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ON INTEGRAL GRAPHS WHICH BELONG TO THE CLASS $\overline{\alpha K_{a,a,\dots,a,b,b,\dots,b}}$

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Let G be a simple graph and let $\overline{G}$ denote its complement. We say that G is integral if its spectrum consists of integral values. Let $K_{xa,yb} = K_{a,a,\dots,a,b,b\dots,b}$ be the complete m-partite graph with $xa + yb$ vertices, where x and y are positive integers and $m = x + y$. In this work we consider integral graphs which belong to the class $\overline{\alpha K_{xa,yb}}$ for any $\alpha > 1$ and $a > b$, where mG denotes the m-fold union of the graph G.

Let G be a simple graph of order n and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be its eigenvalues with respect to its (0,1) adjacency matrix A. The spectrum of G is the set of its eigenvalues and is denoted by $\sigma(G)$. We say that G is integral if its spectrum $\sigma(G)$ consists only of integers [**1**].

An eigenvalue μ of G is main if and only if $\langle \mathbf{j}, \mathbf{P}\mathbf{j}\rangle = n\cos^2\alpha > 0$, where $\mathbf{j}$ is the main vector (with coordinates equal to 1) and $\mathbf{P}$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_A(\mu)$. The quantity $\beta = |\cos\alpha|$ is called the main angle of μ. The main spectrum of G is the set of all its main eigenvalues and is denoted by $\mathcal{M}(G)$.

Let G be a graph with exactly two main eigenvalues μ_1 and μ_2 with $\mu_1 > \mu_2$ and let β_1 and β_2 be the main angles of μ_1 and μ_2, respectively. Then according to [**3**] we have

$$\overline{\mu}_{1,2} = \frac{n-2-\mu_1-\mu_2}{2} \pm \frac{\sqrt{(\mu_1-\mu_2+n)^2 - 4n_1(\mu_1-\mu_2)}}{2}\,, \tag{1}$$

where $\overline{\mu}_1$ and $\overline{\mu}_2$ are the main eigenvalues of its complementary graph $\overline{G}$. Besides, we have [**3**]

$$\overline{n}_{1,2} = \frac{n}{2} \pm \frac{n^2 + (n-2n_1)(\mu_1-\mu_2)}{2\sqrt{(\mu_1-\mu_2+n)^2 - 4n_1(\mu_1-\mu_2)}}\,. \tag{2}$$

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Here, $n_i = n\beta_i^2$ and $\overline{n}_i = n\overline{\beta}_i^2$ $(i = 1, 2)$, $\overline{\beta}_1$ and $\overline{\beta}_2$ denote the main angles of $\overline{\mu}_1$ and $\overline{\mu}_2$, respectively.

Further, let K_n and $K_{m,n}$ denote the complete graph and the complete bipartite graph, respectively. Let $K_{xa,yb} = K_{a,a,\dots,a,b,b\dots,b}$ be the complete m-partite graph with $xa + yb$ vertices, where x and y are positive integers and $m = x + y$. We note that $xK_a \cup yK_b$ with $a > b$ is an integral graph with two main eigenvalues $\mu_1 = (a-1)$ and $\mu_2 = (b-1)$, where mG denotes the m-fold union of the graph G. Applying (1) and (2) to its complement $\overline{xK_a \cup yK_b} = K_{xa,yb}$, keeping in mind that $n_1 = xa$ and $n_2 = yb$, we obtain that $\overline{\mu}_{1,2} = \dfrac{xa + yb - a - b \pm \Delta}{2}$ and $\overline{n}_{1,2} = \dfrac{xa + yb}{2} \pm \dfrac{(xa + yb)^2 - (xa - yb)(a - b)}{2\Delta}$, where

$$\Delta^2 = \big((x+1)a + (y-1)b\big)^2 - 4xa(a-b)\,.$$

Thus, for $\alpha K_{xa,yb}$ we have $\mu_{1,2} = \dfrac{xa + yb - a - b \pm \Delta}{2}$ and

$$n_{1,2} = \frac{(xa+yb)\alpha}{2} \pm \frac{\alpha(xa+yb)^2 - \alpha(xa-yb)(a-b)}{2\Delta}.$$

We note that $\alpha K_{xa,yb}$ is integral if and only if $K_{xa,yb} = \overline{xK_a \cup yK_b}$ is integral. Due to relations (1) and (2) we have recently described all integral graph which belong to the classes $\overline{\alpha K_a \cup \beta K_b}$, $\overline{\alpha K_a \cup \beta K_{b,b}}$ and $\overline{\alpha K_{a,a} \cup \beta K_{b,b}}$ (see [**5**]–[**7**], respectively).

