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CRITICAL SETS IN LATIN SQUARES GIVEN THAT THEY ARE SYMMETRIC

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A uniquely completable (UC) set U is a subset of a Latin square L such that L is the only superset of U which is a Latin square. A critical set C of L is a subset of L such that C is uniquely completable and no subset of C has this property. We show that there is a symmetric Latin square with fixed main diagonal entries for each even number, and obtain a uniquely completable partial symmetric Latin square of order $2n$ for each n and prove that, it is critical set for $n = 3, 4, 5$ and 6, and make a problem.

1. INTRODUCTION

For a graph G denote the vertex set by $V(G)$ and the edge set by $E(G)$. A 1-factor of a graph G is a 1-regular spanning subgraph of G. Let K_n denotes the complete graph with vertex set $\{v_1, \dots, v_n\}$. A 1-factorization of K_{2n} is a collection of pair wise edge-disjoint 1-factors which partition the edge set of K_{2n}. Every complete graph on an even number of vertices has a 1-factorization.

A Latin square L of order n is an $n \times n$ array of entries $\{(i, j; k)\}$ such that each row and column of L contains each of n possible elements exactly once. We may denote a Latin square L by $L = (a_{ij})_{n\times n}$. A uniquely completable (UC) set U is a subset of a Latin square L such that L is the only superset of U which is a Latin square. A critical set C of L is a subset of L such that C is uniquely completable and no subset of C has this property. There are some papers on this concept [**6**–**9**]. Let scs(n) be a function defined the smallest size of a critical set in any $n \times n$ Latin square. It is proved that scs$(3) = 2$, scs$(4) = 4$, scs$(5) = 6$, scs$(6) = 9$, scs$(7) = 12$, scs$(8) = 16$ and for $n \geq 8$, scs$(n) \geq \left\lfloor \dfrac{4n-8}{3} \right\rfloor$. Also it has been shown

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that $\mathrm{scs}(n) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$ which is generally believed to be the correct number [**1,2,4**]. We study critical sets of $2n \times 2n$ Latin squares, given that they are symmetric.

2. CRITICAL SETS IN LATIN SQUARES GIVEN THAT THEY ARE SYMMETRIC

Let S be the set of all symmetric latin squares of order $2n$. In this section we study critical sets in S. We construct a 1-factorization of complete graph K_{2n} for each integer n. Let the vertices of K_{2n} are labelled $v_0, v_1, \ldots, v_{2n-1}$. We place the vertices equidinstantly around a circle with $2n-1$ at the center. Then we observe that $F_0 = \{v_0v_{2n-1}, v_1v_{2n-2}, v_2v_{2n-3}, \ldots, v_{n-1}v_n\}$ is a 1-factor. We construct the factor F_1 by rotating the symbols on the circle counter-clockwise through a $(2n-1)^{th}$ part of a revolution so that the center is fixed. This yields the 1-factor $F_1 = \{v_1v_{2n-1}, v_2v_0, v_3v_{2n-2}, \ldots, v_nv_{n+1}\}$. Note that $F_0 \cap F_1 = \emptyset$. Proceeding in this way, each time rotating the symbols counter-clockwise one position, we get the 1-factors $F_2, F_3, \ldots, F_{2n-2}$ with $F_i = \{v_{i-1}v_{2n-1}, v_{i+1}v_{i-1}, v_{i+2}v_{i-2}, \ldots, v_{i+n-1}v_{i-n+1}\}$ which the symbols other than $2n-1$ are treated integer modulo $2n-1$. It is easy to check that $F_0, F_1, \ldots, F_{2n-2}$ is a 1-factorization of K_{2n}.

Theorem 1. *For any even number n, there exists a symmetric Latin squares with fixed main diagonal entries.*

Proof. Let $F_0, F_1, \ldots, F_{2n-2}$ be the 1-factorization of the complete graph K_{2n} by above construction for each n. We construct a Latin square L of order $2n$ with entries $F_0, \ldots, F_{2n-2}$ and a new entry F_{2n-1} in the following way:

$a_{ii} = 1,\ 1 \leq i \leq 2n$

$a_{ij} = k+2$ if $\{v_i, v_j\} \in E(F_k)$

Then it is easy to check that L is a symmetric Latin square with fixed main diagonal entry. □

Let S' be the set of all symmetric latin squares obtained by above way . We find a UC set for any Latin square in S'.

