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CURIOUS SPECIAL FUNCTION IDENTITIES

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S. Simons [**7**] exibited and proved a most curious identity which may be rewritten using binomial coefficient notation in the form

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}(1+x)^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k. \tag{1}$$

Simons also observed that if we define

$$f(y)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\Big(y-\frac{1}{2}\Big)^k, \tag{2}$$

then

$$f(-y)=(-1)^n f(y), \tag{3}$$

so that $f(y)$ is a polynomial function of y with the same parity as n.

We give a complete generalization of these results to other special functions.

I have exhibited [**5**] a simple proof of (1) and (3) using well-known properties of the Legendre polynomials. It will be useful to recapitulate the proof here.

First, it is known that the Legendre polynomial may be written as

$$P_n(z)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\Big(\frac{z-1}{2}\Big)^k, \tag{4}$$

so that the identity (1), written in Legendre polynomial notation, says that

$$(-1)^n P_n(-2x-1)=P_n(2x+1). \tag{5}$$

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But more generally it is well-known that

$$(-1)^n P_n(-z) = P_n(z),$$

i.e., the LEGENDRE polynomial in z has the same parity as n. This is equivalent to (3).

The identity (1) of SIMONS does not appear in my book [**4**], but will find a place there when a third edition ever appears.

We now generalize (1) as follows.

Theorem 1. *Let a set of polynomials* $\{F_n(z)\}$ *of degree* n *in* z *satisfy*

$$F_n(-z) = (-1)^n F_n(z), \tag{6}$$

and suppose there exists coefficients $A_k(n)$ *such that*

$$F_n(z) = \sum_{k=0}^{n} A_k(n) \Big(\frac{z-1}{2}\Big)^k. \tag{7}$$

then necessarily

$$\sum_{k=0}^{n} (-1)^{n-k} A_k(n)(x+1)^k = \sum_{k=0}^{n} A_k(n) x^k. \tag{8}$$

Proof. Relation (7) may first be rewritten as

$$F_n(2x+1) = \sum_{k=0}^{n} A_k(n) x^k. \tag{9}$$

Then using (6), followed by (7), we find

$$\begin{aligned} F_n(2x+1) &= (-1)^n F_n(-2x-1) \\ &= (-1)^n \sum_{k=0}^{n} A_k(n)(-x-1)^k = \sum_{k=0}^{n} (-1)^{n-k} A_k(n)(x+1)^k, \end{aligned}$$

which proves our theorem. □

All that is necessary to determine curious SIMONS-type identities is then to determine the coefficients $A_k(n)$ corresponding to the function $F_n(x)$. We have only to expand $F_n(2x+1)$ as in (9) and equate coefficients.

The simplest example of our theorem occurs when $F_n(x) = x^n$. It is then easily shown by the binomial theorem that $A_k(n) = \binom{n}{k} 2^k$, so that

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} 2^k (x+1)^k = \sum_{k=0}^{n} \binom{n}{k} 2^k x^k. \tag{10}$$

Of course, this is easily seen directly since each side equals $(2x+1)^n$.

It is natural to consider GEGENBAUER polynomials, which are a form of generalized LEGENDRE polynomials. These are defined by

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{+\infty} C_n^\nu(x) t^n, \tag{11}$$

and $C_n^{1/2}(x) = P_n(x)$. It is also known [**6**, p. 27] that

$$C_n^\nu(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(\nu)_{n-k}}{k!(n-2k)!} (2x)^{n-2k}, \tag{12}$$

so that clearly $C_n^\nu(-x) = (-1)^n C_n^\nu(x)$. Moreover [**6**, p. 279] it is known that

$$C_n^\nu(x) = \sum_{k=0}^{n} \frac{(2\nu)_{n+k}}{k!(n-k)!(\nu+1/2)_k} \left(\frac{x-1}{2}\right)^k. \tag{13}$$

N. B.: The ascending factorial notation is used here. That is to say

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1) \text{ for } n \geq 1, \quad \text{and} \quad (\alpha)_0 = 1 \text{ for } \alpha \neq 0.$$

Theorem 1 then tells us that we have the SIMONS-type identity

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{(2\nu)_{n+k}}{k!(n-k)!(\nu+1/2)_k} (x+1)^k = \sum_{k=0}^{n} \frac{(2\nu)_{n+k}}{k!(n-k)!(\nu+1/2)_k} x^k, \tag{14}$$

which is an extension of the original formula (1) of SIMONS valid for all complex ν, and of course (1) occurs as the special case $\nu = 1/2$.

We offer next a more difficult example using HERMITE polynomials. These may be defined by

$$H_n(x) = (-1)^n e^{x^2} \mathrm{D}_x^n\, e^{-x^2} = \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{k!(n-2k)!} (2x)^{n-2k}. \tag{15}$$

It is clear that $H_n(-x) = (-1)^n H_n(x)$ so they satisfy (6). The first seven values of $H_n(x)$ are as follows:

$$\begin{aligned}
&H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x, \\
&H_4(x) = 16x^4 - 48x^2 + 12, \quad H_5(x) = 32x^5 - 160x^3 + 120x, \\
&H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120.
\end{aligned}$$

Table 1 gives values of the coefficients $A_k(n)$ found by expanding $H_n(2x+1)$ for $0 \leq n \leq 6$. A formula could be worked out for these numbers by use of (9).