We now proceed to establish a characterization of μ-integral graphs which belong to the class $\overline{\alpha K_{xa,yb}}$. We say that a graph G is μ-integral if its main spectrum $\mathcal{M}(G)$ consists only of integral values. In view of this note that $\overline{\alpha K_{xa,yb}}$ is an integral graph if and only if it is μ-integral and its complement $\alpha K_{xa,yb}$ is integral. We also note that $\overline{\alpha K_{xa,yb}}$ is μ-integral if and only if its largest eigenvalue $\overline{\mu}_1 \in \mathbb{N}$. Then according to (1) we get

$$\overline{\mu}_{1,2} = \frac{(xa+yb)\alpha - (x-1)a - (y-1)b - 2 \pm \delta}{2}\,, \tag{3}$$

where $\delta = \sqrt{\big((\alpha-1)(xa+yb) - (a-b)\big)^2 + 4xa(\alpha-1)(a-b)}$. It is clear that $\overline{\alpha K_{xa,yb}}$ is μ-integral if and only if $(\alpha, x, y, a, b, \delta)$ represents a positive integral solution of the Diophantine equation

$$\Big[(\alpha-1)\big(xa+yb\big) - \big(a-b\big)\Big]^2 + 4xa\big(\alpha-1\big)\big(a-b\big) = \delta^2\,. \tag{4}$$

Therefore, the characterization of μ-integral graphs which are related to the class $\overline{\alpha K_{xa,xb}}$ is reduced to the problem of finding the most general integral solution of the equation (4). The general solution of (4) is based on the procedure which is applied in [**4**] for describing μ-integral graphs which belong to the class $\overline{\alpha K_{a,b}}$.

In this work it will be excluded two special cases of the Diophantine equation (4). First, setting $\alpha = 1$ in relation (4) we obtain $\delta^2 = (a-b)^2$, which provides that

$\overline{K_{xa,yb}} = xK_a \cup yK_b$ is integral for any $a, b, x, y \in \mathbb{N}$. Besides, for $a = b$ according to (4) we get $\delta = (\alpha - 1)(x + y)a$, which also implies that $\overline{\alpha K_{xa,ya}}$ is integral for any $\alpha, a, x, y \in \mathbb{N}$. Since these two cases are well-known in the Spectral theory of graphs, in what follows it will be assumed that $\alpha > 1$ and $a > b$.

Next, $\overline{\mu}_1\, \overline{\mu}_2 = \mu_1\, \mu_2 - (n_2 - 1)\, \mu_1 - (n_1 - 1)\, \mu_2 - (n - 1)$ for any G with two main eigenvalues [**3**]. If $G = \alpha K_{xa,yb}$ this relation is transformed into

$$(5) \qquad \big(\overline{\mu}_1 + 1\big)\big(\overline{\mu}_2 + 1\big) = ab\Big[\big(\alpha - 1\big)\big(x + y\big) + 1\Big] .$$

Remark 1. With condition $a > b$ the parameters α, x, y, a, b determine the graph $\alpha K_{xa,yb}$ up to isomorphism, which provides that α, x, y, a, b also uniquely determine the graph $\overline{\alpha K_{xa,yb}}$.

In what follows (m, n) denotes the greatest common divisor of integers $m, n \in \mathbb{N}$ while $m \mid n$ means that m divides n.

Proposition 1. *The linear Diophantine equation $ax + by = c$ has at least one solution if and only if $d \mid c$ where $d = (a, b)$. In that case the most general solution of this equation is given in the form*

$$x = \frac{c}{d}\, x_0 - \frac{b}{d}\, z \quad \textit{and} \quad y = \frac{c}{d}\, y_0 + \frac{a}{d}\, z \qquad (z \in \mathbb{Z})\,,$$

where (x_0, y_0) represents a particular solution[1] *of the equation $ax + by = d$.*

In order to demonstrate a method applied in this paper, we first prove the following result:

