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NO STARLIKE TREES ARE LAPLACIAN COSPECTRAL

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It is proved in this paper that no two starlike trees are Laplacian cospectral, and the starlike trees with maximum degree 3 and 4 are determined by their Laplacian spectrum.

1. INTRODUCTION

Let G be a graph with n vertices and m edges. The degree sequence of G is denoted by $d_1 \geq d_2 \geq \cdots \geq d_n$. Let $A(G)$ and $D(G) = diag(d_i : 1 \leq i \leq n)$ be the adjacency matrix and the degree diagonal matrix of G, respectively. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. It is well known that $L(G)$ is a symmetric, semidefinite matrix. We assume the spectrum of $L(G)$, or the Laplacian spectrum of G, is $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. If more than one graph is involved, we may write $\lambda_i(G)$ in place of λ_i. λ_{n-1} is called the *algebraic connectivity* of G and $\lambda_{n-1} > 0$ if and only if G is connected. The multiplicity of zero as an eigenvalue equals to the number of components of G. The characteristic polynomial of $L(G)$ can be written by

$$P_{L(G)}(x) = |xI - L(G)| = q_0 x^n + q_1 x^{n-1} + \cdots + q_{n-1}x + q_n.$$

We use $\rho(G)$ to denote the adjacency spectral radius of G. Two graphs G and H are said to be *adjacency (Laplacian) cospectral* if they have the same adjacency (Laplacian) spectrum, or in other words, they have equal adjacency (Laplacian) characteristic polynomial. Obviously, two isomorphic graphs are adjacency and Laplacian cospectral.

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For a connected graph G, let $Q(G) = D(G) + A(G)$, we call this matrix Q-*matrix*, its largest eigenvalue is denoted by $\mu(G)$ or μ for simplicity. It is well known that $Q(G)$ is entrywise nonnegative and positive definite, so from the Perron-Frobenius Theorem, there is a unique positive eigenvector corresponding to μ. We call this eigenvector *principal eigenvector*. For the background on the Laplacian eigenvalues of a graph, the reader is referred to [**1**], [**10**], [**11**] and the references therein.

All notations in graph theory that are not defined here, can be founds in [**6**].

A *starlike tree* is a tree with exactly one vertex having degree greater than two. Let P_n denote the path on n vertices. By $S(n_1, n_2, \ldots, n_k)$ we denote the starlike tree which has a vertex v of degree $k \geq 3$ and has the property that

$$S(n_1, n_2, \ldots, n_k) - v = P_{n_1} \cup P_{n_2} \cup \cdots \cup P_{n_k},$$

where $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$. Clearly, $n_1, n_2, \ldots, n_k$ determine the starlike tree up to isomorphism.

In [**2**], the authors raised the following problem: *Which trees are determined by their spectrum*? We now still do not know the answer in the affirmative. In [**2**],[**3**], [**14**], some partial results on this problem were got. In this paper, we will prove that no two starlike trees are Laplacian cospectral, and the starlike trees with maximum degree 3 and 4 are determined by their Laplacian spectrum.

SOME LEMMAS

Lemma 2.1. [**15**] *For a connected graph G, we have $\lambda(G) \leq \mu(G)$, with equality if and only if G is bipartite.*

Lemma 2.2. [**2**] *For $n \times n$ matrices A and B, the following are equivalent*:
(1). *A and B are cospectral*;
(2). *A and B have the same characteristic polynomial*;
(3). *$tr(A^i) = tr(B^i)$, for all $i = 1, 2, \cdots, n$.*

If A is the adjacency matrix of a graph, then $tr(A^i)$ gives the total number of closed walks of length i. So the adjacency cospectral graphs have the same number of edges (for $i = 2$) and triangles (for $i = 3$).

Lemma 2.3. [**12**] *For the characteristic polynomial of the Laplacian matrix of a graph G, we have*

$$q_0 = 1; \quad q_1 = -2m; \quad q_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} {d_i}^2;$$
$$q_3 = \frac{1}{3}\{-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} {d_i}^2 - \sum_{i=1}^{n} {d_i}^3 - 3\sum_{i=1}^{n} {d_i}^2 + tr(A^3)\};$$
$$q_{n-1} = (-1)^{n-1} S(G); \quad q_n = 0,$$

where $S(G)$ is the number of spanning trees of G.

