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TRANSVERSAL SPACES AND FIXED POINT THEOREMS

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In this paper we define Transversal functional probabilistic spaces (upper and lower) as a natural extension of Metric spaces, Probabilistic metric spaces and Fuzzy metric spaces. Also, we formulate and prove some fixed and common fixed point theorems.

1. INTRODUCTION

Transversal spaces were introduced by Tasković in [**11**]. Some of the first results in fixed point theory for mappings defined on transversal spaces are given in [**11**] and [**7**].

Definition 1.1. *Let X be a nonempty set and let $P := (P, \preceq)$ be a partially ordered set. The function $\rho : X \times X \to P$ is called an upper ordered transverse on X if $\rho(x, y) = \rho(y, x)$, and if there exists an upper bisection function $g : P \times P \to P$ such that*

$$\rho(x, y) \preceq \sup\big\{\rho(x, z), \rho(z, y), g\big(\rho(x, z), \rho(z, y)\big)\big\}$$

for all $x, y, z \in X$. An upper ordered transversal space is a triple (X, ρ, g).

Definition 1.2. *The function $\rho : X \times X \to P$ is called a lower ordered transverse on X if $\rho(x, y) = \rho(y, x)$ and if there exists a lower bisection function $d : P \times P \to P$ such that*

$$\inf\big\{\rho(x, z), \rho(z, y), d\big(\rho(x, z), \rho(z, y)\big)\big\} \preceq \rho(x, y)$$

for all $x, y, z \in X$. A lower ordered transversal space is a triple (X, ρ, d).

For $P = [0, +\infty)$ the spaces (X, ρ, g) and (X, ρ, d) we will call upper and lower transversal space.

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For $P = [a, b]$, $0 < a < b$ these spaces we will call the upper or lower transversal interval spaces (see [**12**]). Especially, for $a = 0$ and $b = 1$ we will call this spaces upper and lower transversal probabilistic spaces.

Example 1.1. Every metric space (X, d) can be considered as an upper transversal space. Indeed, for the upper bisection function $g : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ defined by $g(a, b) := a + b$ and the upper transverse function being the metric d of the metric space (X, d), from $d(x, y) \le d(x, z) + d(z, y)$ it follows that the metric space is an upper transversal space. For this choise of the upper transversal function and the upper bisection function we say that the upper transversal space is induced by the metric d.

Example 1.2. Every metric space (X, δ) can be considered as a lower transversal space, too. For the lower bisection function $d : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ defined by $d(a, b) := \big||a| - |b|\big|$ and the lower transverse function being the metric of the metric space (X, δ), from $\delta(x, y) \ge \big|\delta(x, z) - \delta(z, y)\big|$ it follows that every metric space is a lower transversal space. For this choise of the lower transversal function and the lower bisection function we say that the lower transversal space is induced by the metric δ.

Example 1.3. Cicchese in [**2**] and [**3**] defines and considers generalized metric spaces. Here the triangle inequality of the metric spaces is replaced by the following condition:

There exist subset $A \subseteq [0, +\infty)$ which contains an interval $[0, a)$, for some $a > 0$, and a function $\varphi : A \to [0, +\infty)$ such that $\lim\limits_{x \to 0} \varphi(x) = 0$ and for some fixed $\tau \ge 1$ and every $x, y, z \in X$ such that $\rho(x, z) \in A$ holds:

$$\rho(x, y) \le \varphi[\rho(x, z)] + \tau\rho(z, y).$$

It is easy to show that every generalized metric space can be considered as an upper transversal space, where the upper bisection function g is given by $g(p, q) = \varphi(p) + \tau q$, and $P = A$.

2. PRELIMINARIES

We give the definitions of upper and lower transversal functional probabilistic spaces.

Definition 2.1. *Let X be a nonempty set. The symmetric function $\rho : X \times X \times [0, +\infty) \to [0, 1]$ is called an upper functional probabilistic transverse on X if there exists a function $g : [0, 1] \times [0, 1] \to [0, 1]$, called an upper probabilistic bisection function, such that*

$$\rho(p, q)(x) \le \max\big\{\rho(p, s)(x), \rho(s, q)(x), g\big(\rho(p, s)(x), \rho(s, q)(x)\big)\big\} \tag{1}$$

for all $p, q, s \in X$ and for each $x \in [0, +\infty)$. The triple (X, ρ, g) will be called an upper transversal functional probabilistic space.