Theorem 1 may be applied to several other interesting polynomials. For example, it applies to the GOULD-HOPPER polynomial $H_n^r(x,a,p)$ when r is even [**2**]. Also we can use it for the generalized HUMBERT polynomial $P_n(m,x,y,p,C)$ when m is even [**3**].

$n \backslash k$	0	1	2	3	4	5	6	$\cdots$	k
0	1								
1	2	4							
2	2	16	16						
3	−4	24	96	64					
4	−20	−64	192	512	256				
5	−8	−400	−640	1280	2560	1024			
6	184	−192	−4800	−5120	7680	12288	4096		
n									

Table 1. Coefficients $A_k(n)$ found by expanding $H_n(2x+1)$, for $0 \le n \le 6$.

Theorem 2. *As in Theorem* 1*, let* $F_n(-z) = (-1)^n F_n(z)$ *and*

$$F_n(ax+b) = \sum_{k=0}^{n} A_k(n)\, x^k. \tag{16}$$

Then

$$\sum_{k=0}^{n} (-1)^{n-k} A_k(n) \Big(x + \frac{2b}{a}\Big)^k = \sum_{k=0}^{n} A_k(n)\, x^k. \tag{17}$$

Proof. The proof is the same, mutatis mutandis, as that of Theorem 1. The original Theorem 1 where $a = 2$ and $b = 1$ is probably most useful in the sense that various interesting special functions satisfy (7).

Once we have established the identity (8) there are ways to devise variations of it. Thus we have

$$\sum_{k=0}^{n} (-1)^{n-k} A_k(n) k^p (x+1)^k = \sum_{k=0}^{n} A_k(n) k^p x^k, \tag{18}$$

which follows from (8) because $(x\mathrm{D})^p (x+1)^k = (x\mathrm{D})^p x^k = k^p x^k$, so the factor k^p may be introduced on each side. Another variation is

$$\sum_{k=j}^{n} (-1)^{n-k} A_k(n) \binom{k}{j} (x+1)^{k-j} = \sum_{k=j}^{n} A_k(n) \binom{k}{j} x^{k-j}, \tag{19}$$

which follows by taking the j-th derivative of each side of (8).

In (8) let $x = -1$ and we obtain the interesting formula

$$\sum_{k=1}^{n} (-1)^k A_k(n) = \big((-1)^n - 1\big) A_0(n), \quad \text{for all } n \ge 2. \tag{20}$$

This affords a partial check of the n-th row of the array. Thus for the HERMITE polynomial SIMONS array,

$$\begin{aligned}
&64 + 192 - 512 + 256 = 0, \\
&400 - 640 - 1280 + 2560 - 1024 = -2 \cdot (-8) = 16, \\
&192 - 4800 + 5120 + 7680 - 12288 + 4096 = 0, \text{ etc.}
\end{aligned}$$

Starting with the defining characteristic $F_n(-z) = (-1)^n F_n(z)$, as in Theorem 1, it is evident that the general class of polynomials satisfying relation (8) is given by

$$F_n(x) = \sum_{0 \le k \le n/2} C_k(n) x^{n-2k}, \tag{21}$$

where $C_k(n)$ is any arbitrary array. From this it easily follows that

$$A_j(n) = 2^j \sum_{0 \le k \le (n-j)/2} C_k(n) \binom{n-2k}{j}. \tag{22}$$

The simplest instance of this is when $C_k(n) = 1$ identically. Table 2 offers a tabulation of the array in this case In this case we have

$$F_n(x) = \sum_{0 \le k \le n/2} x^{n-2k}, \tag{23}$$

and

$$A_j(n) = 2^j \sum_{0 \le k \le (n-j)/2} \binom{n-2k}{j}. \tag{24}$$

1									
1	2								
2	4	4							
2	8	12	8						
3	12	28	32	16					
3	18	52	88	809	32				
4	24	88	192	256	192	64			
4	32	136	368	640	704	448	128		
5	40	200	640	1376	1984	1856	1024	256	
5	50	280	1040	2656	4736	5824	4736	2304	512

Table 2. Values of $A_j(n) = 2^j \sum_{0 \le k \le (n-j)/2} \binom{n-2k}{j}$, for $0 \le j \le n \le 9$.

Rows of this array checked by the simple formula

$$\sum_{j=0}^{n}(-1)^{n-j}A_j(n) = \left[\frac{n}{2}\right] + 1, \tag{25}$$

where brackets denote the greatest integer function. Another property of these special A's is as follows:

$$\sum_{j=0}^{n} A_j(n) = 3\sum_{j=0}^{n-1} A_j(n-1) \text{ if } n \text{ is odd,}$$

and 1 more than this if n is even.

It is felt that our theorems shed light the form of these kinds of combinatorial or special function identities.

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