Lemma 2.4. [4], [**16**] *Let G be a graph of order n with at least one edge, and the maximum degree of G is Δ, then $\lambda(G) \geq \Delta + 1$. Moreover, if G is connected, then the equality holds if and only if $\Delta = n - 1$.*

Lemma 2.5. [**9**], [**13**] *If G is a graph, then $\lambda(G) \leq max\{d_v + m_v | v \in V(G)\}$, equality holds if and only if G is either a regular bipartite graph or a semiregular bipartite graph.*

Lemma 2.6. [**6**, p. 85] *Let T_1 and T_2 be two trees that have isomorphic line graphs, then $T_1 \cong T_2$.*

Lemma 2.7. [**5**] *Let T be a tree of order n, then $\lambda_i(T) = \rho_i(M(T)) + 2$, $1 \leq i \leq n - 1$, where $\rho_i(M(T))$ is the ith largest adjacency eigenvalue of the line graph of T.*

Lemma 2.8. [**7**] *Let G be bipartite graph and $\mu(G)$ be the spectral radius of $Q(G)$. Let u, v be two vertices of G and d_v be the degree of v, suppose $v_1, v_2, \ldots, v_s \in N(v) \setminus N(u) (1 \leq s \leq d_v)$, $X = (x_1, x_2, \ldots, x_n)$ be the principal eigenvector of $Q(G)$, where x_i corresponds to $v_i, (1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges $(u, v_i), 1 \leq i \leq s$. If $x_u \geq x_v$, then $\mu(G) < \mu(G^*)$.*

Lemma 2.9. *Let $f_1(x) = x - 1$, $f_{i+1}(x) = x - 2 - \frac{1}{f_i}$, $i \geq 1$, then we have $f_i(x) > \frac{x}{x-2} > 1$, if $x > 3 + \sqrt{2}$. Moreover, the sequence $\{f_i(x)\}$ is strictly decreasing.*

Proof. We use the induction on i. When $i = 1$, it is easy to check our result holds. If $f_i(x) > \frac{x}{x-2} > 1$, then

$$f_{i+1}(x) = x - 2 - \frac{1}{f_i} > x - 2 - \frac{x-2}{x} > \frac{x}{x-2}.$$

By the induction hypothesis, we get the first result. Moreover,

$$f_i - f_{i-1} = x - 2 - \frac{1}{f_{i-1}} - x - 2 - \frac{1}{f_{i-2}} = \frac{f_{i-1} - f_{i-2}}{f_{i-1} f_{i-2}},$$

since $f_2 - f_1 < 0$, we get that the sequence $\{f_i(x)\}$ is strictly decreasing. □

Lemma 2.10. *Let u be a vertex of a connected bipartite graph G with at least two vertices. Let $G(k, l)$, $k \geq \ell \geq 1$, be the graph obtained from G by attaching two paths $P_{k+1} = v_1 v_2 \cdots v_k u$ and $P_{l+1} = w_1 w_2 \cdots w_\ell u$ of length k and ℓ, respectively, at u. If $\lambda(G(k, \ell)) > 3 + \sqrt{2}$, then $\lambda\big(G(k, \ell)\big) < \lambda\big(G(k - 1, \ell + 1)\big)$.*

Proof. By Lemma 2.1, we know $\lambda\big(G(k, \ell)\big) = \mu\big(G(k, \ell)\big) = \mu > 3 + \sqrt{2}$. Let X be the principal eigenvector corresponding to μ, and suppose the eigencomponent corresponding to the vertex v is x_v. Our aim is to show that $x_{w_1} > x_{v_2}$, then by