Definition 2.2. *Let X be a nonempty set. The symmetric function $\rho : X \times X \times [0, +\infty) \to [0, 1]$ is called a lower functional probabilistic transverse on X if there* *exists a function* $d : [0,1] \times [0,1] \to [0,1]$, *called a lower probabilistic bisection function, such that*

$$\rho(p,q)(x) \geq \min\left\{\rho(p,s)(x), \rho(s,q)(x), d\big(\rho(p,s)(x), \rho(s,q)(x)\big)\right\} \tag{2}$$

for all $p,q,s \in X$ *and for each* $x \in [0,+\infty)$. *The triple* (X,ρ,d) *we will call lower transversal functional probabilistic space.*

Example 2.1. Every metric space (X,δ) can be considered as a lower transversal functional probabilistic space (X,ρ,d) with the lower probabilistic bisection function $d(a,b) = \min\{a,b\}$ and the lower functional probabilistic transverse $\rho(p,q)(x) = \dfrac{\theta(x)}{\theta(x)+\delta(p,q)}$ where $\theta : [0,+\infty) \to [0,+\infty)$ and $\theta(0) = 0$ is a bijection function such that $\lim\limits_{x\to+\infty} \theta(x) = +\infty$. The triple (X,ρ,d) we will call the lower transversal functional probabilistic space induced by the metric δ.

Before we give few more examples of transversal spaces we introduce Menger probabilistic spaces and Fuzzy metric spaces.

Definition 2.3. *Let* $\mathcal{S}$ *denote the set of all distributions, i.e. the set of all non-decreasing, left-continuous functions* $f : \mathbb{R} \to \mathbb{R}^+$, *satisfying* $\inf\{f(x) : x \in \mathbb{R}\} = 0$ *and* $\sup\{f(x) : x \in \mathbb{R}\} = 1$. *A probabilistic metric space is a pair* $(X,\mathcal{F})$, *where* X *is a nonempty set,* $\mathcal{F} : X \times X \to \mathcal{S}$ *a mapping that to every point* $(p,q) \in X \times X$ *assignes a function from* $\mathcal{S}$, *denoted as* $F_{p,q}(x)$, *which satisfies*

(a) $F_{p,q}(x) = 1$, *for all* $x > 0$ *if and only if* $p = q$,

(b) $F_{p,q}(0) = 0$,

(c) $F_{p,q}(x) = F_{q,p}(x)$,

(d) *From* $F_{p,q}(x) = 1$ *and* $F_{q,s}(y) = 1$ *follows that* $F_{p,s}(x+y) = 1$,

for all $p,q,s \in X$ *and all* $x,y \in \mathbb{R}$.

Definition 2.4. *A binary operation* $t : [0,1]\times[0,1] \to [0,1]$ *is a t-norm if* t *satisfies the following conditions*:

(1) t *is commutative and associative,*

(2) $t(a,1) = a$ *for all* $a \in [0,1]$,

(3) $t(a,b) \leq t(c,d)$ *whenever* $a \leq c$ *and* $b \leq d$, *and* $a,b,c,d \in [0,1]$.

Examples of t-norms are $t(a,b) = \min\{a,b\}$ and $t(a,b) = ab$.

Definition 2.5. *A triple* $(X,\mathcal{F},\mathcal{T})$, *where* $(X,\mathcal{F})$ *is a probabilistic metric space and* $\mathcal{T}$ *a t-norm which satisfies Menger's inequality*

$$F_{p,q}(x+y) \geq \mathcal{T}[F_{p,s}(x), F_{s,q}(y)]$$

for all $p,q,s \in X$ *and all* $x \geq 0, y \geq 0$, *is called a Menger probabilistic metric space.*

Every metric space is a Menger probabilistic metric space when t-norm is given with $\mathcal{T}(a,b) = \min\{a,b\}$. In that case Menger's inequality follows from the triangular inequality, taking $F_{p,q}(x) = H\big(x - d(p,q)\big)$, where

$$H(x) = \begin{cases} 0, & \text{for} \quad x \le 0 \\ 1, & \text{for} \quad x > 0 \end{cases}.$$

Example 2.2. Every Menger probabilistic metric space can be considered as a lower transversal functional probabilistic space. In this case lower probabilistic bisection function is defined with $d(\rho(p,s)(x), \rho(s,q)(x)) = \mathcal{T}\big(\rho(p,s)(x/2), \rho(s,q)(x/2)\big)$, where t-norm satisfies Menger's inequality. We define the lower functional probabilistic transverse as $\rho(p,q)(x) = F_{p,q}(x)$. Now the inequality that defines the lower transversal functional probabilistic space follows from the next inequalities, since

$$\begin{aligned} \rho(p,q)(x) &= F_{p,q}(x) \ge \mathcal{T}[F_{p,s}(x/2), F_{s,q}(x/2)] \\ &\ge \min\big\{F_{p,s}(x), F_{s,q}(x), \mathcal{T}[F_{p,s}(x/2), F_{s,q}(x/2)]\big\} \\ &= \min\big\{\rho(p,s)(x), \rho(s,q)(x), d\big(\rho(p,s)(x), \rho(s,q)(x)\big)\big\} \end{aligned}$$

for all $p, q, s \in X$ and all $x \ge 0$.

Kramosil and Michalek ([**9**]) have defined a notion of fuzzy metric spaces. George and Veeramani ([**4, 5**]) have modified this definition in the following sense:

Definition 2.6. [**13**] *A fuzzy set A in X is a function $A : X \to [0,1]$.*

In fuzzy metric spaces the usual notation of t-norm is $*$ i.e. $a * b = t(a,b)$.

