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SOME FOX-WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTIONS AND ASSOCIATED FAMILIES OF CONVOLUTION OPERATORS

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Here, in this lecture, we aim at presenting a systematic account of the basic properties and characteristics of several subclasses of analytic functions (with *Montel's normalization*), which are based upon some convolution operators on Hilbert space involving the Fox-Wright generalization of the *classical* hypergeometric ${}_qF_s$ function (with q numerator and s denominator parameters). The various results presented in this lecture include (for example) normed coefficient inequalities and estimates, distortion theorems, and the radii of convexity and starlikeness for each of the analytic function classes which are investigated here. We also briefly indicate the relevant connections of the some of the results considered here with those involving the Dziok-Srivastava operator.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Following the usual notations, we let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \qquad (a_1 > 0), \tag{1.1}$$

which are *analytic* in $\mathbb{U} := \mathbb{U}(1)$, where

$$\mathbb{U}(r) := \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < r\}.$$

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For the class $\mathcal{A}$, the normalization:

$$f(0) = f'(0) - 1 = 0, \tag{1.2}$$

is *classical*. As already observed by DZIOK and SRIVASTAVA [**6**], one can obtain interesting results by applying *Montel's normalization* of the form (*cf.* MONTEL [**13**]):

$$f(0) = f'(\rho) - 1 = 0 \tag{1.3}$$

or

$$f(0) = f(\rho) - \rho = 0, \tag{1.4}$$

where ρ is a fixed point of the *punctured* unit disk

$$\mathbb{U}^* := \mathbb{U} \setminus \{0\} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad 0 < |z| < 1\}.$$

The classes of functions with the normalizations (1.3) and (1.4) will henceforth be called the *classes of functions with two fixed points* (see DZIOK and SRIVASTAVA [**6**, p. 8]).

A function f belonging to the class $\mathcal{A}$ is said to be *convex* in $\mathbb{U}(r)$ if and only if (*cf.* [**17**] and [**18**])

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad \big(z \in \mathbb{U}(r);\ 0 < r \leqq 1\big).$$

On the other hand, a function f belonging to the class $\mathcal{A}$ is said to be *starlike* in $\mathbb{U}(r)$ if and only if (*cf.* [**17**] and [**18**])

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad \big(z \in \mathbb{U}(r);\ 0 < r \leqq 1\big).$$

Suppose now that $\mathcal{B}$ is a subclass of the class $\mathcal{A}$. We define the radius of starlikeness $R^*(\mathcal{B})$ and the radius of convexity $R^c(\mathcal{B})$ for the class $\mathcal{B}$ by

$$R^*(\mathcal{B}) := \inf_{f \in \mathcal{B}} \big(\sup\{r \in (0,1] : f \quad \text{is starlike in} \quad \mathbb{U}(r)\}\big)$$

and

$$R^c(\mathcal{B}) := \inf_{f \in \mathcal{B}} \big(\sup\{r \in (0,1] : f \quad \text{is convex in} \quad \mathbb{U}(r)\}\big),$$

respectively.

For two given analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n\, z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n\, z^n,$$

we denote by $f * g$ the *Hadamard product* (or *convolution*) of f and g defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n\, b_n\, z^n =: (g * f)(z). \tag{1.5}$$

For complex parameters

$$\alpha_1, \ldots, \alpha_q \qquad \left(\frac{\alpha_j}{A_j} \neq 0, -1, -2, \ldots;\ j = 1, \ldots, q\right)$$

and

$$\beta_1, \ldots, \beta_s \qquad \left(\frac{\beta_j}{B_j} \neq 0, -1, -2, \ldots;\ j = 1, \ldots, s\right),$$

we define the Fox-Wright generalization ${}_q\Psi_s$ of the hypergeometric ${}_qF_s$ function by (*cf.* Fox [**8**] and Wright ([**20**] and [**21**]; see also [**15**, p. 21] and [**14**, p. 19])

$$(1.6)\quad {}_q\Psi_s\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_q, A_q);\\ \\ (\beta_1, B_1), \ldots, (\beta_s, B_s);\end{matrix}\ z\right] = {}_q\Psi_s\left[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,s}; z\right]$$

$$:= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n)\cdots\Gamma(\alpha_q + A_q n)}{\Gamma(\beta_1 + B_1 n)\cdots\Gamma(\beta_s + B_s n)}\ \frac{z^n}{n!}$$

$$\left(A_j > 0\ \ (j = 1, \ldots, q);\ B_j > 0\ \ (j = 1, \ldots, s);\ 1 + \sum_{j=1}^{s} B_j - \sum_{j=1}^{q} A_j \geqq 0\right)$$

for suitably bounded values of $|z|$. In particular, when

$$A_j = 1\ \ (j = 1, \ldots, q) \qquad \text{and} \qquad B_j = 1\ \ (j = 1, \ldots, s),$$

we have the following obvious relationship:

$$(1.7)\qquad {}_qF_s(\alpha_1, \ldots, \alpha_q;\ \beta_1, \ldots, \beta_s; z) = \omega\, {}_q\Psi_s\left[(\alpha_j, 1)_{1,q}; (\beta_j, 1)_{1,s}; z\right]$$

$$(q \leqq s + 1;\ q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \mathbb{U}),$$

where, *and in what follows*, $\mathbb{N}$ denotes the set of *positive* integers and

$$\omega := \frac{\Gamma(\beta_1)\cdots\Gamma(\beta_s)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_q)}. \tag{1.8}$$

