

SOME PROPERTIES OF C -REFLEXIVE LOCALLY CONVEX SPACES

Stojan Radenović

In this note, we shall prove that the class of C -reflexive spaces is stable with respect to separated quotient, arbitrary product and sum, which is not the case for the closed subspaces and the dense hyperplanes. If the quotient mapping lifts compact disks, then the class of C -reflexive spaces is three-space stable.

In [3] and [4] the classes of different types of semi-reflexive and reflexive locally convex spaces were introduced. In [3] and [5], classic semi-reflexive and reflexive locally convex spaces were studied thoroughly, including the BANACH spaces. In [3], the definitions and some basic properties of the so-called polar semi-reflexive and polar reflexive spaces were given. That kind of reflexivity is called t -reflexive in [4].

In this note, we are considering the class of C -reflexive spaces which was defined in [4]. We shall prove some natural properties of this class of spaces which were not even mentioned there.

If C is the family of all compact disks of a locally convex spaces E , then the topological dual E' is endowed with topology of uniform convergence on the family C , denoted by E'_C , consistent with dual pair $\langle E, E' \rangle$, that is, we have got algebraical equity $(E', E'_C)' = E$.

In the terms of different semi-reflexivities which we come across in mentioned works, you could say that each locally convex space is C -semi-reflexive.

If C is now the family of all compact disks of the space (E', E'_C) , then there exists in E a new locally convex topology, denoted by $E_C = (E'_C)'_C$.

In [4] it is said that locally convex space E is C -reflexive if $E = E_C = (E'_C)'_C$, that is, if starting topology of the space E and the topology E_C of uniform convergence on all E'_C -compact disks are equal. It is clear that the topology E_C is

2000 Mathematics Subject Classification: 46A03, 46A04

Keywords and Phrases: Three-space, C -reflexive, exact sequence, Maskey space.

consistent with the dual pair $\langle E, E' \rangle$ and that it is not weaker than the original topology of the space E .

Henceforward, t will denote the original topology of the space E . We shall begin with the following proposition:

Proposition 1. *A locally convex space (E, t) is C -reflexive if and only if each E'_c -compact disk is t -equicontinuous subset of topological dual E' .*

Proof. Let K be any E'_c -compact disk. If the space (E, t) is C -reflexive, then the polar K° is t -neighbourhood of zero, so K is one t -equicontinuous subset of dual E' , according to [3], 20.8 (1) c). Conversely, let U be arbitrary absolutely convex E_c -neighbourhood of zero. Then, there exists E'_c -compact disk K , such that $U \supset K^\circ$. It follows that $U^\circ \subset K^{\circ\circ} = \overline{K}^{\sigma(E', E)} = \overline{K}^{E'_c} = K$. Since U° is one $\sigma(E', E)$ -closed disk, then U° is also E'_c -closed, so U° is E'_c -compact, because K is E'_c -compact disk. According to hypothesis U° is t -equicontinuous subset of E' , so $U^{\circ\circ}$ is t -neighbourhood of zero. Since, $U^{\circ\circ} = \overline{U}^t = \overline{U}^{E_c} \subset 2U$, it follows that U is t -neighbourhood of zero, i.e. $t = E_c$.

Corollary 2. *If (E, t) is C -reflexive space, then the completion $(\widehat{E}, \widehat{t})$ is also C -reflexive space.*

Proof. According to proposition 1, let K be one \widehat{E}'_c -compact disk. K is evidently also E'_c -compact, that is, t -equicontinuous subset. Since the topologies t and \widehat{t} yield the same equicontinuous subsets of $(E, t)' = (\widehat{E}, \widehat{t})'$ the proof follows.

Proposition 3. *For each locally convex space (E, t) the spaces (E', E'_c) and $(E, E_c) = (E, (E'_c)')$ are C -reflexive.*

Proof. Let K be any $(E'_c)'_c$ -compact disk, that is, one E_c -compact disk. Since topology E_c is not weaker than the original topology t , K is then one t -compact disk. Therefore, K is E'_c -equicontinuous subset of E , that is, according to proposition 1, (E', E'_c) is C -reflexive space. Proof for the space (E, E_c) is the similarly.