Theorem 2. *For each integer n there exists a symmetric Latin square of order $2n$ with a UC set of size $n^2 - n + 2$.*

Proof. Consider the following partial Latin square L:

$a_{1j} = j,\ j = 1, 2, \ldots, 2n-2, j \neq n$

$a_{ii} = 1, 2 \leq i \leq 2n, i \neq 2n-3$

$$a_{ij} = \begin{cases} 2n-k,\ i,j > 1,\ i \neq j,\ i+j = 2n-2k+1,\ k > 1 \\ n-k,\quad i,j > 1,\ i+j = 2n-2k,\ k > 0 \end{cases} ,i < j$$

$a_{(2n-1)2n} = n$

$a_{(2n-2)2n} = 2n-1$

Then the integers $1, 2, \ldots, 2n-1$ occur in the first row or $(2n)^{th}$ column, so the cell a_{12n} will be forced by $2n$. Now the integers $1, 2, \ldots, 2n-2, 2n$ occur in the first row or $(2n-1)^{th}$ column, so the cell $a_{1(2n-1)}$ will be forced by $2n-1$. Similarly the cell $a_{2(2n)}$ and then the cells $a_{2(2n-1)}, a_{2(2n-2)}, a_{2(2n-3)}$ will be forced. We continue with the third row for $a_{3(2n)}, a_{3(2n-1)}, a_{3(2n-2)}, a_{3(2n-3)}, a_{3(2n-4)}$. Continuing this way we obtain a symmetric Latin square. So L is uniquely completable to a symmetric Latin square of S'. □

Now we check whether the UC sets in above theorem are critical sets. For each $2n$, we denote the UC set of order $2n$ in above theorem by L'_{2n}. Also we denote the unique symmetric Latin square in S' containing L'_{2n} by L_{2n}.

Theorem 3. *For each positive integer n, any entry of the main diagonal of L'_{2n} is necessary for the completation.*

Proof. Let L be a Latin square of order $2n$ which obtained from L_{2n} by changing the entries

1) $a_{11} \leftrightarrow a_{1(2n)}, a_{(2n)1} \leftrightarrow a_{(2n)(2n)}$,

2) $a_{11} \leftrightarrow a_{1(2n-1)}, a_{(2n-1)1} \leftrightarrow a_{(2n-1)(2n-1)}$ or

3) $a_{ii} \leftrightarrow a_{i(2n-2)}, a_{(2n-2)i} \leftrightarrow a_{(2n-2)(2n-2)}$, $2 \le i \le 2n-3$,

then it is easily seen that L is a symmetric Latin square. □

Corollary 4. *L'_6 is a critical set.*

Proof. $L'_6 =$

	2		4		
	1				
		1			
					5
				1	3
					1

Since we have each of the symbols $2, 4, 3$ and 5 just one time in L'_6 and the symbol 6 is absent, the proof is obvious.

Proposition 5. *L'_8 is a critical set.*

Proof. Consider $L'_8 =$

	2	3		5	6		
	1	6	3				
		1					
			1				
				1			
							7
						1	4
							1

and let $G_8 = L'_8 - \{x\}$ where $x \in \{a_{13}, a_{16}, a_{23}, a_{24}\}$. It is sufficient to show that G_8 is not a UC-set in S because we have each of the other entries just one time. For this consider the following Latin squares:

1	2	3	4	5	6	7	8
2	1	**8**	3	7	4	6	5
3	8	1	7	4	2	5	6
4	3	7	1	6	5	8	2
5	7	4	6	1	8	2	3
6	4	2	5	8	1	3	7
7	6	5	8	2	3	1	4
8	5	6	2	3	7	4	1