Theorem 1. *If $\overline{\alpha K_{xa,yb}}$ is integral with $\overline{\mu}_1 = (ab - 1)$ then it belongs to the class of μ-integral graphs*

$$(6) \qquad \overline{(\ell m + 1)K_{[\frac{kn}{\tau}\, x_0 - \frac{\ell n}{\tau}\, z]a\,,[\frac{kn}{\tau}\, y_0 + \frac{km}{\tau}\, z]\, b}}\,,$$

where (i) *$a = km + 1$ and $b = \ell n + 1$ such that $(m, n) = 1$ and $km > \ell n$;* (ii) *$\tau = (km, \ell n)$ such that $\tau \mid nk$;* (iii) *(x_0, y_0) is a particular solution of the linear Diophantine equation $(km)x + (\ell n)y = \tau$ and* (iv) *z is any integer such that*

$$\Big(\frac{kn}{\tau}\, x_0 - \frac{\ell n}{\tau}\, z\Big) \geq 1 \quad \textit{and} \quad \Big(\frac{kn}{\tau}\, y_0 + \frac{km}{\tau}\, z\Big) \geq 1.$$

Proof. If $(\overline{\mu}_1 + 1) = ab$ using (3) and (5) we easily get (i) $(\overline{\mu}_2 + 1) = (\alpha - 1)(x + y) + 1$ and (ii) $\delta = ab - (\alpha - 1)(x + y) - 1$. Using (i) and (ii) it is not difficult to see that (4) is transformed to

$$(7) \qquad \big(a - 1\big)\big(b - 1\big) = \big(\alpha - 1\big)\Big[\big(a - 1\big)x + \big(b - 1\big)y\Big] .$$

[1] A particular solution of the equation $ax + by = d$ may be obtained by using the EUCLID algorithm. In that case the coefficients a and b uniquely determine x_0 and y_0.

Setting $(\alpha-1, b-1)=\ell$ we have $\alpha-1=\ell m$ and $b-1=\ell n$ such that $(m,n)=1$. In view of this it follows that $m \mid a-1$. Setting $a-1=km$ relation (7) is reduced to the linear Diophantine equation $(km)x+(\ell n)y=kn$. This equation has at least one solution if and only if $(km,\ell n)=\tau \mid kn$. In that case, according to Proposition 1, we get $x=\frac{kn}{\tau}x_0-\frac{\ell n}{\tau}z$ and $y=\frac{kn}{\tau}y_0+\frac{km}{\tau}z$, where $(km)x_0+(\ell n)y_0=\tau$. □

In what follows we show that there exists an one-to-one correspondence between the μ-integral graphs $\overline{\alpha K_{xa,yb}}$ with $\overline{\mu}_1=(ab-1)$ and the parameters k,ℓ,m,n.

Proposition 2. *If $\overline{\alpha K_{xa,yb}}$ is μ-integral with $\overline{\mu}_1=(ab-1)$ then it uniquely determines the parameters k,ℓ,m,n.*

Proof. Suppose that k_1,ℓ_1,m_1,n_1 and k_2,ℓ_2,m_2,n_2 determine the same μ-integral graph $\overline{\alpha K_{a,b}}$ with the largest eigenvalue $\overline{\mu}_1=(ab-1)$. Then according to Remark 1 and using that $(\alpha-1,b-1)=\ell$ we get $\ell_1=\ell_2$. Since $\alpha-1=\ell m$ and $b-1=\ell n$ we obtain $m_1=m_2$ and $n_1=n_2$. Since $a-1=km$ we obtain $k_1=k_2$. □

Remark 2. If (x_0,y_0) is obtained by using the EUCLID algorithm then a fixed μ-integral graph $\overline{\alpha K_{xa,yb}}$ with the largest eigenvalue $\overline{\mu}_1=(ab-1)$ also uniquely determines the parameters x_0,y_0,z.

Theorem 2. *If $\overline{\alpha K_{xa,yb}}$ is integral then it belongs to the class of μ-integral graphs*

$$\overline{\big(kmn+1\big)K_{[\,(rst)\,x_0-(mqt)z]a\,,\,[\,(rst)y_0+(nps)z]b}}\,, \tag{8}$$

where (i) $a=\Big(\frac{knprs+pq}{\tau}\Big)z^+$ *and* $b=\Big(\frac{kmqrt+pq}{\tau}\Big)z^+$ *such that* $(knprs+pq, kmqrt+pq)=\tau$ *and* $(\tau,pq)=1$, $(k,pqrst)=1$, $(mqt,nps)=1$ *and* $nps>mqt$, $(r,pq)=1$ *and* $z^+\in\mathbb{N}$; (ii) (x_0,y_0) *is a particular solution of the linear Diophantine equation* $(nps)x+(mqt)y=1$ *and* (iii) z *is any integer such that* $(rst)x_0-(mqt)z\ge 1$ *and* $(rst)y_0+(nps)z\ge 1$.

Proof. We note first that if $\overline{\alpha K_{xa,yb}}$ is integral then according to (3) and (4) it turns out that $\overline{\alpha K_{x(az^+)\,,\,y(bz^+)}}$ is integral for any $z^+\in\mathbb{N}$. Consequently, without loss of generality we can assume that $(a,b)=1$.