Definition 2.7. [**4**] *A triple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions*:

(Fm-1) $M(x,y,t) > 0$,

(Fm-2) $M(x,y,t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,

(Fm-3) $M(x,y,t) = M(y,x,t)$,

(Fm-4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$ *for all* $x,y,z \in X$ *and* $t,s > 0$,

(Fm-5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous.*

Example 2.3. Every fuzzy metric space can be considered as a lower transversal functional probabilistic space. The proof of this fact is similar to the proof given in Example 2.2.

Definition 2.8. *Let (X, ρ, d) be a lower transversal functional probabilistic space.*
(a) *A sequence $(p_n)_{n\in\mathbb{N}}$ in (X, ρ, d) converges to a point $p \in X$ if for any $\varepsilon > 0$ and any $\lambda \in (0,1)$ there exists an integer n_0 such that*

$$\rho(p, p_n)(\varepsilon) > 1 - \lambda \text{ for all } n \ge n_0.$$

(b) *A sequence $(p_n)_{n\in\mathbb{N}}$ is said to be Cauchy if for any $\varepsilon > 0$ and any $\lambda \in (0,1)$ there exists an integer n_0 such that*

$$\rho(p_m, p_n)(\varepsilon) > 1 - \lambda \quad \text{for all} \quad m, n \ge n_0.$$

(c) *A lower transversal functional probabilistic space will be called complete if every Cauchy sequence is convergent in* X.

Throughout this paper we consider lower transversal functional probabilistic spaces with the lower functional probabilistic transverse $\rho(p,q)(x)$ which satisfies the following conditions

(T1) $\rho(p,q)(x)$ is a left-continuous function for $x \in (0,\infty)$ and right-continuous at the point $x = 0$,

(T2) $\rho(p,q)(x) = 1$ for all $x > 0$ iff $p = q$,

(T3) $\rho(p,q)(x)$ is a non-decreasing function,

(T4) $\lim\limits_{x\to+\infty} \rho(p,q)(x) = 1$ for all $p, q \in X$.

Also, we assume that the lower probabilistic bisection function $d(x,y)$ satisfies:

(B1) $d(x,y)$ is a non-decreasing and continuous function,

(B2) $d(x,x) \geq x$,

(B3) $\lim\limits_{x\to 1} d(a,x) = a$.

Lemma 2.1. *Let* (X,ρ,d) *be a lower transversal functional probabilistic space, with the lower functional probabilistic transverse satisfying* (T1)–(T4) *and lower bisection function satisfying* (B1)–(B3), $\liminf\limits_{n\to+\infty} p_n = p$, $\liminf\limits_{n\to+\infty} q_n = q$, *then*

$$\liminf_{n\to+\infty} \rho(p_n,q_n)(x) = \rho(p,q)(x).$$

Proof. The proof follows from the fact that $\rho(p,q)(x)$ is a left-continuous function and the fact that (T1)–(T4) and (B1)–(B3) are satisfied. The body of the proof is similar to analogous result for probabilistic metric spaces (see [**10**]). □

Definition 2.9. *Let* (X,ρ,d) *be a lower transversal functional probabilistic space and* $A \subseteq X$. *Let the mappings* $\delta_A(t) : (0,\infty) \to [0,1]$ *be defined as*

$$\delta_A(t) = \inf_{p,q\in A} \sup_{\varepsilon< t} \rho(p,q)(\varepsilon).$$

The constant $\delta_A = \sup\limits_{t>0} \delta_A(t)$ *will be called lower transversal functional probabilistic diameter of set* A.

Definition 2.10. *If* $\delta_A = 1$ *we will call the set* A *strongly bounded set in lower transversal functional probabilistic space.*

Lemma 2.2. *Let* (X,ρ,d) *be a lower transversal functional probabilistic space. A set* $A \subseteq X$ *is strongly bounded iff for each* $r \in (0,1)$ *there exists* $t > 0$ *such that* $\rho(p,q)(t) > 1 - r$ *for all* $p, q \in A$.

Proof. The proof follows from the definitions of $\sup A$ and $\inf A$ of non-empty sets.