1	2	3	4	5	**8**	7	6
2	1	6	3	7	4	8	5
3	6	1	7	4	2	5	8
4	3	7	1	8	5	6	2
5	7	4	8	1	6	2	3
8	4	2	5	6	1	3	7
7	8	5	6	2	3	1	4
6	5	8	2	3	7	4	1

1	2	**8**	4	5	6	7	3
2	1	6	3	7	4	5	8
8	6	1	7	4	3	2	5
4	3	7	1	8	5	6	2
5	7	4	8	1	2	3	6
6	4	3	5	2	1	8	7
7	5	2	6	3	8	1	4
3	8	5	2	6	7	4	1

1	2	3	4	5	6	7	8
2	1	6	**8**	7	4	3	5
3	6	1	7	4	8	5	2
4	8	7	1	2	5	6	3
5	7	4	2	1	3	8	6
6	4	8	5	3	1	2	7
7	3	5	6	8	2	1	4
8	5	2	3	6	7	4	1

□

Proposition 6. *L'_{10} is a critical set.*

Proof. Consider $L'_{10} =$

	2	3	4		6	7	8		
	1	7	3	8	4				
		1	8	4					
			1						
				1					
					1				
						1			
									9
								1	5
									1

and let $G_{10} = L'_{10} - \{x\}$ which $x \in \{a_{13}, a_{14}, a_{17}, a_{18}, a_{23}, a_{24}, a_{25}, a_{26}, a_{34}, a_{35}\}$. It is sufficient to show that G_{10} is not a UC-set in S. For this order see appendix. □

Proposition 7. *L'_{12} is a critical set.*

Proof. Consider $L'_{12} =$

	2	3	4	5		7	8	9	10		
	1	8	3	9	4	10	5				
		1	9	4	10	5					
			1	10	5						
				1							
					1						
						1					
							1				
								1			
											11
										1	6
											1

and let $G_{12} = L'_{12} - \{x\}$ which

$x \in \{a_{13}, a_{14}, a_{15}, a_{18}, a_{19}, a_{1(10)}, a_{23}, a_{24}, a_{25}, a_{26}, a_{27}, a_{28}, a_{34}, a_{35}, a_{36}, a_{37}, a_{45}, a_{46}\}$.

It is sufficient to show that G_{12} is not a UC-set in S. For this see appendix. $\square$

Here we have the following:

Problem. Is L'_{2n} a critical set in S for each $n \geq 3$?

APPENDIX

$2n = 10$:

1	2	3	4	5	6	7	8	9	10
2	1	**10**	3	8	4	9	5	7	6
3	10	1	8	4	9	5	7	6	2
4	3	8	1	9	5	10	6	2	7
5	8	4	9	1	7	6	2	10	3
6	4	9	5	7	1	2	10	3	8
7	9	5	10	6	2	1	3	8	4
8	5	7	6	2	10	3	1	4	9
9	7	6	2	10	3	8	4	1	5
10	6	2	7	3	8	4	9	5	1

1	2	3	4	5	6	**10**	8	9	7
2	1	7	3	8	4	9	5	10	6
3	7	1	8	4	9	5	10	6	2
4	3	8	1	9	5	7	6	2	10
5	8	4	9	1	10	6	2	7	3
6	4	9	5	10	1	2	7	3	8
10	9	5	7	6	2	1	3	8	4
8	5	10	6	2	7	3	1	4	9
9	10	6	2	7	3	8	4	1	5
7	6	2	10	3	8	4	9	5	1

1	2	**10**	4	5	6	7	8	9	3
2	1	7	3	8	4	9	5	6	10
10	7	1	8	4	9	5	6	3	2
4	3	8	1	9	5	6	2	10	7
5	8	4	9	1	3	2	10	7	6
6	4	9	5	3	1	10	7	2	8
7	9	5	6	2	10	1	3	8	4
8	5	6	2	10	7	3	1	4	9
9	6	3	10	7	2	8	4	1	5
3	10	2	7	6	8	4	9	5	1