REMARK 4. If (E, t) is locally convex space, then

$$t \leq E_c = (E'_c)'_c \leq \tau(E, E')$$

where $\tau(E, E')$ is MACKAY topology associated to space (E, t) .

Proposition 5. *If (E, t) is Mackey locally convex space, then it is C -reflexive.*

Proof. According to previous remark, $((\tau(E, E'))'_c)'_c = E_c \geq \tau(E, E')$, that is $E_c = \tau(E, E')$, since $\tau(E, E')$ is the strongest locally convex topology on E consistent with dual pair $\langle E, E' \rangle$, $E' = (E, t)'$.

EXAMPLE 6. There exists C -reflexive space which is not MACKAY.

Proof. For each reflexive BANACH space E of infinite dimension the space (E', E'_c) is such. Indeed, then $E'_c \neq \tau(E', E) = \beta(E', E)$ where $\beta(E', E)$ is usual strong

topology in dual E' of the space E . According to the proposition 3, (E', E'_c) is C -reflexive space. It is not MACKEY space, because the starting space E is of the infinite dimension-the unit ball cannot be a compact disk. The mentioned example generalizes the example 4.1 from [4].

In a general theory of locally convex spaces, the following matters were studied:

- If a space (E, t) possesses a certain property, for instance P , does its subspace possess the same property as well?
- If a space (E, t) possesses a property P and if F is its closed subspace, does the quotient $(E/F, t/F)$ possess the property P ?
- If the outer members F and E/F of the short exact sequence

$$0 \rightarrow F \xrightarrow{j} E \xrightarrow{q} E/F \rightarrow 0$$

possess the property P , does the middle member E possess the property P ? If the answer is affirmative, it is then said that property P is three-space stable, and if it is negative, it is three-space unstable.

Further in this note, we shall discuss the mentioned matters, if the property P is “ C -reflexive”.

Proposition 7. *Let F be a closed subspace of C -reflexive locally convex space (E, t) . Then, the quotient space $(E/F, t/F)$ is also C -reflexive.*

Proof. First

$$(1) \quad t/F = E_c/F$$

and

$$(2) \quad t/F \leq (E/F)_c = ((E/F)'_c)'$$

where, as we previously denoted $E_c = (E'_c)'$. Then, according to

$$\sigma(F^\circ, E/F) \leq E'_c|F^\circ \leq (E/F)'_c \leq \tau(F^\circ, E/F)$$

it is also fulfilled

$$(3) \quad (E/F)_c \leq E_c/F.$$

Now, from (1), (2) and (3) it follows

$$t/F = (E/F)_c = ((E/F)'_c)'$$

that is $(E/F, t/F)$ is C -reflexive locally convex space.

“ C -reflexivity” is one of rare properties among “semi-reflexive” and “reflexive” ones, which is inherited on separated quotient. The known KÖTHE-GROTHEN-DIECK-FRECHET-MONTEL space ([3], 31.5) shows that classical semi-reflexivity and reflexivity are not inherited on quotient space.

If F is a closed subspace of finite codimension, we have the result:

Proposition 8. *Let F be closed subspace of finite codimension of C -reflexive locally convex space (E, t) . Then $(F, t|F)$ is C -reflexive.*

Proof. It is sufficient to assume that F is a closed hyperplane, and then apply mathematical induction. If this is the case, then there exists $x_0 \in E \setminus F$ so that $(E, t) = (F, t|F) \oplus (L, t|L)$, where L is the linear span of the point x_0 . Then $(F, t|F) \simeq (E/L, t/L)$. Since L is one dimensional subspace of E , therefore, closed in (E, t) , then according to proposition 7 the space $(E/L, t/L)$ is C -reflexive, so $(F, t|F)$ is such as well.

If F is a dense hyperplane, we have the following counterexample:

EXAMPLE 9. Let G be a dense hyperplane in the algebraic dual X^* of the infinite dimensional vector space X and let $x^* \in X^* \setminus G$. Then the space X provided with the MACKAY topology $t := \tau(X, G)$ is C -reflexive as a MACKAY space. $F := \ker x^*$ is a dense hyperplane in (X, t) , which is not C -reflexive with respect to the relative topology $t|F$.