Moreover, in terms of Fox's H-function [**9**], we have (*cf.*, *e.g.*, [**14**, p. 19])

$${}_q\Psi_s\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_q, A_q);\\ \\ (\beta_1, B_1), \ldots, (\beta_s, B_s);\end{matrix}\ z\right]$$

$$= H^{1,q}_{q,s+1}\left[-z \left|\ \begin{matrix}(1 - \alpha_1, A_1), \ldots, (1 - \alpha_q, A_q)\\ \\ (0, 1), (1 - \beta_1, B_1), \ldots, (1 - \beta_s, B_s)\end{matrix}\right.\right].$$

It should be remarked in passing that a *further* generalization of Fox's H-function is provided by the $\bar{H}$-function which was encountered in the physics literature while investigating and illustrating the use of certain Feynman integrals that arise naturally in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions (see, for example, [**16**]).

Other interesting and useful special cases of the Fox-Wright generalized hypergeometric ${}_q\Psi_s$ function defined by (1.6) include (for example) the generalized Bessel function $J_\nu^\mu(z)$ defined by (*cf.* Wright [**19**])

$$J_\nu^\mu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(\mu n+\nu+1)} = {}_0\Psi_1\left[\text{—};(\nu+1,\mu);-z\right],$$

which, for $\mu = 1$, corresponds essentially to the classical Bessel function $J_\nu(z)$, and the generalized Mittag-Leffler function $E_{\lambda,\mu}(z)$ defined by

$$E_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n+\mu)} = {}_1\Psi_1\left[(1,1);(\mu,\lambda);z\right],$$

whose *further* special cases appeared recently as solutions of several families of fractional differential equations with physical applications (see, for details, Gorenflo *et al.* [**10**]; see also the recent monograph on the subject of *Fractional Differential Equations* [**11**]).

Now let $q, s \in \mathbb{N}$ and suppose that the parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ are *also* positive real numbers. Then, corresponding to a function

$$\vartheta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s};z\right]$$

defined by

$$\vartheta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s};z\right] := \omega z\,{}_q\Psi_s\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s};z\right],$$

we consider a linear operator

$$\Theta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s}\right] : \mathcal{A} \longrightarrow \mathcal{A}$$

defined by the following Hadamard product (or convolution) (*cf.* Dziok *et al.* [**3**, p. 45 *et seq.*]):

$$\Theta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s}\right] f(z) := \vartheta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s};z\right] * f(z). \tag{1.9}$$

Remark 1. The linear operator $\Theta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s}\right]$ includes (as its special cases) various other linear operators which were investigated, in a *unified* manner, by Dziok and Srivastava ([**4**], [**5**] and [**6**]), who made appropriate use of the hypergeometric ${}_qF_s$ function (in place of the Fox-Wright ${}_q\Psi_s$ function) in the definition (1.9) (see also [**2**] and [**12**]). Indeed, by setting

$$A_j = 1 \quad (j = 1, \ldots, q) \qquad \text{and} \qquad B_j = 1 \quad (j = 1, \ldots, s)$$

in the definition (1.9), we are led immediately to the aforementioned Dziok-Srivastava operator

$$\Theta\left[(\alpha_j, 1)_{1,q};\ (\beta_j, 1)_{1,s}\right],$$

which contains, as its *further* special cases, such other linear operators of Geometric Function Theory as the *Hohlov operator*, the *Carlson-Shaffer operator*, the *Ruscheweyh derivative operator*, the *generalized Bernardi-Libera-Livingston operator*, the *fractional derivative operator*, and so on (see, for the *precise* relationships, Dziok and Srivastava [**4**, pp. 3-4]).

For convenience, we write

$$\Theta[\alpha_1] f(z) := \Theta[(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s)] f(z). \tag{1.10}$$

Let $\mathcal{H}$ be a *complex Hilbert space* and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all *bounded linear operators* on $\mathcal{H}$. For a complex-valued function f analytic in a domain $\mathbb{E}$ of the complex z-plane containing the spectrum $\sigma(\mathbb{P})$ of the bounded linear operator $\mathbb{P}$, let $f(\mathbb{P})$ denote the operator on $\mathcal{H}$ defined by [**1**, p. 568]

$$f(\mathbb{P}) = \frac{1}{2\pi i} \int_{\mathcal{C}} (z\mathbb{I} - \mathbb{P})^{-1}\ f(z)\, \mathrm{d}z,$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}$ and $\mathcal{C}$ is a positively-oriented simple rectifiable closed contour containing the spectrum $\sigma(\mathbb{P})$ in the interior domain. The operator $f(\mathbb{P})$ can also be defined by the following series:

$$f(\mathbb{P}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\ \mathbb{P}^n,$$

which converges in the normed topology (*cf.* [**7**]).

