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INEQUALITIES INVOLVING INVERSE CIRCULAR AND INVERSE HYPERBOLIC FUNCTIONS

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Inequalities connecting inverse circular and inverse hyperbolic functions are established. These results are otained with the aid of an elementary transcendental function which belongs to the family of R-hypergeometric functions discussed in detail in Carlson's monograph [**2**].

1. INTRODUCTION AND NOTATION

In this paper we offer several inequalities involving inverse circular and inverse hyperbolic functions. The main results are derived from the inequalities satisfied by the R-hypergeometric function $R_C(\cdot\,,\cdot)$. Let $x \ge 0$ and $y > 0$. Following [**2**]

$$R_C(x,y) = \frac{1}{2}\int_0^\infty (t+x)^{-1/2}(t+y)^{-1}\mathrm{d}t. \tag{1.1}$$

It is well-known that $R_C(\lambda x, \lambda y) = \lambda^{-1/2}R_C(x,y)$ $(\lambda > 0)$, i.e., R_C is a homogeneous function of degree $-1/2$ in its variables and also that $R_C(x,x) = x^{-1/2}$ and

$$R_C(0,y) = \frac{\pi}{2\sqrt{y}} \qquad (y > 0). \tag{1.2}$$

For later use let us record the following formula

$$R_C(x,y) = \begin{cases} (y-x)^{-1/2}\arccos{(x/y)^{1/2}}, & x < y \\ (x-y)^{-1/2}\operatorname{arccosh}{(x/y)^{1/2}}, & x > y \end{cases} \tag{1.3}$$

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(see [**2**, (6.9-15)]). Other inverse circular and inverse hyperbolic functions also admit representations in terms of the R_C function [**2**, Ex. 6.9-16]

$$\arcsin x = xR_C(1-x^2, 1), \qquad |x| \le 1 \tag{1.4}$$

$$\arctan x = xR_C(1, 1+x^2), \qquad x \in \mathbb{R} \tag{1.5}$$

$$\operatorname{arcsinh} x = xR_C(1+x^2, 1), \qquad x \in \mathbb{R} \tag{1.6}$$

$$\operatorname{arctanh} x = xR_C(1, 1-x^2), \qquad |x| < 1. \tag{1.7}$$

Bounds for the inverse circular and inverse hyperbolic functions can be obtained using the following inequalities

$$\frac{3}{x_n + 2y_n} \le R_C(x^2, y^2) \le (x_n y_n^2)^{-1/3}, \quad n \ge 0 \tag{1.8}$$

(see [**5**, (3.10) and (2.2)]) where the sequences $\{x_n\}_0^\infty$ and $\{y_n\}_0^\infty$ are generated using the Schwab-Borchardt algorithm

$$x_0 = x,\ y_0 = y,\ x_{n+1} = (x_n + y_n)/2,\ y_{n+1} = \sqrt{x_{n+1} y_n},\ n = 0, 1, \ldots$$

(see [**1**], [**2**]). It has been shown in [**5**, 3.3] that the sequences $\{3/(x_n + 2y_n)\}_0^\infty$ and $\{(x_n y_n^2)^{-1/3}\}_0^\infty$ converge monotonically to the common limit $R_C(x^2, y^2)$. It is worth mentioning that Carlson's inequalities

$$\frac{6(1-x)^{1/2}}{2\sqrt{2} + (1+x)^{1/2}} < \arccos x < \frac{\sqrt[3]{4}\,(1-x)^{1/2}}{(1+x)^{1/6}}, \qquad 0 < x < 1$$

(see, e.g., [**4**, 3.4.30]) follow from (1.8) with $n = 1$ and (1.3) used with $x := x^2$ and $y = 1$. Lower bounds for the function $\arcsin x$ (see [**4**, 3.4.31]) can be derived using the first inequality in (1.8) with $n = 0$, $n = 1$ and $x_0 = (1-x^2)^{1/2}$ followed by application of (1.4). We omit further details.

For later use, let us record three inequalities

$$y\big(R_C(y^2, x^2)\big)^{-1} \le \big(R_C(x^2, y^2)\big)^{-2} \le \frac{1}{2}\Big(\big(R_C(y^2, x^2)\big)^{-2} + y^2\Big), \tag{1.9}$$

$$\big(R_C(x^2, y^2)\big)^2 \le \frac{R_C(y, A)}{A\sqrt{y}}, \tag{1.10}$$

and

$$\big(R_C(x, A)\big)^2 \le R_C(y^2, x^2) \qquad (A = (x+y)/2) \tag{1.11}$$

which have been established in [**6**, Theorem 3.1].

The main results of this note are contained in the next section.

2. MAIN RESULTS

Our first result reads as follows.