Example 2.4. Let (X,ρ,d) be a lower transversal functional probabilistic space induced by the metric δ introduced in the Example 2.1. $A \subseteq X$ is metrically bounded if and only if it is strongly bounded in the lower transversal functional probabilistic space (X, ρ, d). To prove that, suppose that $A \subseteq X$ is metrically bounded, i.e. $\delta(p, q) < k$, for some $k \in \mathbb{R}$ and all $p, q \in A$. Let $r \in (0, 1)$ be arbitrary. For the lower functional probabilistic transverse $\rho(p, q)(x)$ it follows that $\rho(p, q)(x) > \dfrac{\theta(x)}{\theta(x) + k}$ for all $p, q \in A$. From $\dfrac{\theta(x)}{\theta(x) + k} > 1 - r$ we get that $\theta(x) > \dfrac{k(1-r)}{r}$. Since θ is a bijection function it follows that there exists $x \in (0, +\infty)$ such that $\theta(x) > \dfrac{k(1-r)}{r}$, i.e. $\rho(p, q)(x) > 1 - r$. From Lemma 2.2. it follows that A is strongly bounded set in the lower transversal functional probabilistic space (X, ρ, d). Conversely, if A is a strongly bounded set in (X, ρ, d) then for arbitrary $r \in (0, 1)$ there exists $x > 0$ such that $\rho(p, q)(x) = \dfrac{\theta(x)}{\theta(x) + \delta(p, q)} > 1 - r$ for all $p, q \in A$. From these inequalities it follows that $\delta(p, q) < \dfrac{r}{1-r}\,\theta(x)$ for all $p, q \in A$ i.e. the set A is metrically bounded. This completes the proof. $\square$

Definition 2.11. *Let (X, ρ, d) be a lower transversal functional probabilistic space. A subset $F \subseteq X$ will be called closed if for every sequence $\{p_n\}_{n \in \mathbb{N}} \subseteq F$ such that $p_n \to p_0$ as $n \to \infty$ it follows that $p_0 \in F$. The minimal closed set containing F will be called the closure of F and it will be denoted by $\overline{F}$.*

Definition 2.12. *Let (X, ρ, d) be a lower transversal functional probabilistic space. A collection of sets $\{F_n\}_{n \in \mathbb{N}}$ is said to have lower transversal diameter zero iff for each pair $\lambda \in (0, 1)$ and $x > 0$ there exists $n \in \mathbb{N}$ such that $\rho(p, q)(x) > 1 - \lambda$ for all $p, q \in F_n$.*

Theorem 2.1. *Let (X, ρ, d) be a complete lower transversal functional probabilistic space and let $\{F_n\}_{n \in \mathbb{N}}$ be a nested sequence of nonempty closed sets. If this sequence of sets has lower transversal diameter zero then it has a nonempty intersection.*

Proof. Let $\{F_n\}_{n \in \mathbb{N}}$ be a nested sequence of nonempty closed sets with a lower transversal diameter zero. Let $p_n \in F_n, n \in \mathbb{N}$. Since $\{F_n\}_{n \in \mathbb{N}}$ has lower transversal diameter zero, for $\lambda \in (0, 1)$ and $x > 0$ there exists $n_0 \in \mathbb{N}$ such that $\rho(p, q)(x) > 1 - \lambda$ for all $p, q \in F_{n_0}$. Therefore, $\rho(p_n, p_m)(x) > 1 - \lambda$ for all $n, m \geq n_0$. Since $p_n \in F_n \subset F_{n_0}$ and $p_m \in F_m \subset F_{n_0}$ it follows that $\{p_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Because (X, ρ, d) is a complete lower transversal functional probabilistic space then p_n converges to some $p \in X$. Now for each n it follows that $p_k \in F_n$ for all $k \geq n$. Therefore $p \in \overline{F_n} = F_n$ for every n and hence $p \in \bigcap_{n \in \mathbb{N}} F_n$. This completes the proof.

Remark 2.1. If the space (X, ρ, d) satisfies (T2) then the element $p \in \bigcap_{n \in \mathbb{N}} F_n$ is unique. Let us suppose that there exist $p, q \in \bigcap_{n \in \mathbb{N}} F_n$. From the fact that the family $\{F_n\}_{n \in \mathbb{N}}$ has lower transversal diameter zero it follows that $\rho(p, q)(x) > 1 - 1/n$ for each n and for fixed $x > 0$. this implies that $\rho(p, q)(x) = 1$. From (T2) it follows that $p = q$.

Definition 2.13. *Two self-mappings S and T defined on a lower transversal functional probabilistic space (X, ρ, d) are compatible if*

$$\lim_{n \to +\infty} \rho(STp_n, TSp_n)(x) = 1 \text{ for all } x > 0,$$

whenever $(p_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that sequences* $(Sp_n)_{n\in\mathbb{N}}$ *and* $(Tp_n)_{n\in\mathbb{N}}$ *converge to some point* $p \in X$*. Than we say that the pair* $\{S, T\}$ *is compatible.*

Remark 2.2. Let S and T be compatible self-mappings defined on a lower transversal functional probabilistic space (X, ρ, d). From Definition 2.13. by taking $p_n = z$ for all $n \in \mathbb{N}$ and for some point $z \in X$ it follows: If $Sz = Tz$ for some $z \in X$ then $STz = TSz$.