Let $\mathcal{E}(q, s; A, B; \mathbb{P})$ denote the class of functions f of the form:

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n\ z^n \qquad \left(a_1 > 0;\ a_n \geqq 0;\ n \in \mathbb{N} \setminus \{1\}\right), \tag{1.11}$$

which also satisfy the following subordination condition:

$$\alpha_1 \frac{\Theta[\alpha_1 + 1] f(\mathbb{P})}{\Theta[\alpha_1] f(\mathbb{P})} + A_1 - \alpha_1 \prec A_1\ \frac{1 + A\mathbb{P}}{1 + B\mathbb{P}} \quad (0 \leqq B \leqq 1;\ \ -B \leqq A < B) \tag{1.12}$$

for all operators $\mathbb{P}$ such that $\mathbb{P} \neq \mathbb{O}$ and $\|\mathbb{P}\| < 1$, $\mathbb{O}$ being the null operator on $\mathcal{H}$.

Finally, for a real parameter ρ $(0 < |\rho| < 1)$, we define the following subclasses of the class $\mathcal{E}\left(q, s; A, B; \mathbb{P}\right)$:

(1.13) $$\mathcal{E}_{\rho}\left(q, s; A, B; \mathbb{P}\right) := \left\{f : f \in \mathcal{E}\left(q, s; A, B; \mathbb{P}\right) \quad \text{and satisfies (1.4)}\right\}$$

and

(1.14) $$\mathcal{E}^{*}_{\rho}\left(q, s; A, B; \mathbb{P}\right) := \left\{f : f \in \mathcal{E}\left(q, s; A, B; \mathbb{P}\right) \quad \text{and satisfies (1.3)}\right\}.$$

In particular, for $q = s + 1$ and $\alpha_{s+1} = A_{s+1} = 1$, we write

$$\begin{aligned} \mathcal{E}\left(s; A, B; \mathbb{P}\right) &= \mathcal{E}\left(s+1, s; A, B; \mathbb{P}\right), \\ \mathcal{E}_{\rho}\left(s; A, B; \mathbb{P}\right) &= \mathcal{E}_{\rho}\left(s+1, s; A, B; \mathbb{P}\right), \end{aligned}$$

and

(1.15) $$\mathcal{E}^{*}_{\rho}\left(s; A, B; \mathbb{P}\right) = \mathcal{E}^{*}_{\rho}\left(s+1, s; A, B; \mathbb{P}\right).$$

In this lecture, we propose to present a systematic investigation of such basic properties and charateristics of each of the analytic function classes which we have introduced here as (for example) the normed coefficient estimates, distortion theorems, and the radii of convexity and starlikeness. We also briefly indicate the relevant connections of some of the results considered here with those involving the aforementioned Dziok-Srivastava operator.

2. A SET OF COEFFICIENT INEQUALITIES AND COEFFICIENT ESTIMATES

We begin by stating and proving the following result involving coefficient inequalities and estimates (*cf.* Dziok *et al.* [**3**]).

Theorem 1. *A function f of the form* (1.11) *belongs to the class $\mathcal{E}\left(q, s; A, B; \mathbb{P}\right)$ if and only if*

(2.1) $$\sum_{n=2}^{\infty} \delta_n \; a_n \leqq a_1 \; \delta_1 \qquad \big(\delta_n := \left[(B+1)\, n - (A+1)\right] \sigma_n\big),$$

where σ_n is given by

(2.2) $$\sigma_n := \frac{\Gamma\left[\alpha_1 + A_1\left(n-1\right)\right] \cdots \Gamma\left[\alpha_q + A_q\left(n-1\right)\right]}{(n-1)! \cdot \Gamma\left[\beta_1 + B_1\left(n-1\right)\right] \cdots \Gamma\left[\beta_s + B_s\left(n-1\right)\right]} \qquad (n \in \mathbb{N}).$$

Proof. Let a function f of the form (1.11) belong to the class $\mathcal{E}\left(q, s; A, B; \mathbb{P}\right)$. Then, in view of (1.12), we have

$$\alpha_1 \frac{\Theta\left[\alpha_1 + 1\right] f\left(\mathbb{P}\right)}{\Theta\left[\alpha_1\right] f\left(\mathbb{P}\right)} + A_1 - \alpha_1 = A_1 \frac{1 + Aw\left(\mathbb{P}\right)}{1 + Bw\left(\mathbb{P}\right)} \qquad \left(0 \leqq B \leqq 1; \; -B \leqq A < B\right),$$

where $w(\mathbb{O}) = \mathbb{O}$ ($\mathbb{O}$ being the null operator on $\mathcal{H}$) and $\|w(\mathbb{P})\| < 1$ for all operators $\mathbb{P} \neq \mathbb{O}$. It follows that

$$\left\| \frac{\alpha_1 \{\Theta[\alpha_1+1] f(\mathbb{P}) - \Theta[\alpha_1] f(\mathbb{P})\}}{\alpha_1 \; B\Theta[\alpha_1+1] f(\mathbb{P}) - \{AA_1 + (\alpha_1 - A_1) B\} \Theta[\alpha_1] f(\mathbb{P})} \right\| < 1. \tag{2.3}$$

Making use of (1.6), (1.9), and (1.10), the normed inequality (2.3) simplifies to the form:

$$\left\| \frac{\sum_{n=2}^{\infty} (n-1) \sigma_n \; a_n \; \mathbb{P}^{n-1}}{a_1 \; \delta_1 - \sum_{n=2}^{\infty} (Bn - A) \sigma_n \; a_n \; \mathbb{P}^{n-1}} \right\| < 1, \tag{2.4}$$

where δ_1 and σ_n are defined by (2.1) and (2.2), respectively.