Theorem. *The following inequalities*

$$\left(\frac{\arcsin x}{x}\right)^2 \le \frac{\operatorname{arctanh} x}{x} \le \left(\frac{\arcsin x}{x\sqrt{1-x^2}}\right)^{1/2}, \quad (|x|<1) \tag{2.1}$$

and

$$\left(\frac{\operatorname{arcsinh} x}{x}\right)^2 \le \frac{\arctan x}{x} \le \left(\frac{\operatorname{arcsinh} x}{x\sqrt{1+x^2}}\right)^{1/2}, \quad (x\in\mathbb{R}) \tag{2.2}$$

hold true. Inequalities (2.1) *and* (2.2) *become equalities if* $x=0$*.*

Proof. For the proof of inequalities (2.1) we shall employ the following one

$$R_C^2(x^2,y^2) \le \frac{R_C(y^2,x^2)}{y} \le \frac{1}{y}\left(\frac{R_C(x^2,y^2)}{x}\right)^{1/2}. \tag{2.3}$$

The first inequality in (2.3) follows from the first inequality in (1.9) while the second one is obtained from the first inequality by interchanging x with y, i.e., by letting $x := y$ and $y := x$. Substituting $x^2 := 1-x^2$ and $y=1$ in (2.3) we obtain the desired result using (1.4) and (1.7). In order to prove the inequalities (2.2) it suffices to use (2.3) with $x^2 := 1+x^2$ and $y=1$ followed by application of (1.6) and (1.5). □

Companion inequalities to (2.1) and (2.2) are contained in the following.

Theorem 2. *Let* $|x|<1$*. Then*

$$\left(\frac{\operatorname{arctanh} u}{u}\right)^2 \le \frac{\arcsin x}{x} \le \left(\frac{\operatorname{arctanh} u}{u(1-u^2)}\right)^{1/2}, \tag{2.4}$$

where $u=\sqrt{\frac{1}{2}\left(1-\sqrt{1-x^2}\right)}$*. If* $x\in\mathbb{R}$*, then*

$$\left(\frac{\arctan v}{v}\right)^2 \le \frac{\operatorname{arcsinh} x}{x} \le \left(\frac{\arctan v}{v(1+v^2)}\right)^{1/2}, \tag{2.5}$$

where $v=\sqrt{\frac{1}{2}\left(\sqrt{1+x^2}-1\right)}$*. Equalities hold in* (2.4) *and* (2.5) *if* $x=0$*.*

Proof. There is nothing to prove when $x=0$. Since all members of (2.4) and (2.5) are even functions in x, we will always assume that $x>0$. Inequalities (2.4) and (2.5) follow easily from the following one

$$\big(R_C(x,A)\big)^2 \le R_C(y^2,x^2) \le \left(\frac{R_C(x,A)}{A\sqrt{x}}\right)^{1/2}, \tag{2.6}$$

where $A = (x + y)/2$ is the arithmetic mean of two positive numbers x and y. The first inequality in (2.6) is (1.11) while the second one follows from (1,10) after interchanging x with y. Letting $y^2 = 1 - x^2$ and $x = 1$ in (2.6) we obtain

$$\big(R_C(1, A)\big)^2 \le R_C(1 - x^2, 1) \le \left(\frac{R_C(1, A)}{A}\right)^{1/2},$$

where $A = \frac{1}{2}\big(1 + \sqrt{1 - x^2}\big)$. Writing $A = 1 - u^2$ we obtain

$$\big(R_C(1, 1 - u^2)\big)^2 \le R_C(1 - x^2, 1) \le \left(\frac{R_C(1, 1 - u^2)}{1 - u^2}\right)^{1/2}.$$

Application of (1.4) and (1.7) completes the proof of (2.4). Inequalities (2.5) can be established in an analogous manner. We use (2.6) with $y^2 = 1 + x^2$, $x = 1$ to obtain

$$R_C^2(1, 1 + v^2) \le R_C(1 + x^2, 1) \le \left(\frac{R_C(1, 1 + v^2)}{1 + v^2}\right)^{1/2}.$$

Making use of (1.5) and (1.6) we obtain the desired result. □

Our next result reads as follows.

Theorem 3. *The following inequalities*

$$\left(\frac{\arcsin x}{\operatorname{arctanh} x}\right)^2 + \left(\frac{\arcsin x}{x}\right)^2 \ge 2, \qquad (|x| < 1) \tag{2.7}$$

$$\left(\frac{\operatorname{arcsinh} x}{\arctan x}\right)^2 + \left(\frac{\operatorname{arcsinh} x}{x}\right)^2 \ge 2, \qquad (x \in \mathbb{R}) \tag{2.8}$$

$$\left(\frac{\arccos x}{\operatorname{arccosh}(1/x)}\right)^2 + \left(\frac{\arccos x}{\sqrt{1 - x^2}}\right)^2 \ge 2, \qquad (|x| < 1, x \ne 0) \tag{2.9}$$

and

$$\left(\frac{\operatorname{arccosh} x}{\arccos(1/x)}\right)^2 + \left(\frac{\operatorname{arccosh} x}{\sqrt{x^2 - 1}}\right)^2 \ge 2, \qquad (|x| \ge 1) \tag{2.10}$$

are valid. Inequalities (2.7) *and* (2.8) *become equalities if* $x = 0$. *Equalities hold in* (2.9) *and* (2.10) *if* $x = 1$.