Lemma 2.8, we can get the desired result. From $\mu X = QX$, on the path P_{k+1}, we can get that

$$x_{v_2} = f_1(\mu)x_{v_1}, \ldots, x_{v_i} = f_{i-1}(\mu)x_{v_{i-1}}.$$

So,

$$x_u = f_2(\mu)f_3(\mu)\cdots f_k(\mu)x_{v_2}.$$

Similarly,

$$x_u = f_1(\mu)f_2(\mu)\cdots f_\ell(\mu)x_{w_1}.$$

Combining the above two relations, we have

$$\frac{x_{w_1}}{x_{v_2}} = \frac{f_{\ell+1}(\mu)\cdots f_k(\mu)}{f_1(\mu)}.$$

By Lemma 2.9,

$$\begin{aligned} f_{\ell+1}(\mu)f_{\ell+2}(\mu) &= \left(\mu - 2 - \frac{1}{f_{\ell+1}(\mu)}\right)f_{\ell+1}(\mu) = (\mu-2)f_{\ell+1}(\mu) - 1 \\ &> (\mu-2)\frac{\mu}{\mu-2} - 1 = f_1(\mu). \end{aligned}$$

So $x_{w_1} > x_{v_2}$, and the result holds. □

By Lemma 2.10, we can get the following corollaries.

Corollary 2.11. *Of all trees with diameter d, T_d has the maximal Laplacian spectral radius, where T_d is obtained by attaching $n-d-1$ pendent vertices on the vertex $\left[\frac{d+1}{2}\right]$ of the path P_{d+1}.*

Corollary 2.12. *Of all trees of order n, the star has the maximal Laplacian spectral radius and the path has the minimal Laplacian spectral radius.*

Corollary 2.13. *The Laplacian spectral radius of T_d, $2 \le d \le n-1$ can be ordered by*

$$\lambda(P_n) = \lambda(T_{n-1}) < \cdots < \lambda(T_2) = \lambda(S_{1,n-1}).$$

3. MAIN RESULTS

Let

$$S_1 = S(n_1, n_2, \ldots, n_k), \quad n_1 \ge n_2 \ge \cdots \ge n_k \ge 1, \quad k \ge 3, \quad \sum_{i=1}^{k} n_i = n-1,$$

$$S_2 = S(m_1, m_2, \ldots, m_\ell), \quad m_1 \ge m_2 \ge \cdots \ge m_\ell \ge 1, \quad \ell \ge 3, \quad \sum_{j=1}^{\ell} m_j = n-1$$

be two starlike trees.

Theorem 3.1. *If S_1 and S_2 are Laplacian cospectral, then $k = \ell$.*

Proof. By Lemma 2.3, S_1 and S_2 have the same number of vertices and edges. Consider the line graph G_1 and G_2 of S_1 and S_2, so G_1 and G_2 have the same number of vertices. By Lemma 2.7, G_1 and G_2 are adjacency cospectral. By item (3) of Lemma 2.2, G_1 and G_2 have the same number of edges, that is

$$\binom{k}{2} + (n_1 - 1) + \cdots + (n_k - 1) = \binom{\ell}{2} + (m_1 - 1) + \cdots + (m_\ell - 1),$$

i.e., $\binom{k}{2} - k = \binom{\ell}{2} - \ell$, this implies our result. □

Theorem 3.2. *No two non isomorphic starlike trees are Laplacian cospectral.*

Proof. Let S_1 and S_2 be two non isomorphic starlike trees as we said above. We will prove that if S_1 and S_2 are Laplacian cospectral, then $S_1 \cong S_2$. If S_1 and S_2 are Laplacian cospectral, then by Theorem 3.1, we know $k = \ell$. If $n_i \neq m_i$ for some i, then by $k, \ell \geq 3$, S_1 or S_2 contains $K_{1,3}^*$ as a subgraph, where $K_{1,3}^*$ is obtained by subdividing an edge of the star $K_{1,3}$, note that $\lambda(K_{1,3}^*) > 3 + \sqrt{2}$, then by Lemma 2.10, we get $\lambda(S_1) \neq \lambda(S_2)$, this contradicts to the fact that they are Laplacian cospectral. So $n_i = m_i$ for all i and we complete the proof. □

In the rest of this paper, we will obtain that some classes of starlike trees are determined by their Laplacian spectrum.