Putting $\mathbb{P} = r\mathbb{I}$ $(0 < r < 1)$, we find from (2.4) that

$$\sum_{n=2}^{\infty} (n-1) \sigma_n \; a_n \; r^{n-1} \leqq a_1 \; \delta_1 - \sum_{n=2}^{\infty} (Bn - A) \sigma_n \; a_n \; r^{n-1} \qquad (0 < r < 1),$$

which, upon letting $r \to 1-$, yields the assertion (2.1) of Theorem 1.

Conversely, let a function f of the form (1.11) satisfy the condition (2.1). Then it is sufficient to prove that

$$\begin{aligned} &\|\alpha_1 \Theta[\alpha_1+1] f(\mathbb{P}) - \Theta[\alpha_1] f(\mathbb{P})\| \\ &\qquad - \|\alpha_1 B\Theta[\alpha_1+1] f(\mathbb{P}) - \{AA_1 + (\alpha_1 - A_1) B\} \Theta[\alpha_1] f(\mathbb{P})\| < 0. \end{aligned}$$

Choosing $\mathbb{P} = r\mathbb{I}$ $(0 < r < 1)$, we have

$$\begin{aligned} &\|\alpha_1 \Theta[\alpha_1+1] f(\mathbb{P}) - \Theta[\alpha_1] f(\mathbb{P})\| \\ &\qquad\qquad - \|\alpha_1 B\Theta[\alpha_1+1] f(\mathbb{P}) - \{AA_1 + (\alpha_1 - A_1) B\} \Theta[\alpha_1] f(\mathbb{P})\| \\ &\quad = \left\| \sum_{n=2}^{\infty} (n-1) \sigma_n \; a_n \; \mathbb{P}^n \right\| - \left\| a_1 \; \delta_1 - \sum_{n=2}^{\infty} (Bn - A) \sigma_n \; a_n \; \mathbb{P}^n \right\| \\ &\quad \leqq \sum_{n=2}^{\infty} (n-1) \sigma_n \; a_n \; r^n - \left(a_1 \; \delta_1 - \sum_{n=2}^{\infty} (Bn - A) \sigma_n \; a_n \; r^n \right) \\ &\quad = \sum_{n=2}^{\infty} \delta_n \; a_n \; r^n - a_1 \; \delta_1 \\ &\quad < \sum_{n=2}^{\infty} \delta_n \; a_n - a_1 \; \delta_1 \leqq 0, \end{aligned}$$

which shows that f belongs to the class $\mathcal{E}(q, s; A, B; \mathbb{P})$. This evidently completes the proof of Theorem 1.

Corollary 1. *A function f of the form* (1.11) *belongs to the class $\mathcal{E}_\rho(q, s; A, B; \mathbb{P})$ if and only if it satisfies* (1.4) *and*

$$\sum_{n=2}^{\infty} \left(\delta_n - \delta_1 \; \rho^{n-1} \right) a_n \leqq \delta_1, \tag{2.5}$$

where δ_n *is defined by* (2.1).

Corollary 2. *A function* f *of the form* (1.11) *belongs to the class* $\mathcal{E}_\rho^*(q,s;A,B;\mathbb{P})$ *if and only if it satisfies* (1.3) *and*

$$\sum_{n=2}^{\infty}\left(\delta_n - n\ \delta_1\ \rho^{n-1}\right)a_n \leqq \delta_1, \tag{2.6}$$

where δ_n *is defined by* (2.1).

Corollary 1 and Corollary 2 can be obtained by observing that, for a function f of the form (1.11) with the normalization (1.4), we have

$$a_1 = 1 + \sum_{n=2}^{\infty} a_n\ \rho^{n-1}, \tag{2.7}$$

and that, for a function f of the form (1.11) with the normalization (1.3), we have

$$a_1 = 1 + \sum_{n=2}^{\infty} n a_n\ \rho^{n-1}. \tag{2.8}$$

By applying (2.7) and (2.8), the inequality (2.1) yields the assertions (2.5) and (2.6), respectively.

The following lemmas are easy consequences of Corollary 1 and Corollary 2.

