Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 17 (2006), 84–87.
Available electronically at **http: //matematika.etf.bg.ac.yu**

# QUASINORMABILITY OF $\mathrm{D}_b$ SPACES

*Stojan Radenović*

In this short paper we shall prove that a space of a $\mathrm{D}_b$ type ( g(DF) or quasi-DF by some authors) is quasinormable.

In a general theory of locally convex spaces there are five significant classes of spaces with fundamental sequences of the family of all bounded subsets or some of its subfamilies. These are spaces do (DF), (dF), $\mathrm{D}_b$, dual-metric and (df) type.

(DF) and (df) classes were defined by Grothendieck [**2**], (dF) by Brauner [**1**], $\mathrm{D}_b$ (g(DF) or almost-DF) were defined by Noureddine [**6**], Ruess [**8**] and Mazon [**5**], whereas, the class of dual-metric spaces was defined by Pietsch [**7**].

Let us remind ourselves of these classes of spaces.

Locally convex space $(X, t)$ is the space of (DF) type (resp. dual-metric;(df)) if it has a fundamental sequence of bounded subsets and if each $X'_b$-bounded subset, which is a countable union of $t$-equicontinuous subsets (resp. $X'_b$-bounded sequence; $X'_b$-convergent sequence) is $t$-equicontinuous subset.

Space $(X, t)$ is a (dF) Brauner space if it is $p$-reflexive with a fundamental sequence of compact subsets.

A barrel $T$ of the space $(X, t)$ is $b$-barrel if its intersection with $t$-bounded absolutely convex subset is relative $t$-neighborhood of zero. Locally convex space $(X, t)$ is b-barrelled if each b-barrel in it is a $t$-neighborhood of zero.

Then, it is said that the space $(X, t)$ of the type $D_b$ if it is b-barrelled with fundamental sequence of bounded subsets.

Classes of spaces (DF) and (dF) are incomparable. The space of (DF) type (resp. (dF)) is dual-metric (resp. $\mathrm{D}_b$), whereas classes of $\mathrm{D}_b$ and dual-metric are incomparable. Class of spaces (df) comprises all the others. For each five classes of spaces different inheritance properties were studied (subspace, quotient, natural topology, three-space problem, . . .).

2000 Mathematics Subject Classification: 46A03, 46A04
Keywords and Phrases: DF, $\mathrm{D}_b$, dF-spaces, quasinormable, locally convex spaces.
Suported by Ministry of Science, Technolology and Development of Serbia.

Knowing whether some locally convex space is quasinormable in GROTHENDIECK sense or not is of a special interest for that space. In the paper [**2**] where he defined spaces of (DF) type and quasinormable spaces, GROTHENDIECK proved that some spaces of (DF) type are quasinormable. It was not until later (1973) that KATS [**3**] proved that every space of (DF) type is quasinormable. The same author used a fine counterexample [**4**] to show that dual-metric space need not be quasinormable.

In this work, we shall prove, inspired by KATS's result, that each space of $D_b$ type is quasinormable.

Let us first remined ourselves of the definition of quasinormability in GROTHENDIECK sense.

**Definition.** *Locally convex space $(X, t)$ is quasinormable if for each $t$-equicontinuous subset $E$ of $X'$ there exists $t$-neighborhood of zero $U$ in $X$, such that $E \subset U^0$ and that on $E$ strong topology $X'_b$ and topology of Banach space $X'_{U^0}$ are equal, that is, that $X'_b|\,E = X'_{U^0}|\,E$.*

In the following proof of the theorem, we are using the result of NOUREDDINE:

**Lemma.** *Let $(X, t)$ be a $b$-barrelled space. For each sequence $H_n$ of $t$-equicontinuous subsets of $X'$ such that $\bigcup_{n=1}^{+\infty} H_n$ is strongly bounded subset and for each zero sequence of scalar $\varepsilon_n$, $\bigcup_{n=1}^{+\infty} \varepsilon_n H_n$ is $t$-equicontinuous subset of $X'$.*

**Theorem.** *Each locally convex space of $D_b$ type is quasinormable.*

**Proof.** Let $E$ arbitrary $t$-equicontinuous (absolutely convex $\sigma(X', X)$-closed can be taken) subset of $X'$ and let

$$U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$$

be fundamental system of absolutely convex $\sigma(X', X)$-closed $X'_b$-neighborhoods of zero. Subsets $n^2E \cap U_{n^2}$, $n = 1, 2, 3, \ldots$ are $t$-equicontinuous and

$$F = \bigcup_{n=1}^{+\infty} \left(n^2E \cap U_{n^2}\right)$$

is $X'_b$-bounded. Indeed, subset $\bigcup_{n=N}^{+\infty} \left(n^2E \cap U_{n^2}\right)$, for some fixed $N$ is contained in $U_{N^2}$ :

$$\left(N^2E \cap U_{N^2}\right) \cup \left(\left(N+1\right)^2 E \cap U_{(N+1)^2}\right) \cup \cdots \subset U_{N^2} \cup U_{(N+1)^2} \cup \cdots \subset U_{N^2}.$$