Lemma 2.3. *Let* S *and* T *be compatible self-mappings defined on a lower transversal functional probabilistic space* (X, ρ, d) *with the lower bisection function which satisfies* (B1)–(B3) *and let* Sp_n *and* Tp_n *converge to some point* $z \in X$ *for a sequence* $\{p_n\}_{n\in\mathbb{N}}$ *in* X*. If* S *is continuous then*

$$\lim_{n\to+\infty} TSp_n = Sz \quad \text{for all} \quad x > 0.$$

Proof. Let $\lambda \in (0, 1)$ and $x > 0$ be arbitrary. Since S and T are compatible then $\rho(TSp_n, STp_n)(x) > 1-\lambda$. Also, Sp_n and Tp_n converge to z, so $\rho(Tp_n, z)(x) > 1-\lambda$ and $\rho(Sp_n, z)(x) > 1 - \lambda$. From (B1)–(B3) and continuity of S, using (2) we have that the following inequalities hold.

$$\begin{aligned}\rho(TSp_n, Sz)(x) \geq\ & \min\big\{\rho(TSp_n, STp_n)(x), \rho(STp_n, Sz)(x),\\ & \qquad d\big(\rho(TSp_n, STp_n)(x), \rho(STp_n, Sz)(x)\big)\big\}\\ >\ & \min\big\{1-\lambda, d(1-\lambda, 1-\lambda)\big\} = 1-\lambda.\end{aligned}$$

Taking that $\lambda \to 0$ and $n \to +\infty$ we get

$$\liminf_{n\to+\infty} \rho(TSp_n, Sz)(x) = 1$$

i.e. $\lim\limits_{n\to+\infty} TSp_n = Sz$. This completes the proof.

3. MAIN RESULTS

In this section we present a common fixed point theorem for compatible mappings, with nonlinear contractive conditions. One of the first results that includes nonlinear contractive conditions for mappings defined on metric spaces was given by Boyd and Wong in [**1**]. Also, nonlinear contractive conditions for mappings defined on intuitionistic fuzzy metric spaces are obtained by Ješić and Babačev in [**8**]. These results are proved for commuting mappings and R-weakly commuting mappings (see [**8**]). It is easy to see that compatible mappings define a larger class of functions than the class of R-weakly commuting mappings. Indeed, every pair of commuting or R-weakly commuting mappings is compatible, but the converse is not true. We start this section with one lemma.

Lemma 3.1. *Let* (X, ρ, d) *be a lower transversal functional probabilistic space with the lower functional probabilistic transverse satisfying* (T1)–(T4)*. Let* φ :

$(0,+\infty) \to (0,+\infty)$ *be a continuous, non-decreasing function which satisfies* $\varphi(x) < x$ *for all* $x > 0$. *Then the following statement holds.*

If for $p, q \in X$ *it holds that* $\rho(p,q)\big(\varphi(x)\big) \geq \rho(p,q)(x)$ *for all* $x > 0$ *then* $p = q$.

Proof. First, note that $\lim\limits_{n\to+\infty} \varphi^n(x) = 0$ for all $x \in (0,1)$ where φ^n denotes the n-th iteration of φ. For fixed $p, q \in X$ there exists

$$\lim_{x\to 0+} \rho(p,q)(x) = \rho(p,q)(0).$$

Suppose $\rho(p,q)\big(\varphi(x)\big) \geq \rho(p,q)(x)$ for all $x > 0$. Because $\rho(p,q)\big(\varphi(x)\big) \leq \rho(p,q)(x)$, by induction it follows that $\rho(p,q)\big(\varphi^n(x)\big) = \rho(p,q)(x)$ for all $x > 0$. Taking limits as $n \to +\infty$ we have that $\rho(p,q)(x) = \rho(p,q)(0)$ for all $x > 0$ and $\lim\limits_{x\to+\infty} \rho(p,q)(x) = \rho(p,q)(0)$. From (T4) it follows that $\rho(p,q)(0) = 1$ i.e. $\rho(p,q)(x) = 1$ for all $x > 0$. This means that $p = q$.

Theorem 3.1. *Let* A, B, S *and* T *be self-mappings defined on complete lower transversal functional probabilistic space* (X, ρ, d) *such that* $A(X)$ *and* $B(X)$ *are strongly bounded sets, with the lower functional probabilistic transverse which satisfies* (T1)–(T4) *and lower bisection function which satisfies* (B1)–(B3)*, satisfying the conditions*:

(a) $A(X) \subseteq T(X), B(X) \subseteq S(X)$,

(b) *one of* A, B, S, T *is continuous*,

(c) *the pairs* $\{A, S\}$ *and* $\{B, T\}$ *are compatible*,

(d) *there exists a continuous, non-decreasing function* $\varphi : (0,\infty) \to (0,+\infty)$ *which satisfies* $\varphi(x) < x$ *for all* $x > 0$ *and*

$$\rho(Ap, Bq)\big(\varphi(x)\big) \geq \rho(Sp, Tq)(x), \text{ for all } x > 0 \text{ and } p, q \in X. \tag{3}$$