Lemma 1. *If there exists a positive integer* n_0 $(n_0 \in \mathbb{N}\setminus\{1\})$ *such that*

$$\delta_{n_0} - \delta_1\ \rho^{n_0-1} \leqq 0, \tag{2.9}$$

then the function

$$f_{n_0}(z) = \left(1 + a\rho^{n_0-1}\right)z - az^{n_0}$$

belongs to the class $\mathcal{E}_\rho(q,s;A,B;\mathbb{P})$ *for any positive real number* a. *Moreover, for all* n $(n \in \mathbb{N}\setminus\{1\})$ *such that*

$$\delta_n - \delta_1\ \rho^{n-1} > 0,$$

the functions

$$f_n(z) = \left(1 + a\rho^{n_0-1} + b\rho^{n-1}\right)z - az^{n_0} - bz^n \tag{2.10}$$

$$\left(n \in \mathbb{N}\setminus\{1\};\ b := \frac{\delta_1 + a\left(\delta_1\ \rho^{n_0-1} - \delta_{n_0}\right)}{\delta_n - \delta_1\ \rho^{n-1}}\right)$$

belong to the class $\mathcal{E}_\rho(q,s;A,B;\mathbb{P})$.

Lemma 2. *If there exists a positive integer* n_0 $(n_0 \in \mathbb{N}\setminus\{1\})$ *such that*

$$\delta_{n_0} - n_0\ \delta_1\ \rho^{n_0-1} \leqq 0,$$

then the function

$$f_{n_0}(z) = \left(1 + an_0\ \rho^{n_0-1}\right) z - az^{n_0}$$

belongs to the class $\mathcal{E}_\rho(q, s; A, B; \mathbb{P})$ *for any positive real number* a. *Moreover, for all* n $(n \in \mathbb{N} \setminus \{1\})$ *such that*

$$\delta_n - n\delta_1\ \rho^{n-1} > 0,$$

the functions

$$f_n(z) = \left(1 + an_0\ \rho^{n_0-1} + bn\rho^{n-1}\right) z - az^{n_0} - bz^n \tag{2.11}$$

$$\left(n \in \mathbb{N} \setminus \{1\}\,;\ b := \frac{\delta_1 + a\left(n_0\ \delta_1\ \rho^{n_0-1} - \delta_{n_0}\right)}{\delta_n - n\delta_1\ \rho^{n-1}}\right)$$

belong to the class $\mathcal{E}^*_\rho(q, s; A, B; \mathbb{P})$.

Applying Lemma 1 and Corollary 1, we obtain

Corollary 3. *If there exists a positive integer* n_0 $(n_0 \in \mathbb{N} \setminus \{1\})$ *such that*

$$\delta_{n_0} - \delta_1\ \rho^{n_0-1} < 0,$$

then the coefficients a_n *of a function* f *of the form* (1.11) *and belonging to the class* $\mathcal{E}_\rho(q, s; A, B; \mathbb{P})$ *are unbounded. Moreover, all of these coefficients* a_n *are unbounded also when*

$$\delta_n - \delta_1\ \rho^{n-1} = 0 \qquad (n \in \mathbb{N} \setminus \{1\}).$$

In all other cases, if a function f *of the form* (1.11) *belongs to the class* $\mathcal{E}_\rho(q, s; A, B; \mathbb{P})$, *then*

$$a_n \leqq \frac{\delta_1}{\delta_n - \delta_1\ \rho^{n-1}} \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{2.12}$$

The result is sharp for the functions given by

$$f_n(z) = \frac{\delta_n\ z - \delta_1\ z^n}{\delta_n - \delta_1\ \rho^{n-1}} \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{2.13}$$

Applying Lemma 2 and Corollary 2, we have

Corollary 4. *If there exists a positive integer* n_0 $(n_0 \in \mathbb{N} \setminus \{1\})$ *such that*

$$\delta_{n_0} - n_0\ \delta_1\ \rho^{n_0-1} < 0,$$

then the coefficients a_n *of a function* f *of the form* (1.11) *and belonging to the class* $\mathcal{E}^*_\rho(q, s; A, B; \mathbb{P})$ *are unbounded. Moreover, all of these coefficients* a_n *are unbounded also when*

$$\delta_n - n\delta_1\ \rho^{n-1} = 0 \qquad (n \in \mathbb{N} \setminus \{1\}).$$

In all other cases, if a function f *of the form* (1.11) *belongs to the class* $\mathcal{E}_{\rho}^{*}(q,s;A,B;\mathbb{P})$, *then*

$$a_n \leqq \frac{\delta_1}{\delta_n - n\delta_1\ \rho^{n-1}} \qquad (n \in \mathbb{N}\setminus\{1\}). \tag{2.14}$$

The result is sharp for the functions given by

$$f_n(z) = \frac{\delta_n\ z - \delta_1\ z^n}{\delta_n - n\delta_1\ \rho^{n-1}} \qquad (n \in \mathbb{N}\setminus\{1\}). \tag{2.15}$$

Each of the following results (Corollary 5 and Corollary 6) follows from Corollary 3 and Corollary 4 above.

Corollary 5. *For* δ_n *given by* (2.1), *let the sequence* $\left\{\delta_n - \delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty}$ *be positive. If a function* f *of the form* (1.11) *belongs to the class* $\mathcal{E}_{\rho}(q,s;A,B;\mathbb{P})$, *then the assertion* (2.12) *holds true for all* n $(n \in \mathbb{N}\setminus\{1\})$. *The result is sharp for the functions given by* (2.13).