Proof. Inequalities (2.7)–(2.19) can be regarded as special cases of the inequality

$$\big(R_C(x^2, y^2)\big)^2\big(R_C^{-2}(y^2, x^2) + y^2\big) \ge 2 \qquad (x > 0,\ y > 0) \tag{2.11}$$

which follows from the second inequality in (1.9). Equality holds in (2.11) if $x = y$. In order to prove (2.7) we put $x^2 := 1 - x^2$ and $y = 1$ in (2.11) and next we use (1.4) and (1.7) Similarly, letting $x^2 := 1 + x^2$ and $y = 1$ in (2.11) and applying (1.5) and (1.6) we obtain the inequalities (2.8). For the proof of the inequalities (2.9) we use (2.11) with $y = 1$ together with two formulas

$$R_C(x^2, 1) = \frac{\arccos x}{\sqrt{1 - x^2}}$$

and

$$R_C(1, x^2) = \frac{\operatorname{arccosh}(1/x)}{\sqrt{1-x^2}} \qquad (|x| \le 1) \tag{2.12}$$

which follow easily from (1.3). If $|x| \ge 1$, then

$$R_C(x^2, 1) = \frac{\operatorname{arccosh} x}{\sqrt{x^2-1}}$$

and

$$R_C(1, x^2) = \frac{\arccos(1/x)}{\sqrt{x^2-1}}\,. \tag{2.13}$$

Letting $y = 1$ in (2.11) and next using the last two formulas we obtain the inequalities (2.10). □

We shall prove now the following.

Theorem 4. *If $0 < y \le 1 \le x$, then*

$$\frac{\operatorname{arccosh} x}{\sqrt{x^2-1}} \le \frac{\arccos y}{\sqrt{1-y^2}} \tag{2.14}$$

with equality if $x = y = 1$. Also, if $0 \le x \le 1$, then

$$\sqrt{1-x^2}\,\operatorname{arctanh} x \le \sqrt{1+x^2}\,\arctan x \tag{2.15}$$

with the inequality reversed if $-1 < x \le 0$. Inequality (2.15) becomes an equality if $x = 0$.

Proof. B. C. Carlson and J. L. Gustafson [**3**] have proven a result which in a particular case states that the function R_C is strictly totally positive. Thus if $0 \le x_1 < x_2$ and $0 < y_1 < y_2$, then

$$R_C(x_1, y_2)R_C(x_2, y_1) < R_C(x_1, y_1)R_C(x_2, y_2).$$

Letting above $x_1 = 0$, $x_2 = x > 0$ and next using (1.2) we obtain

$$\sqrt{y_1}\, R_C(x, y_1) < \sqrt{y_2}\, R_C(x, y_2). \tag{2.16}$$

Assume that $0 < y < 1 < x$. Putting in (2.16) $y_1 = 1/x^2$, $y_2 = 1/y^2$, and $x = 1$ we obtain

$$\frac{1}{x} R_C\left(1, \frac{1}{x^2}\right) < \frac{1}{y} R_C\left(1, \frac{1}{y^2}\right). \tag{2.17}$$

Application of (2.12), with $x := 1/x$, to the first member of (2.17) and use of (2.13), with $x := 1/y$, on the second member of (2.17) completes the proof of (2.14). In order to establish the inequality (2.15) we use (2.16) with $y_1 = 1 - x^2$, $y_2 = 1 + x^2$ $(0 < x < 1)$, and $x = 1$, to obtain

$$\sqrt{1 - x^2}\, R_C(1, 1 - x^2) < \sqrt{1 + x^2}\, R_C(1, 1 + x^2).$$

Making use of (1.7) and (1.5) we obtain the assertion. The proof is complete. □

We close this section with the following.

Theorem 5. *Let* $f(t)$ *denote one of the following functions* $\operatorname{arcsin} t$, $\arctan t$, $\operatorname{arcsinh} t$, $\operatorname{arctanh} t$ *and let* x *and* y *belong to the domain of* $f(t)$. *If* $z^2 = (x^2 + y^2)/2$, *then the following inequality*

$$\left(\frac{f(z)}{z}\right)^2 \le \frac{f(x)}{x}\,\frac{f(y)}{y} \tag{2.18}$$

is valid.

Proof. It follows from Proposition 2.1 in [**6**] that the function $R_C(\cdot\,,\cdot)$ is logarithmically convex in its variables

$$\left(R_C\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)\right)^2 \le R_C(x_1, y_1) R_C(x_2, y_2) \tag{2.19}$$

$(x_1 \ge 0,\ x_2 \ge 0,\ y_1 > 0,\ y_2 > 0)$. Letting in (2.19) $x_1 = 1 - x^2$, $x_2 = 1 - y^2$, $y_1 = y_2 = 1$ and next using (1.4) we obtain the desired result when $f(t) = \arcsin t$. The remaining cases can be established in the same way. □

Using (1.3) and (2.19) one can establish inequalities similar to (2.18) when $f(t) = \arccos t$ and $f(t) = \operatorname{arccosh} t$. We omit further details.

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