Indeed, as $(G, \sigma(G, F))$ is barrelled space ([3], 27.1. page 369) all bounded sets in (X, t) are finite dimensional; in particular, $(F, t|F)'_c = (F, t|F)'_\sigma = (G, \sigma(G, F))$. Moreover, it is clear from linear algebra that G is equal to the algebraic dual of F . Therefore $(G, \sigma(G, F))$ is MONTEL space which implies that $((F, t|F)'_c)'_c = (G, \sigma(G, F))'_c = (G, \sigma(G, F))'_\beta = (F, \beta(F, G))$, where $\beta(F, G)$ is the strongest locally convex topology on F , hence complete and thus different from $t|F$ (for details see [3], 5. the spaces ω_d and φ_d , pages 287, 288).

Corollary 10. *The class of C -reflexive locally convex spaces is not stable with respect to closed subspaces.*

Proof. Let (F, t) be locally convex space which is not C -reflexive space (for instance: the previous counterexample). According to KOMURA's result [2], (F, t) is the closed subspace of some barrelled space (which is C -reflexive, because it is MACKAY). It follows that closed subspace of C -reflexive does not need to be C -reflexive, i.e. the class of C -reflexive locally convex spaces is not stable with respect to closed subspaces.

REMARK 11. From ([4], example 4.5), it follows that the semi-reflexive space $(\ell_2, \sigma(\ell_2, \ell_2))$ is not C -reflexive. But, this is true for each reflexive BANACH space of infinite dimension.

Indeed, let $(E, \|\cdot\|)$ be such space. Then,

$$\left((E, \sigma(E, E'))'_c \right)'_c = (E', \tau(E', E))'_c = (E', \beta(E', E))'_c = (E, t).$$

According to ([3], 21. 5. (4)), the space (E, t) is complete and thus different from $(E, \sigma(E, E'))$, i.e. the space $(E, \sigma(E, E'))$ is not C -reflexive.

As for the last matter of C -reflexive spaces, we firstly introduce:

Definition 12. We say that quotient mapping

$$q : (E, t) \rightarrow (E/F, t/F)$$

lifts compact disks if for each t/F -compact disk K_1 , there exists t -compact disk K_2 so that

$$K_1 \subset q(K_2).$$

Since $q(K_2) = \overline{q(K_2)}^{t/F}$, then the terms “lifting” and “lifting with closure” of compact disks are equivalent as opposed to the same terms taking bounded disks instead of compact ones.

From the general theory of locally convex spaces [3] and [5], we know that (we used while proving proposition 7)

$$E'_c|F^\circ \leq (E/F)'_c,$$

as well as that the topology $E'_c|F^\circ$ is consistent with dual pair $\langle F^\circ, E/F \rangle \simeq \langle (E/F)', E/F \rangle$. Based on that, it follows that the quotient mapping lifts compact disks if and only if

$$E'_c|F^\circ = (E/F)'_c.$$

Now, we formulate the main result of this note:

Proposition 13. *If in short exact sequence*

$$0 \rightarrow (F, t|F) \xrightarrow{j} (E, t) \xrightarrow{q} (E/F, t/F) \rightarrow 0$$

the outer members $(F, t|F)$ and $(E/F, t/F)$ are C -reflexive and if the quotient mapping q lifts compact disks, then the middle member (E, t) is also C -reflexive space.

Proof. According to the assumption it follows that $t|F = F_c$, $t/F = (E/F)_c$ and $E'_c|F^\circ = (E/F)'_c$. The first two equalities are the consequence of the fact that subspace and quotient are C -reflexive spaces, and last equality is equivalent with the condition that the quotient mapping lifts compact disks. In order to prove that the space (E, t) is C -reflexive, we shall prove the equality topologies t and E_c on subspace F and quotient E/F , that is

$$t|F = E_c|F \wedge t/F = E_c/F.$$

Then, according to result of DIEROLF-SCHAWANENGEL from [1], Lemma 1, on minimal topological groups, it follows that $t = E_c$, that is, the space (E, t) is C -reflexive.

Since $t \leq E_c = (E'_c)'_c$, then $t|F = F_c = (F'_c)'_c \leq E_c|F$ on one side; on the other side $F_c \geq E_c|F$, because the quotient image of each E'_c -compact disk is E'_c/F° -compact disk ($F'_c \leq E'_c/F^\circ$). Therefore, $t|F = E_c|F$.