Corollary 6. *For* δ_n *given by* (2.1), *let the sequence* $\left\{\delta_n - n\delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty}$ *be positive. If a function* f *of the form* (1.11) *belongs to the class* $\mathcal{E}_{\rho}^{*}(q,s;A,B;\mathbb{P})$, *then the assertion* (2.14) *holds true for all* n $(n \in \mathbb{N}\setminus\{1\})$. *The result is sharp for the functions given by* (2.15).

Remark 2. For

$$q = s+1, \quad \alpha_{s+1} = A_{s+1} = 1, \quad \beta_1 \leqq \alpha_1 + 1, \quad A_1 \leqq \alpha_1,$$
$$\beta_j \leqq \alpha_j \quad (j = 2,\ldots,s), \quad \text{and} \quad B_j = A_j \quad (j = 1,\ldots,s),$$

the sequences

$$\left\{\delta_n - \delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty} \qquad \text{and} \qquad \left\{\delta_n - n\delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty}$$

are positive and nondecreasing. Moreover, if $\beta_1 \leqq \alpha_1$, then the sequences

$$\left\{\frac{\delta_n - n\delta_1\ \rho^{n-1}}{n}\right\}_{n=2}^{\infty} \qquad \text{and} \qquad \left\{\frac{\delta_n - \delta_1\ \rho^{n-1}}{n}\right\}_{n=2}^{\infty}$$

are positive and nondecreasing.

3. DISTORTION THEOREMS AND THEIR APPLICATIONS

In this section, we first state and prove the following distortion theorem (*cf.* Dziok *et al.* [**3**]).

Theorem 2. *Let a function* f *of the form* (1.11) *belong to the class* $\mathcal{E}_{\rho}(q,s;A,B;\mathbb{P})$. *Also let* δ_n *be defined by* (2.1). *If the sequence* $\left\{\delta_n - \delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty}$ *is positive and nondecreasing, then*

$$\mathcal{J}(r) \leqq \|f(\mathbb{P})\| \leqq \frac{\delta_2\ r + \delta_1\ r^2}{\delta_2 - \delta_1\ \rho} \qquad \big(\|\mathbb{P}\| = r\ \ (0 < r < 1)\big), \tag{3.1}$$

where

$$\mathcal{J}(r) = \begin{cases} r & (r \leqq \rho) \\ \dfrac{\delta_2\ r - \delta_1\ r^2}{\delta_2 - \delta_1\ \rho} & (r > \rho). \end{cases} \tag{3.2}$$

If the sequence

$$\left\{ \frac{\delta_n - \delta_1\ \rho^{n-1}}{n} \right\}_{n=2}^{\infty}$$

is positive and nondecreasing, then

$$a_1 - \frac{2\delta_1\ r}{\delta_2 - \delta_1\ \rho} \leqq \|f'(\mathbb{P})\| \leqq \frac{\delta_2\ r + 2\delta_1\ r}{\delta_2 - \delta_1\ \rho} \qquad \big(\|\mathbb{P}\| = r\ \ (0 < r < 1)\big). \tag{3.3}$$

The result is sharp, with the extremal function f_2 *given by* (2.13) (*with* $n = 2$) *and* $f(z) = z$.

Proof. Let a function f of the form (1.11) belong to the class $\mathcal{E}_\rho(q, s; A, B; \mathbb{P})$. If the sequence $\left\{\delta_n - \delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty}$ is positive and nondecreasing, by Corollary 1, we have

$$\sum_{n=2}^{\infty} a_n \leqq \frac{\delta_1}{\delta_2 - \delta_1\ \rho}. \tag{3.4}$$

Moreover, if the sequence

$$\left\{ \frac{\delta_n - \delta_1\ \rho^{n-1}}{n} \right\}_{n=2}^{\infty}$$

is positive and nondecreasing, by Corollary 2, we have

$$\sum_{n=2}^{\infty} n a_n \leqq \frac{2\delta_1}{\delta_2 - 2\delta_1\ \rho}. \tag{3.5}$$

Using (2.7) and (3.4), we find for

$$\mathbb{P} = r\mathbb{I} \qquad (0 < r < 1)$$

that

$$\begin{aligned} \|f(\mathbb{P})\| = \left\| a_1\ \mathbb{P} - \sum_{n=2}^{\infty} a_n\ \mathbb{P}^n \right\| &\leqq r\left(a_1 + \sum_{n=2}^{\infty} a_n\ r^{n-1} \right) \\ &\leqq r\left(1 + \sum_{n=2}^{\infty} a_n\ \rho^{n-1} + \sum_{n=2}^{\infty} a_n\ r^{n-1} \right) \\ &\leqq r\left(1 + (\rho + r) \sum_{n=2}^{\infty} a_n \right) \leqq \frac{\delta_2\ r + \delta_1\ r^2}{\delta_2 - \delta_1\ \rho} \end{aligned} \tag{3.6}$$

and

$$\left\| f\left(\mathbb{P}\right) \right\| = \left\| a_1 \ \mathbb{P} - \sum_{n=2}^{\infty} a_n \ \mathbb{P}^n \right\| \geqq r \left(a_1 - \sum_{n=2}^{\infty} a_n \ r^{n-1} \right) \tag{3.7}$$

$$= r \left(1 + \sum_{n=2}^{\infty} a_n \left(\rho^{n-1} - r^{n-1} \right) \right).$$