Then A, B, S *and* T *have a unique common fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point. From (a) it follows that there exists $x_1 \in X$ such that $A(x_0) = T(x_1)$ and for this point x_1 we have that there exists $x_2 \in X$ such that $B(x_1) = S(x_2)$. By induction we can construct a sequence $\{z_n\}_{n\in\mathbb{N}}$ as follows

$$z_{2n-1} = Tx_{2n-1} = Ax_{2n-2}, \quad z_{2n} = Sx_{2n} = Bx_{2n-1}. \tag{4}$$

Let us consider nested sequence of nonempty closed sets defined by

$$F_n = \overline{\{z_n, z_{n+1}, \ldots\}},\ n \in \mathbb{N}.$$

We shall prove that the family $\{F_n\}_{n\in\mathbb{N}}$ has a lower transversal diameter zero.

To this end, let $\lambda \in (0,1)$ and $x > 0$ be arbitrary. From $F_k \subseteq \overline{A(X)} \cup \overline{B(X)}$ it follows that F_k is an strongly bounded set for arbitrary $k \in \mathbb{N}$. Hence, there exists $x_0 > 0$ such that

$$\rho(p,q)(x_0) > 1 - \lambda \text{ for all } p, q \in F_k. \tag{5}$$

From $\lim_{n\to+\infty}\varphi^n(x_0)=0$ we conclude that there exists $m\in\mathbb{N}$ such that $\varphi^m(x_0)<x$. Let $n=m+k$ and $p,q\in F_n$ be arbitrary. There exist sequences $\{z_{n(i)}\}$, $\{z_{n(j)}\}$ in F_n $\quad(n(i),n(j)\geq n\quad i,j\in\mathbb{N})$ such that $\lim_{i\to+\infty}z_{n(i)}=p$ and $\lim_{j\to+\infty}z_{n(j)}=q$.

Case I. Suppose that $n(i)\in 2\mathbb{N}-1$ and $n(j)\in 2\mathbb{N}$ or vice versa for $i,j\in\mathbb{N}$ large enough i.e. $z_{n(i)}=Ax_{n(i)-1}$ and $z_{n(j)}=Bx_{n(j)-1}$.
From (3) we have that

$$
\begin{aligned}
\rho(z_{n(i)},z_{n(j)})\big(\varphi(x)\big)&=\rho(Ax_{n(i)-1},Bx_{n(j)-1})\big(\varphi(x)\big)\geq\rho(Sx_{n(i)},Tx_{n(j)})(x)\\
&=\rho(Ax_{n(i)-1},Bx_{n(j)-1})(x)=\rho(z_{n(i)-1},z_{n(j)-1})(x).
\end{aligned}
\tag{6}
$$

Thus, by induction we get

$$\rho(z_{n(i)},z_{n(j)})\big(\varphi^m(x)\big)\geq\rho(z_{n(i)-m},z_{n(j)-m})(x).$$

Since $\varphi^m(x_0)<x$ and because $\rho(p,q)(\cdot)$ is a non-decreasing function, from the last inequalities it follows that

$$\rho(z_{n(i)},z_{n(j)})(x)\geq\rho(z_{n(i)},z_{n(j)})\big(\varphi^m(x_0)\big)\geq\rho(z_{n(i)-m},z_{n(j)-m})(x_0).$$

As $\{z_{n(i)-m}\}$, $\{z_{n(j)-m}\}$ are sequences in F_k from (5) it follows that

$$\rho(z_{n(i)-m},z_{n(j)-m})(x_0)>1-r\quad\text{for all}\quad i,j\in\mathbb{N}.\tag{7}$$

Case II. If both of $n(i)$ and $n(j)$ are from set $2\mathbb{N}-1$ we have

$$
\begin{aligned}
\rho(z_{n(i)},z_{n(j)})\big(\varphi(x)\big)&=\rho(Ax_{n(i)-1},Ax_{n(j)-1})\big(\varphi(x)\big)\\
&\geq\min\big\{\rho(Ax_{n(i)-1},Bx_{n(\ell)-1})\big(\varphi(x)\big),\rho(Bx_{n(\ell)-1},Ax_{n(j)-1})\big(\varphi(x)\big),\\
&\qquad d\big(\rho(Ax_{n(i)-1},Bx_{n(\ell)-1})\big(\varphi(x)\big),\rho(Bx_{n(\ell)-1},Ax_{n(j)-1})\big(\varphi(x)\big)\big)\big\}
\end{aligned}
$$

for arbitrary $n(\ell)\geq n$ and $n(\ell)\in 2\mathbb{N}$. Since d is non-decreasing and satisfies (B2) it follows that $d(x,y)\geq\min\{x,y\}$. Applying this fact in previous inequalities we get

