal{D}'(\mathbb{R}^n_+)$. Here $\|\cdot\|$ denotes the norms $|\langle\cdot\,,\varphi\rangle| \le \epsilon\|\varphi\|_{L^1}$ for all test functions φ. For the proof of the result we refer the reader to [**4**, **5**].

Theorem 5.1. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfy the inequality* (5.1). *Then there exists a unique quadratic function*

$$q(x) = \sum_{1\le j\le k\le n} a_{jk}\, x_j x_k$$

such that

$$\|u - q(x)\| \le \frac{\varepsilon}{2}\,.$$

Theorem 5.2. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ satisfy the inequality* (5.2) *Then either u is a bounded measurable function satisfying*

$$\|u\|_{L^\infty} \le \frac{1}{2}(1+\sqrt{1+2\epsilon})$$

or else

$$u = \cos(c\cdot x)$$

for some $c \in \mathbb{C}^n$.

Theorem 5.3. *Let $u, v, w \in \mathcal{D}'(I^n)$ be a solution of the inequality* (5.3) *with $v \ne -1$ and $w \ne -1$. Then either u, v and w are all bounded functions or of the forms*

$$\begin{aligned} u &= \alpha\beta(1+x_1)^{a_1}\cdots(1+x_n)^{a_n} - 1,\\ v &= \alpha(1+x_1)^{a_1}\cdots(1+x_n)^{a_n} - 1,\\ w &= \beta(1+x_1)^{a_1}\cdots(1+x_n)^{a_n} - 1 + r(x), \end{aligned}$$

where α, β, $a_j \in \mathbb{C}$, $j = 1, \ldots, n$ with $\alpha\beta \ne 0$ and $r(x)$ is a bounded measurable function on I^n such that $\|r\|_{L^\infty} \le \epsilon$. In particular, if $v = -1\,(w = -1)$, w is arbitrary (v is arbitrary) and u is a bounded measurable function such that $\|u(x)+1\|_{L^\infty(I^n)} \le \epsilon$.

As a direct consequence of the above result we get the following.

Theorem 5.4. *Let $u \in \mathcal{D}'(I^n)$ be a solution of the inequality* (5.4). *Then either u is a bounded measurable function such that*

$$\|u(x) + 1\|_{L^\infty} \le \frac{1}{2}\,(1 + \sqrt{1 + 4\epsilon})$$

or else

$$u = (1 + x_1)^{a_1} \cdots (1 + x_n)^{a_n} - 1$$

for some $a \in \mathbb{C}^n$.

Finally we state the stability of logarithmic functional equations (5.5), (5.6) and the multiplicative functional equations (5.7), (5.8).

Theorem 5.5. *Let $u, v, w \in \mathcal{D}'(\mathbb{R}_+^n)$ satisfy the inequality* (5.5). *Then u, v, w are of the forms*

$$\begin{aligned} u &= \ln(x_1^{a_1} \cdots x_n^{a_n}) + \alpha + \beta + r_1(x), \\ v &= \ln(x_1^{a_1} \cdots x_n^{a_n}) + \alpha + r_2(x), \\ w &= \ln(x_1^{a_1} \cdots x_n^{a_n}) + \beta + r_3(x), \end{aligned}$$

where α, β, $a_j \in \mathbb{C}$, $j = 1, \ldots, n$ and r_1, r_2, r_3 are bounded measurable functions on $\mathbb{R}_+^n$ such that

$$\|r_1\|_{L^\infty} \le 3\epsilon, \ \ \|r_2\|_{L^\infty} \le 4\epsilon, \ \ \|r_3\|_{L^\infty} \le 4\epsilon.$$

Theorem 5.6. *Let $u \in \mathcal{D}'(\mathbb{R}_+^n)$ satisfy the inequality* (5.6). *Then u is of the form*

$$u = \ln(x_1^{a_1} \cdots x_n^{a_n}) + r(x)$$

where $a_j \in \mathbb{C}$, $j = 1, \ldots, n$ and r is a bounded measurable function on $\mathbb{R}_+^n$ such that

$$\|r\|_{L^\infty} \le \epsilon.$$

Theorem 5.7. *Let $u, v, w \in \mathcal{D}'(\mathbb{R}_+^n)$ satisfy the inequality* (5.7) *with $v \ne 0$, $w \ne 0$. Then u, v and w are all bounded measurable functions or of the forms*

$$\begin{aligned} u &= \alpha\beta x_1^{a_1} \cdots x_n^{a_n}, \\ v &= \alpha\, x_1^{a_1} \cdots x_n^{a_n}, \\ w &= \beta x_1^{a_1} \cdots x_n^{a_n}, \end{aligned}$$

where α, β, $a_j \in \mathbb{C}$, $j = 1, \ldots, n$ with $\alpha\beta \ne 0$.

Theorem 5.8. *Let $u \in \mathcal{D}'(\mathbb{R}_+^n)$ satisfy the inequality* (5.8). *Then u is a bounded measurable function such that*

$$\|u(x)\|_{L^\infty} \le \frac{1}{2}(1 + \sqrt{1 + 4\epsilon})$$

or else

$$u = x_1^{a_1} \cdots x_n^{a_n},$$

where $a_j \in \mathbb{C}$, $j = 1, \ldots, n$.

Acknowledgement. The author was supported by the Korean Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2007-521-C00016).

REFERENCES

1. J. Aczél, J. Dhombres: *Functional equations in several variables.* Cambridge University Press, New York-Sydney, 1989.
2. J. A. Baker: *Distributional methods for functional equations.* Aequationes Math., **62** (2001), 136–142.
3. J. A. Baker: *On a functional equation of Aczél and Chung.* Aequationes Math., **46** (1993), 99–111.
4. J. Chung: *Distributional method for a class of functiuonal equations and their stabilities.* Acta Math. Sinica, **23** (2007), 2017–2026.
5. J. Chung: *A distributional version of functional equations and their stabilities.* Nonlinear Analysis, **62** (2005), 1037–1051.
6. E. Y. Deeba, E. L. Koh: *The Pexider functional equations in distributions.* Canad. J. Math., **42** (1990), 304–314.
7. E. Y. Deeba, Shishen Xie: *Distributional analog of a functional equation.* Applied Mathematics Letters, **16** (2003), 669–673.
8. L. Hörmander: *The analysis of linear partial differential operators I.* Springer-Verlag, Berlin–New York, 1983.
9. Th. M. Rassias: *On the stability of linear mapping in Banach spaces.* Proc. Amer. Math. Soc., **72** (1978), 297–300.
10. L. Schwartz: *Théorie des distributions.* Hermann, Paris, 1966.

Department of Mathematics,
Kunsan National University,
Kunsan 573-701,
Republic of Korea
E–mail: jychung@kunsan.ac.kr

(Received September 25, 2008)