If $r \leqq \rho$, then we have $\left\| f\left(\mathbb{P}\right) \right\| \geqq r$. If $r > \rho$, then the sequence$\left\{ \rho^{n-1} - r^{n-1} \right\}_{n=2}^{\infty}$ is negative and decreasing. Hence, by (3.7), we obtain

$$\left\| f\left(\mathbb{P}\right) \right\| \geqq r \left(1 + (\rho - r) \sum_{n=2}^{\infty} a_n \right) \geqq \frac{\delta_2 \ r - \delta_1 \ r^2}{\delta_2 - \delta_1 \ \rho},$$

which, in conjunction with (3.6), yields the assertion (3.1) of Theorem 2.

Similarly, by using (3.5) in conjunction with (2.7), we arrive at the assertion (3.3) of Theorem 2.

The proof of the following result is analogous to that of Theorem 2.

Theorem 3. *Let a function f of the form* (1.11) *belong to the class $\mathcal{E}_{\rho}^{*}\left(q, s; A, B; \mathbb{P}\right)$. Also let δ_n be defined by* (2.1). *If the sequence $\left\{ \delta_n - n\delta_1 \ \rho^{n-1} \right\}_{n=2}^{\infty}$ is positive and nondecreasing, then*

$$a_1 \ r - \frac{\delta_1 \ r^2}{\delta_2 - \delta_1 \ \rho} \leqq \left\| f\left(\mathbb{P}\right) \right\| \leqq \frac{\delta_2 \ r + \delta_1 \ r^2}{\delta_2 - n\delta_1 \ \rho} \qquad \left(\left\| \mathbb{P} \right\| = r \ \ (0 < r < 1) \right). \tag{3.8}$$

If the sequence

$$\left\{ \frac{\delta_n - n\delta_1 \ \rho^{n-1}}{n} \right\}_{n=2}^{\infty}$$

is positive and nondecreasing, then

$$\mathcal{J}'\left(r\right) \leqq \left\| f'\left(\mathbb{P}\right) \right\| \leqq \frac{\delta_2 + 2\delta_1 \ r}{\delta_2 - n\delta_1 \ \rho} \qquad \left(\left\| \mathbb{P} \right\| = r \ \ (0 < r < 1) \right), \tag{3.9}$$

where $\mathcal{J}\left(r\right)$ is defined by (3.2). *The result is sharp, with the extremal function f_2 given by* (2.15) *with $n = 2$ and $f\left(z\right) = z$.*

Applying Lemma 1, we deduce the following result.

Corollary 7. *If there exists an integer n_0 $(n_0 \in \mathbb{N} \setminus \{1\})$ such that* (2.9) *holds true, then $\left\| f\left(\mathbb{P}\right) \right\|$ and $\left\| f'\left(\mathbb{P}\right) \right\|$ $\left(\left\| \mathbb{P} \right\| = r \ \ (0 < r < 1) \right)$ for functions of the class $\mathcal{E}_{\rho}\left(q, s; A, B; \mathbb{P}\right)$ are unbounded.*

Next, by applying Lemma 2, we have

Corollary 8. *If there exists an integer n_0 $(n_0 \in \mathbb{N} \setminus \{1\})$ such that* (2.10) *holds true, then $\left\| f\left(\mathbb{P}\right) \right\|$ and $\left\| f'\left(\mathbb{P}\right) \right\|$ $\left(\left\| \mathbb{P} \right\| = r \ \ (0 < r < 1) \right)$ for functions of the class $\mathcal{E}_{\rho}^{*}\left(q, s; A, B; \mathbb{P}\right)$ are unbounded.*

By virtue of Remark 2, Theorem 2 and Theorem 3 give the following results.

Corollary 9. *Let a function f of the form* (1.11) *belong to the class* $\mathcal{E}_\rho\,(s; A, B; \mathbb{P})$. *If*

$$\beta_1 \leqq \alpha_1 + 1, \quad A_1 \leqq \alpha_1, \quad \beta_j \leqq \alpha_j \quad (j = 2, \ldots, s)\,, \quad \textit{and} \quad B_j = A_j \quad (j = 1, \ldots, s)\,,$$

then the assertion (3.1) *holds true. Further, if* $\beta_1 \leqq \alpha_1$, *then the assertion* (3.3) *holds true.*

Corollary 10. *Let a function f of the form* (1.11) *belong to the class* $\mathcal{E}^*_\rho\,(s; A, B; \mathbb{P})$. *If*

$$\beta_1 \leqq \alpha_1 + 1, \quad A_1 \leqq \alpha_1, \quad \beta_j \leqq \alpha_j \quad (j = 2, \ldots, s)\,, \quad \textit{and} \quad B_j = A_j \quad (j = 1, \ldots, s)\,,$$

then the assertion (3.8) *holds true. Further, if* $\beta_1 \leqq \alpha_1$, *then the assertion* (3.9) *holds true.*

4. COMPUTATION OF THE ASSOCIATED RADII OF CONVEXITY AND STARLIKENESS

Our first set of results involving the radius of starlikeness can be stated as Theorem 4 below (*cf.* Dziok *et al.* [**3**]).