1	2	3	4	5	6	7	8	9	10
2	1	7	**6**	8	4	9	5	10	3
3	7	1	8	4	9	5	10	6	2
4	6	8	1	9	5	10	3	2	7
5	8	4	9	1	10	3	2	7	6
6	4	9	5	10	1	2	7	3	8
7	9	5	10	3	2	1	6	8	4
8	5	10	3	2	7	6	1	4	9
9	10	6	2	7	3	8	4	1	5
10	3	2	7	6	8	4	9	5	1

1	2	3	**10**	5	6	7	8	9	4
2	1	7	3	8	4	9	5	10	6
3	7	1	8	4	9	5	10	6	2
10	3	8	1	9	5	4	6	2	7
5	8	4	9	1	10	6	2	7	3
6	4	9	5	10	1	2	7	3	8
7	9	5	4	6	2	1	3	8	10
8	5	10	6	2	7	3	1	4	9
9	10	6	2	7	3	8	4	1	5
4	6	2	7	3	8	10	9	1	5

1	2	3	4	5	6	7	8	9	10
2	1	7	3	8	**10**	9	5	4	6
3	7	1	8	4	9	5	10	6	2
4	3	8	1	9	5	10	6	2	7
5	8	4	9	1	7	6	2	10	3
6	10	9	5	7	1	2	4	3	8
7	9	5	10	6	2	1	3	8	4
8	5	10	6	2	4	3	1	7	9
9	4	6	2	10	3	8	7	1	5
10	6	2	7	3	8	4	9	5	1

1	2	3	4	5	6	7	8	9	10
2	1	7	3	8	4	9	5	10	6
3	7	1	8	**10**	9	5	4	6	2
4	3	8	1	9	5	10	6	2	7
5	8	10	9	1	7	6	2	4	3
6	4	9	5	7	1	2	10	3	8
7	9	5	10	6	2	1	3	8	4
8	5	4	6	2	10	3	1	7	9
9	10	6	2	4	3	8	7	1	5
10	6	2	7	3	8	4	9	5	1

1	2	3	4	5	6	7	**10**	9	8
2	1	7	3	8	4	9	5	6	10
3	7	1	8	4	9	5	6	10	2
4	3	8	1	9	5	10	7	2	6
5	8	4	9	1	10	6	2	3	7
6	4	9	5	10	1	2	8	7	3
7	9	5	10	6	2	1	3	8	4
10	5	6	7	2	8	3	1	4	9
9	6	10	2	3	7	8	4	1	5
8	10	2	6	7	3	4	9	5	1

1	2	3	4	5	6	7	8	9	10
2	1	7	3	**6**	4	9	5	10	8
3	7	1	8	4	9	5	10	2	6
4	3	8	1	9	5	10	7	6	2
5	6	4	9	1	10	8	2	7	3
6	4	9	5	10	1	2	3	8	7
7	9	5	10	8	2	1	6	3	4
8	5	10	7	2	3	6	1	4	9
9	10	2	6	7	8	3	4	1	5
10	8	6	2	3	7	4	9	5	1

1	2	3	4	5	6	7	8	9	10
2	1	7	3	8	4	9	5	10	6
3	7	1	**2**	4	9	5	10	6	8
4	3	2	1	9	5	10	6	8	7
5	8	4	9	1	10	6	7	3	2
6	4	9	5	10	1	8	2	7	3
7	9	5	10	6	8	1	3	2	4
8	5	10	6	7	2	3	1	4	9
9	10	6	8	3	7	2	4	1	5
10	6	8	7	2	3	4	9	5	1

$2n = 12$:

1	2	**12**	4	5	6	7	8	9	10	11	3
2	1	8	3	9	4	10	5	11	6	7	12
12	8	1	9	4	10	5	11	6	7	3	2
4	3	9	1	10	5	11	6	8	2	12	7
5	9	4	10	1	11	6	7	12	3	2	8
6	4	10	5	11	1	3	2	7	12	8	9
7	10	5	11	6	3	1	12	2	8	9	4
8	5	11	6	7	2	12	1	3	9	4	10
9	11	6	8	12	7	2	3	1	4	10	5
10	6	7	2	3	12	8	9	4	1	5	11
11	7	3	12	2	8	9	4	10	5	1	6
3	12	2	7	8	9	4	10	5	11	6	1