The equality of topologies $E'_c|F^\circ$ and $(E/F)'_c$ brings to relation $E_c/F \leq t/F$. Indeed, if K is any E'_c -compact disk, then $K \cap F^\circ$ is one $E'_c|F^\circ$ -compact, that is $(E/F)'_c$ -compact disk, so

$$(K \cap F^\circ)^\circ = K^\circ + F^{\circ\circ} = K^\circ + F$$

is one $((E/F)'_c)'_c = (E/F)_c = t/F$ -neighbourhood of zero. The reverse relation $t/F \leq E_c/F$ is apparent. Therefore, $t/F = E_c/F$. The proof that the space (E, t) is C -reflexive is complete.

For now, we do not know whether in the previous proposition we can let go off the assumption of lifting compact disks.

For arbitrary product and direct sum of C -reflexive locally convex spaces, we give the following results:

Proposition 14. *Let $(E_i)_{i \in I}$ be an arbitrary family of C -reflexive locally convex spaces. Then the product $\prod_{i \in I} E_i$ and the direct sum $\bigoplus_{i \in I} E_i$ are C -reflexive spaces.*

Proof. Let us denote with $E = \prod_{i \in I} E_i$. Then, as is well known ([3], 22,5 (2)) $E' = \bigoplus_{i \in I} E'_i$. It is clear that compact disks K of the shape $K = \prod_{i \in I} K_i$, K_i is compact disk in E_i , make the base of the set of compact disks of the space E . Therefore, according to ([3], 22.5 (1), the first part), it follows $E'_c = \bigoplus_{i \in I} (E_i)'_c$.

Let K now be arbitrary E'_c -compact disk. It is also E'_c -bounded and that is why there exists a finite subset $J \subset I$, so that

$$K \subset \bigoplus_{i \in J} pr_i(K)$$

(pr_i is the projection of E' on E'_i). Since the sets $pr_i(K)$, $i \in J$, are $(E_i)'_c$ -compact, and therefore are also E_i -equicontinuous, then $\bigoplus_{i \in J} pr_i(K)$ is also one E -equicontinuous set, so K is such as well. The space E is then C -reflexive if such are the spaces E_i .

If now $E = \bigoplus_{i \in I} E_i$ is direct sum of C -reflexive locally convex spaces E_i , let us prove that E is also C -reflexive space. It is known ([3], 22.5 (4)) that $E' = \prod_{i \in I} E'_i$. Since the base of the set of all compact disks of the space E includes the sets of the shape $\bigoplus_{i \in J \subset I} pr_i(K)$, where K is a compact subset of E , pr_i is the projection from E on E_i and J is the finite subset of I . Therefore, according to ([3], 22.5 (1), the second part), it follows

$$E'_c = \prod_{i \in I} (E_i)'_c.$$

Let $K \subset E'$ be now a E'_c -compact disk, that is $K \subset \prod_{i \in I} K_i$, where K_i are compact disks in space $(E_i)'_c$. Since the spaces E_i are C -reflexive, then K_i are equicontinuous,

therefore $\prod_{i \in I} K_i$ is such, that is, the subset K is E -equicontinuous, i.e. the space E is C -reflexive.

REFERENCES

1. S. DIEROLF, U. SCHWANENGEL: *Examples of locally compact non-compact minimal topological groups*. Pac. J. Math., **82** (1979), 349–355.
2. Y. KOMURA: *On linear topological spaces*. Kumamoto J. Sci., **5** 1962, 148–157.
3. G. KÖTHE: *Topological Vector Spaces I*, 2nd ed. Springer, Berlin-Heidelberg-New York, 1983.
4. S. H. KYE: *Several reflexivities in topological vector spaces*. Journal of Mathematical Analysis and Application, **139** (1989), 477–482.
5. R. MEISE, D. VOGT: *Introduction to Functional Analysis*. Clarendon Press, Oxford, 1997.

University of Belgrade,
Faculty of Mechanical Engineering,
Kraljice Marije 16,
11120 Beograd,
Serbia
E-mail: radens@beotel.yu; sradenovic@mas.bg.ac.yu

(Received July 10, 2006)