Set $\bigcup_{n=1}^{N-1} \left(n^2E \cap U_{n^2}\right)$ is $X'_b$-bounded as finite union of bounded sets. Therefore, $U_{N^2}$ absorbs $F = \bigcup_{n=1}^{+\infty} \left(n^2E \cap U_{n^2}\right)$. Since $N$ is fixed natural number, each $U_{n^2}$ absorbs $F = \bigcup_{n=1}^{+\infty} \left(n^2E \cap U_{n^2}\right)$. This means that $F = \bigcup_{n=1}^{+\infty} \left(n^2E \cap U_{n^2}\right)$ is strongly bounded, that is, $X'_b$-bounded.

According to Lemma for $\varepsilon_n = \dfrac{1}{n}$ subset

$$B = \bigcup_{n=1}^{+\infty} \frac{1}{n}\left(n^2 E \cap U_{n^2}\right) = \bigcup_{n=1}^{+\infty}\Big(nE \cap \frac{1}{n}\,U_{n^2}\Big)$$

is $t$-equicontinuous. Therefore, there exists a $t$-neighborhood of zero $U$ such that

$$\bigcup_{n=1}^{+\infty}\Big(nE \cap \frac{1}{n}\,U_{n^2}\Big) \subset U^0.$$

Let us be reminded that the basis of neighborhoods of zero in an absolutely convex subset $E$ in the relative topology from $X'_{U^0}$ comprise sets of the shape $\varepsilon U^0 \cap E, \varepsilon > 0$. So, if $V$ is one neighborhood of zero from $X'_{U^0}|\,E$, then there exists $\varepsilon > 0$ so that $V \supset \varepsilon U^0 \cap E$. Let us now take a natural number $N$, such that $\dfrac{1}{N} < \varepsilon$. From this it follows

$$\varepsilon U^0 \supset \frac{1}{N}\,U^0 \supset \frac{1}{N}\bigcup_{n=1}^{+\infty}\Big(nE \cap \frac{1}{n}\,U_{n^2}\Big) \supset \frac{1}{N}\Big(NE \cap \frac{1}{N}\,U_{n^2}\Big) = E \cap \frac{1}{N^2}U_{N^2}.$$

Now, $\varepsilon U^0 \cap E \supset E \cap \dfrac{1}{N^2}\,U_{N^2}$, that is, $V \supset E \cap \dfrac{1}{N^2}\,U_{N^2}$, i.e. $X'_{U^0}|\,E \le X'_b|\,E$. Since the relation $X'_b|\,E \le X'_{U^0}|\,E$ is apparent, it means that

$$X'_b|\,E = X'_{U^0}|\,E,$$

that is, space $(X, t)$ is quasinormable. □

Based on Kats's results, previous theorem and relations of the mentioned five classes of the spaces, it follows that spaces (DF), (dF) and $D_b$ are quasinormable, and dual-metric and (df) are not.

**Acknowledgement.** We tank the referee for valuable suggestions.

## REFERENCES

1. K. Brauner: *Duals of Fréchet spaces and generalization of the Banach-Dieudonné theorem.* Duke Math. J., **40** (1973), 845–856.
2. A. Grothendieck: *Sur les espaces (F) et (DF).* Summa Brasil. Math., **3** (1954), 57–123.
3. M. Kats: *Each (DF)-space is quasinormable.* Funkcionalna analiza sa primenama, t. **7**, sveska 2, 1973, 85–86.
4. M. P. Kats: *An example of a dual-metric space which is not a (DF)-space.* Vestnik Moskovskogo Universiteta, **4**-1975, 36–38.
5. J. Mazón: *Some classes of locally convex spaces.* Arch. Math., **38** (1982), 131-137.

6. K. Noureddine: *Nouvelles classes d'espaces localment convexes.* Publ. Dep. Math. Lyon, **10**, 3 (1973), 105–123.
7. A. Pietsch: *Nuclear Locally Convex Spaces.* Berlin-Heidelberg-New York: Springer 1972.
8. W. Ruess: *On the locally convex structure of strict topologies.* Math. Z., **153** (1977), 179-192.

Faculty of Mechanical Engineering,
Kraljice Marije 16, 11000 Beograd,
Serbia
E-mails: radens@beotel.yu
sradenovic@mas.bg.ac.yu

(Received September 27, 2005)