1	2	3	**11**	5	6	7	8	9	10	4	12
2	1	8	3	9	4	10	5	11	6	12	7
3	8	1	9	4	10	5	11	6	12	7	2
11	3	9	1	10	5	8	6	12	7	2	4
5	9	4	10	1	11	6	12	7	2	8	3
6	4	10	5	11	1	12	7	2	9	3	8
7	10	5	8	6	12	1	2	4	3	11	9
8	5	11	6	12	7	2	1	3	4	9	10
9	11	6	12	7	2	4	3	1	8	10	5
10	6	12	7	2	9	3	4	8	1	5	11
4	12	7	2	8	3	11	9	10	5	1	6
12	7	2	4	3	8	9	10	5	11	6	1

1	2	3	4	**11**	6	7	8	9	10	12	5
2	1	8	3	9	4	10	5	11	6	7	12
3	8	1	9	4	10	5	11	6	12	2	7
4	3	9	1	10	5	8	6	12	7	11	2
11	9	4	10	1	12	6	7	5	2	8	3
6	4	10	5	12	1	11	2	7	3	9	8
7	10	5	8	6	11	1	12	2	9	3	4
8	5	11	6	7	2	12	1	3	4	10	9
9	11	6	12	5	7	2	3	1	8	4	10
10	6	12	7	2	3	9	4	8	1	5	11
12	7	2	11	8	9	3	10	4	5	1	6
5	12	7	2	3	8	4	9	10	11	6	1

1	2	3	4	5	6	7	**12**	9	10	11	8
2	1	8	3	9	4	10	5	11	6	7	12
3	8	1	9	4	10	5	11	6	7	12	2
4	3	9	1	10	5	11	6	8	12	2	7
5	9	4	10	1	11	6	7	12	2	8	3
6	4	10	5	11	1	12	2	7	8	3	9
7	10	5	11	6	12	1	8	2	3	9	4
12	5	11	6	7	2	8	1	3	9	4	10
9	11	6	8	12	7	2	3	1	4	10	5
10	6	7	12	2	8	3	9	4	1	5	11
11	7	12	2	8	3	9	4	10	5	1	6
8	12	2	7	3	9	4	10	5	11	6	1

1	2	3	4	5	6	7	8	**12**	10	11	9
2	1	8	3	9	4	10	5	11	6	7	12
3	8	1	9	4	10	5	11	6	12	2	7
4	3	9	1	10	5	11	6	8	7	12	2
5	9	4	10	1	11	6	12	7	2	8	3
6	4	10	5	11	1	12	7	2	9	3	8
7	10	5	11	6	12	1	2	3	8	9	4
8	5	11	6	12	7	2	1	9	3	4	10
12	11	6	8	7	2	3	9	1	4	10	5
10	6	12	7	2	9	8	3	4	1	5	11
11	7	2	12	8	3	9	4	10	5	1	6
9	12	7	2	3	8	4	10	5	11	6	1

1	2	3	4	5	6	7	8	9	**12**	11	10
2	1	8	3	9	4	10	5	11	6	7	12
3	8	1	9	4	10	5	11	6	2	12	7
4	3	9	1	10	5	11	6	12	7	8	2
5	9	4	10	1	11	6	12	7	8	2	3
6	4	10	5	11	1	12	7	2	9	3	8
7	10	5	11	6	12	1	2	8	3	4	9
8	5	11	6	12	7	2	1	3	10	9	4
9	11	6	12	7	2	8	3	1	4	10	5
12	6	2	7	8	9	3	10	4	1	5	11
11	7	12	8	2	3	4	9	10	5	1	6
10	12	7	2	3	8	9	4	5	11	6	1