$$
\begin{aligned}
&\rho(z_{n(i)},z_{n(j)})\big(\varphi(x)\big)\\
&\quad\geq\min\big\{\rho(Ax_{n(i)-1},Bx_{n(\ell)-1})\big(\varphi(x)\big),\rho(Bx_{n(\ell)-1},Ax_{n(j)-1})\big(\varphi(x)\big)\big\}\\
&\quad\geq\min\big\{\rho(Sx_{n(i)},Tx_{n(\ell)})(x),\rho(Sx_{n(\ell)},Tx_{n(j)})(x)\big\}\\
&\quad\geq\min\big\{\rho(Ax_{n(i)-1},Bx_{n(\ell)-1})(x),\rho(Ax_{n(\ell)-1},Bx_{n(j)-1})(x)\big\}\\
&\quad=\min\big\{\rho(z_{n(i)-1},z_{n(\ell)-1})(x),\rho(z_{n(\ell)-1},z_{n(j)-1})(x)\big\}.
\end{aligned}
$$

By induction we conclude that

$$\rho(z_{n(i)},z_{n(j)})\big(\varphi^m(x)\big)\geq\min\big\{\rho(z_{n(i)-m},z_{n(\ell)-m})(x),\rho(z_{n(\ell)-m},z_{n(j)-m})(x)\big\}.$$

Finally, we get that

$$\begin{aligned}\rho(z_{n(i)}, z_{n(j)})(x) &\geq \rho(z_{n(i)}, z_{n(j)})\big(\varphi^m(x_0)\big)\\ &\geq \min\big\{\rho(z_{n(i)-m}, z_{n(\ell)-m})(x_0), \rho(z_{n(\ell)-m}, z_{n(j)-m})(x_0)\big\}.\end{aligned}$$

Since $\{z_{n(i)-m}\}$, $\{z_{n(j)-m}\}$ and $\{z_{n(\ell)-m}\}$ are sequences from F_k we have

$$\rho(z_{n(i)-m}, z_{n(\ell)-m})(x_0) > 1-\lambda \quad \text{and} \quad \rho(z_{n(\ell)-m}, z_{n(j)-m})(x_0) > 1-\lambda. \tag{8}$$

Analogous, we can prove the inequality (7) in the case when $n(i), n(j) \in 2\mathbb{N}$.

Finally, from (7) and (8) we conclude that in both cases it is satisfied

$$\rho(z_{n(i)}, z_{n(j)})(x) > 1-r$$

for all $i, j \in \mathbb{N}$. Taking the $\liminf$ as $i, j \to +\infty$, and applying Lemma 2.1. we get that $\rho(p,q)(x) > 1-\lambda$ for all $p, q \in F_n$ i.e. family $\{F_n\}_{n\in\mathbb{N}}$ has lower transversal diameter zero.

Applying Theorem 2.1. we conclude that this family has nonempty intersection, which consists of exactly one point z. Since the family $\{F_n\}_{n\in\mathbb{N}}$ has lower transversal diameter zero and $z \in F_n$ for all $n \in \mathbb{N}$ then for each $\lambda \in (0,1)$ and each $x > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that $\rho(z_n, z)(x) > 1-\lambda$. From the last it follows that for each $\lambda \in (0,1)$ it holds

$$\liminf_{n\to+\infty} \rho(z_n, z)(x) > 1-\lambda$$

Letting $\lambda \to 0$ we get

$$\liminf_{n\to+\infty} \rho(z_n, z)(x) = 1$$

i.e. $\lim\limits_{n\to+\infty} z_n = z$. From the definition of sequences $\{Ax_{2n-2}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}$ and $\{T_{2n-1}\}$ it follows that all these sequences converge to z.

Let us prove that z is a common fixed point of mappings A, B, S and T. For this purpose, let us first suppose that S is continuous. Then it holds that $\lim\limits_{n\to+\infty} SSx_{2n} = Sz$. From compatibility of $\{A, S\}$ and Lemma 2.3. it follows that $\lim\limits_{n\to+\infty} ASx_{2n} = Sz$. Using the condition (3) we get that the following inequality holds.

$$\rho(ASx_{2n}, Bx_{2n-1})\big(\varphi(x)\big) \geq \rho(SSx_{2n}, Tx_{2n-1})(x).$$

Taking the $\liminf$ as $n \to +\infty$ we get

$$\rho(Sz, z)\big(\varphi(x)\big) \geq \rho(Sz, z)(x).$$

From the Lemma 3.1., it follows that $Sz = z$. Using the condition (3) again we have that

$$\rho(Az, Bx_{2n-1})\big(\varphi(x)\big) \geq \rho(Sz, Tx_{2n-1})(x)$$

and taking the $\liminf$ as $n \to \infty$ we get

$$\rho(Az, z)\big(\varphi(x)\big) \geq \rho(Sz, z)(x) = \rho(z, z)(x) = 1.$$

This implies $Az = z$. Since $A(X) \subseteq T(X)$, there exists a point $u \in X$ such that $z = Az = Tu$ and it holds that

$$\rho(z, Bu)\big(\varphi(x)\big) = \rho(Az, Bu)\big(\varphi(x)\big) \geq \rho(Sz, Tu)(x) = \rho(z, z)(x) = 1,$$

which means that $Bu = z$. From the compatibility of $\{B, T\}$ and Remark 2.2 it follows that $Tz = TBu = BTu = Bz$. Also, from (3) it holds that

$$\rho(Ax_{2n}, Bz)\big(\varphi(x)\big) \geq \rho(Sx_{2n}, Tz)(x).$$

Taking the $\liminf$ as $n \to +\infty$ and using Lemma 3.1., we get that $Bz = z$. Therefore, z is a common fixed point of A, B, S and T. If T is a continuous function then the proof that z is a common fixed point of A, B, S and T is analogue to previous.