Theorem 4. *If a function f of the form* (1.11) *belongs to the class* $\mathcal{E}\,(q, s; A, B; \mathbb{P})$, *then f is starlike in the disk*

$$\left\| R^*\big(\mathcal{E}\,(q, s; A, B; \mathbb{P})\big) \right\| < r_1 := \inf_{n \in \mathbb{N}\setminus\{1\}} \left(\frac{\delta_n}{n\delta_1}\right)^{1/(n-1)}, \tag{4.1}$$

where δ_n is defined by (2.1). *The result is sharp for the function f^*_a given by*

$$f^*_a\,(z) = a\left(z - \frac{\delta_1}{\delta_n}\, z^n\right) \qquad (a > 0)\,. \tag{4.2}$$

Proof. It suffices to show that

$$\left\| \frac{\mathbb{P}\, f'\,(\mathbb{P})}{f\,(\mathbb{P})} - 1 \right\| < 1 \qquad \big(\mathbb{P} = r_1 \mathbb{I}\ \ (0 < r_1 < 1)\big). \tag{4.3}$$

Since

$$\left\| \frac{\mathbb{P}\, f'\,(\mathbb{P})}{f\,(\mathbb{P})} - 1 \right\| = \left\| \frac{\sum\limits_{n=2}^{\infty} (n-1)\, a_n\ \mathbb{P}^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty} a_n\ \mathbb{P}^{n-1}} \right\|,$$

the condition (4.3) holds true if

$$\left\| \frac{\mathbb{P}\, f'\,(\mathbb{P})}{f\,(\mathbb{P})} - 1 \right\| \leqq \frac{\sum\limits_{n=2}^{\infty} (n-1)\, a_n\ r_1^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty} a_n\ r_1^{n-1}} \leqq 1,$$

that is, if

$$\sum_{n=2}^{\infty} na_n \; r_1^{n-1} \leqq a_1. \tag{4.4}$$

By Theorem 1, we also have

$$\sum_{n=2}^{\infty} \frac{\delta_n \; a_n}{\delta_1} \leqq a_1, \tag{4.5}$$

where δ_n is defined by (2.1). Comparing (4.4) and (4.5), we obtain the desired result (4.1). The sharpness of the result (4.1) can easily be verified for the function f_a^* given by (4.2).

Theorem 5. *If a function f of the form* (1.11) *belongs to the class* $\mathcal{E}(q, s; A, B; \mathbb{P})$, *then f is convex in the disk*

$$R^c\big(\mathcal{E}(q, s; A, B; \mathbb{P})\big) < r_2 := \inf_{n \in \mathbb{N} \setminus \{1\}} \left(\frac{\delta_n}{n^2 \; \delta_1} \right)^{1/(n-1)}, \tag{4.6}$$

where δ_n is defined by (2.1). *The result is sharp for the function f_a^c given by*

$$f_a^c(z) = a \left(z - \frac{n\delta_1}{\delta_n} \; z^n \right) \qquad (a > 0). \tag{4.7}$$

Proof. It suffices to show that

$$\left\| \frac{\mathbb{P} \; f''(\mathbb{P})}{f'(\mathbb{P})} \right\| < 1 \qquad \big(\mathbb{P} = r_2 \; \mathbb{I} \;\; (0 < r_2 < 1)\big). \tag{4.8}$$

Since

$$\left\| \frac{\mathbb{P} \; f''(\mathbb{P})}{f'(\mathbb{P})} \right\| = \left\| -\frac{\sum_{n=2}^{\infty} n(n-1) \, a_n \; \mathbb{P}^{n-1}}{a_1 - \sum_{n=2}^{\infty} na_n \; \mathbb{P}^{n-1}} \right\|,$$

the condition (4.8) holds true if

$$\left\| \frac{\mathbb{P} \; f''(\mathbb{P})}{f'(\mathbb{P})} \right\| \leqq \frac{\sum_{n=2}^{\infty} n(n-1) \, a_n \; r_2^{n-1}}{a_1 - \sum_{n=2}^{\infty} na_n \; r_2^{n-1}} \leqq 1,$$

that is, if

$$\sum_{n=2}^{\infty} n^2 \; a_n \; r_2^{n-1} \leqq a_1. \tag{4.9}$$

By comparing (4.9) with (4.5) again, we arrive at the desired result (4.6), with the extremal function f_a^c given by (4.7).

Remark 3. Just as we pointed out in Remark 1, the various results presented in this lecture would provide interesting extensions and generalizations of those considered earlier for *simpler* analytic function classes. The details involved in the derivations of such specializations of the results presented here are fairly straightforward.

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