Theorem 3.3. [**14**] *The star is determined by its Laplacian spectrum.*

Proof. Since the line graph of a star is a complete graph and the complete graph is determined by its adjacency spectrum (see [**2**]), together with Lemmas 2.6 and 2.7, we get the result. □

Theorem 3.4. *The starlike trees with maximum degree 3 are determined by their Laplacian spectrum.*

Proof. Let S be a starlike tree as in Section 1 and G be a graph that are Laplacian cospectral with S. By Lemma 2.2, G and S share the same characteristic polynomial and by Lemma 2.3, G and S have the same number of vertices and edges. Since S is connected, $\lambda_{n-1}(S) > 0$, so $\lambda_{n-1}(G) > 0$, G must be connected and is a tree. By Lemmas 2.5, 24, we have $4 \leq \lambda(S) < 5$. So by Lemma 2.4, the maximum degree of G is not greater than 3. Suppose the number of vertices of degree 1, 2, 3 are x_1, x_2, x_3, respectively. By Lemma 2.3, from q_1, q_2, we have

$$x_1 + x_2 + x_3 = n,\ x_1 + 2x_2 + 3x_3 = 2(n-1),\ x_1 + 4x_2 + 9x_3 = 3 + 4(n-4) + 9.$$

Solving the above equations, we have $x_1 = 3, x_2 = n - 4, x_3 = 1$. That is, G and S have the same degree sequence, so G is a starlike tree. By Theorem 3.2, we have $G \cong S$, this is the desired result. □

Theorem 3.5. *The starlike trees with maximum degree* 4 *are determined by their Laplacian spectrum.*

Proof. Note that in Lemma 2.3, for the adjacency matrix A of a tree, $tr(A^3) = 0$. The remaining work is the same as in Theorem 3.4. □

REFERENCES

1. D. Cvetković, M. Doob, H. Sachs: *Spectra of graphs.* Academic Press, New York, 1980.
2. E. R. van Dam, W. H. Haemers: *Which graphs are determined by their spectrum*? Linear Algebra Appl., **373** (2003), 241–272.
3. M. Doob, W. H. Haemers: *The complement of the path is determined by its spectrum.* Linear Algebra Appl., **356** (2002), 57–65.
4. R. Grone, R. Merris: *The Laplacian spectrum of a graph* II. SIAM J. Matrix Anal. Appl., **11** (1990), 218–238.
5. I. Gutman, V. Gineityte, M. Lepović: *The high-band in the photoelectron spectrum of alkaners and its dependence on molecular structure.* J. Serb. Chem. Soc., **64** (1999), 673–680.
6. F. Harary: *Graph Theory.* Addison Wesley, 1969.
7. Y. Hong, X. D. Zhang: *Sharp upper and lower bound for the largest eigenvalue of the Laplacian matrices of trees.* Discrete Math., **296** (2005), 187–197.
8. M. Lepović, I. Gutman: *No starlike trees are cospectral.* Discrete Math., **242** (2002), 291–295.
9. R. Merris: *A note on the Laplacian graph eigenvalues.* Linear Algebra Appl., **285** (1998), 33–35.
10. R. Merris: *Laplacian matrices of graphs: a survey.* Linear Algebra Appl., **197–198** (1994), 143–176.
11. B. Mohar: *The Laplacian spectrum of graphs.* Graph theory, Combinartorics and applications, Vol **2**, Edt. Y. Alavi, etc, Wiley, 1991, pp. 871–898.
12. C. S. Oliveira, N. M. M. de Abreu, S. Jukiewilz: *The characteristic polynomial of the Laplacian graphs in (a,b)-linear classes.* Linear Algebra Appl., **356** (2002), 113–121.
13. Y. L. Pan: *Sharp upper bounds for the Laplacian graph eigenvalues.* Linear Algebra Appl., 355 (2002) 287–295.
14. X. L. Shen, Y. P. Hou, Y. P. Zhang: *Graph Z_n and some graphs related to Z_n are determined by their spectrum.* Linear Algebra Appl., **404** (2005), 58–68.
15. J. Shu, Y. Hong, K. Wenren: *A sharp upper bound on the largest eigenvalue of the Laplacian matrix of a graph.* Linear Algebra Appl., **347** (2002), 123–129.
16. X. D. Zhang: *The spectral radius of triangle-free graphs.* Australasian Journal of Combin., **26** (2002), 33–39.

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