Now, suppose that A is a continuous function. Then $\rho(AAx_{2n}, Az)(x) > 1-\lambda$. From compatibility of $\{A, S\}$ and Lemma 2.3. it follows that $\rho(SAx_{2n}, Az)(x) > 1 - \lambda$. Using the condition (3) we get that

$$\rho(AAx_{2n}, Bx_{2n-1})\big(\varphi(x)\big) \geq \rho(SAx_{2n}, Tx_{2n-1})(x).$$

Taking the $\liminf$ as $n \to +\infty$ we obtain

$$\rho(Az, z)\big(\varphi(x)\big) \geq \rho(Az, z)(x).$$

From the Lemma 3.1., it follows that $Az = z$. Since $A(X) \subseteq T(X)$, there exists a point $v \in X$ such that $z = Az = Tv$. From $\rho(Az, Bv)\big(\varphi(x)\big)$ we have that

$$\rho(AAx_{2n}, Bv)\big(\varphi(x)\big) \geq \rho(SAx_{2n}, Tv)(x).$$

Taking the $\liminf$ as $n \to \infty$ we get

$$\rho(z, Bv)\big(\varphi(x)\big) = \rho(Az, Bv)\big(\varphi(x)\big) \geq \rho(Az, Tv)(x) = \rho(z, z)(x) = 1,$$

and so $z = Bv$. Since pair $\{B, T\}$ is compatible, using the Remark 2.2. we get $Tz = TBv = BTv = Bz$. Also, using (3) we get that

$$\rho(Ax_{2n}, Bz)\big(\varphi(x)\big) \geq \rho(Sx_{2n}, Tz)(x).$$

Taking the $\liminf$ as $n \to \infty$ we obtain

$$\rho(z, Bz)\big(\varphi(x)\big) \geq \rho(z, Tz)(x) = \rho(z, Bz)(x).$$

Hence, that $z = Bz = Tz$. Since $B(X) \subseteq S(X)$, there exists a point $w \in X$ such that $z = Bz = Sw$. Using (3) we get that

$$\rho(Aw, z)(\varphi(x)) = \rho(Aw, Bz)\big(\varphi(x)\big)$$

$$\geq \rho(Sw, Tz)(x) = \rho(Sw, Bz)(x) = \rho(z, z)(x) = 1,$$

which means that $Aw = z$. Since $\{A, S\}$ are compatible and $z = Aw = Sw$, from the Remark 2.2. we have that $Az = ASw = SAw = Sz$. Therefore, z is a common fixed point of A, B, S and T. If B is continuous then the proof is analogous.

Let us prove that z is a unique common fixed point. For this purpose let us suppose that there exists another common fixed point, denoted by y. From (3) follows that

$$\rho(z,y)\big(\varphi(x)\big) = \rho(Az,By)\big(\varphi(x)\big) \geq \rho(Sz,Ty)(x) = \rho(z,y)(x).$$

Finally, applying Lemma 3.1. it follows that $z = y$. This completes the proof. □

As a consequence of the previous theorem we get a version of the Banach contraction theorem with nonlinear contractive conditions for mappings defined on lower transversal functional probabilistic spaces.

Theorem 3.2. *Let A be a self-mapping defined on a complete lower transversal functional probabilistic space (X, ρ, d) such that $A(X)$ is strongly bounded set, with the lower functional probabilistic transverse which satisfies* (T1)–(T4) *and lower bisection function which satisfies* (B1)–(B3) *and there exists some continuous, non-decreasing function $\varphi : (0, +\infty) \to (0, +\infty)$, which satisfies $\varphi(x) < x$ for all $x > 0$ and*

$$\rho(Ap,Aq)\big(\varphi(x)\big) \geq \rho(p,q)(x), \text{ for all } x > 0 \text{ and } p, q \in X. \tag{9}$$

Then A has a unique fixed point.

Proof. Taking that $A = B$ and $S = T = I$ identical mapping, since A commutes with I all the conditions of Theorem 3.1 are satisfied, i.e. the statement follows from Theorem 3.1. □

Also, since fuzzy metric spaces and probabilistic Menger metric spaces are transversal, from Theorem 3.1 we get similar results for mappings defined on these spaces, which are improvements of results in fixed point theory with linear contractive conditions (see [**6**]).

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