1	2	3	4	5	6	7	8	9	10	11	12
2	1	**12**	3	9	4	10	5	11	6	7	8
3	12	1	9	4	10	5	11	6	7	8	2
4	3	9	1	10	5	11	6	8	12	2	7
5	9	4	10	1	11	6	7	2	8	12	3
6	4	10	5	11	1	8	12	7	2	3	9
7	10	5	11	6	8	1	2	12	3	9	4
8	5	11	6	7	12	2	1	3	9	4	10
9	11	6	8	2	7	12	3	1	4	10	5
10	6	7	12	8	2	3	9	4	1	5	11
11	7	8	2	12	3	9	4	10	5	1	6
12	8	2	7	3	9	4	10	5	11	6	1

1	2	3	4	5	6	7	8	9	10	11	12
2	1	8	**12**	9	4	10	5	11	6	3	7
3	8	1	9	4	10	5	11	6	7	12	2
4	12	9	1	10	5	11	6	7	2	8	3
5	9	4	10	1	11	6	7	3	12	2	8
6	4	10	5	11	1	12	2	8	3	7	9
7	10	5	11	6	12	1	3	2	8	9	4
8	5	11	6	7	2	3	1	12	9	4	10
9	11	6	7	3	8	2	12	1	4	10	5
10	6	7	2	12	3	8	9	4	1	5	11
11	3	12	8	2	7	9	4	10	5	1	6
12	7	2	3	8	9	4	10	5	11	6	1

1	2	3	4	5	6	7	8	9	10	11	12
2	1	8	3	**12**	4	10	5	11	6	9	7
3	8	1	9	4	10	5	11	6	7	12	2
4	3	9	1	10	5	11	6	7	12	2	8
5	12	4	10	1	11	6	7	8	2	3	9
6	4	10	5	11	1	2	9	12	8	7	3
7	10	5	11	6	2	1	12	3	9	8	4
8	5	11	6	7	9	12	1	2	3	4	10
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12	7	2	8	9	3	4	10	5	11	6	1

1	2	3	4	5	6	7	8	9	10	11	12
2	1	8	3	9	**7**	10	5	11	6	12	4
3	8	1	9	4	10	5	11	6	12	7	2
4	3	9	1	10	5	11	6	12	8	2	7
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6	7	10	5	11	1	12	4	2	9	3	8
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9	11	6	12	7	2	8	3	1	4	10	5
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12	4	2	7	3	8	9	10	5	11	6	1

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3	8	1	9	4	10	5	11	6	12	7	2
4	3	9	1	10	5	6	7	11	2	12	8
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12	7	2	8	3	9	10	4	5	11	6	1

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2	1	8	3	9	4	10	**12**	11	6	5	7
3	8	1	9	4	10	5	11	6	12	7	2
4	3	9	1	10	5	11	6	2	7	12	8
5	9	4	10	1	11	6	7	12	8	2	3
6	4	10	5	11	1	12	3	7	2	8	9
7	10	5	11	6	12	1	2	8	9	3	4
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3	8	1	**2**	4	10	5	11	6	12	7	9
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3	8	1	9	**2**	10	5	11	6	12	7	4
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12	7	4	8	3	2	9	10	5	11	6	1

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2	1	8	3	9	4	10	5	11	6	12	7
3	8	1	9	4	<u>**2**</u>	5	11	6	12	7	10
4	3	9	1	10	5	11	6	12	7	2	8
5	9	4	10	1	11	6	12	7	8	3	2
6	4	2	5	11	1	12	7	10	9	8	3
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12	7	10	8	2	3	4	9	5	11	6	1

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2	1	8	3	9	4	10	5	11	6	12	7
3	8	1	9	4	10	<u>**2**</u>	11	6	12	7	5
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5	9	4	10	1	11	6	12	7	8	2	3
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3	8	1	9	4	10	5	11	6	12	7	2
4	3	9	1	<u>**8**</u>	5	11	6	12	7	2	10
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6	4	10	5	11	1	12	7	2	8	3	9
7	10	5	11	6	12	1	2	3	9	4	8
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4	3	9	1	10	<u>**2**</u>	11	6	12	7	8	5
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12	7	2	5	3	9	8	4	10	11	6	1

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