Setting $(\overline{\mu}_1+1)=\theta ab$ where $\theta=\frac{\tau}{\beta}$ such that $(\tau,\beta)=1$, by using (3) and (5) we obtain

$$\overline{\mu}_2+1=\frac{(\alpha-1)(x+y)+1}{\theta} \quad \text{and} \quad \delta=\theta ab-\frac{(\alpha-1)(x+y)+1}{\theta}\,. \tag{9}$$

Then by a straightforward calculation it is not difficult to see that equation (4) is reduced to the form $(\theta a-1)(\theta b-1)=(\alpha-1)[(\theta a-1)x+(\theta b-1)y]$. We now arrive at

$$\big(\tau a-\beta\big)\big(\tau b-\beta\big)=\big(\alpha-1\big)\beta\Big[\big(\tau a-\beta\big)x+\big(\tau b-\beta\big)y\Big]\,. \tag{10}$$

Let $(\tau a-\beta,\tau b-\beta)=\gamma$. Then $\tau a=\gamma\rho+\beta$ and $\tau b=\gamma\varphi+\beta$ where $(\rho,\varphi)=1$. In view of this and according to (10), we easily get $\gamma\rho\varphi=(\alpha-1)\beta(\rho x+\varphi y)$. We note that $(\gamma\rho+\beta, \gamma\varphi+\beta)=\tau$ because $(a,b)=1$. Besides, since $(\tau,\beta)=1$ and $(a,b)=1$ we have $(\beta,\gamma)=1$. Consequently, it turns out that $\beta \mid \rho\varphi$. Let $(\beta,\rho)=p$ and let $\beta=pq$ and $\rho=p\pi$. Then $(q,\pi)=1$, $(p,\gamma)=1$ and $(q,\gamma)=1$. Thus, it must be $q \mid \varphi$. Setting $\varphi=q\omega$ we get $(p,q)=1$, $(p,\omega)=1$ and $(\pi,\omega)=1$. So we obtain that

$$\gamma\pi\omega=(\alpha-1)\big[(p\pi)x+(q\omega)y\big]\,. \tag{11}$$

Further, if we set $(\alpha-1,\omega)=m$ then $\alpha-1=m\nu$ and $\omega=mt$ so that $(t,\nu)=1$. Setting $(\nu,\pi)=n$ we get $\nu=kn$ and $\pi=ns$ so that $(k,s)=1$. In view of this it follows that $k \mid \gamma$. Setting $\gamma=kr$ we arrive at $\alpha=kmn+1$, $a=\dfrac{knprs+pq}{\tau}$ and $b=\dfrac{kmqrt+pq}{\tau}$. Besides, we note that (11) is transformed in the following linear Diophantine equation $(nps)x+(mqt)y=rst$. Since $(nps,mqt)=1$ this equation has at least one solution. The general solution of this equation is $x=(rst)x_0-(mqt)z$ and $y=(rst)y_0+(nps)z$, where $(nps)x_0+(mqt)y_0=1$. □

Proposition 3. *If $\overline{\alpha K_{xa,yb}}$ is a μ-integral graph then it uniquely determines the parameters k, m, n, p, q, r, s, t,τ and z^+.*

Proof. Assume that k_1, m_1, n_1, p_1, q_1, r_1, s_1, t_1, τ_1, z_1^+ and k_2, m_2, n_2, p_2, q_2, r_2, s_2, t_2, τ_2, z_2^+ determine the same μ-integral graph $\overline{\alpha K_{xa,yb}}$.

Since $\left(\dfrac{knprs+pq}{\tau}, \dfrac{kmqrt+pq}{\tau}\right)=1$ it follows that $(a,b)=z^+$. From this and according to Remark 1 we have $z_1^+=z_2^+$. Therefore, without loss of generality we may suppose $(a,b)=1$. Since $(\overline{\mu}_1+1)=\theta ab$ and $(\tau,\beta)=1$ we obtain $\tau_1=\tau_2$ and $\beta_1=\beta_2$, that is $p_1q_1=p_2q_2$. Keeping in mind that $(\tau a-\alpha\beta, \tau b-\alpha\beta)=\gamma$ we get $\gamma_1=\gamma_2$. So from $\tau a=\gamma\rho+\beta$ and $\tau b=\gamma\varphi+\beta$ we get $\rho_1=\rho_2$ and $\varphi_1=\varphi_2$. Since $(\beta,\rho)=p$ it turns out that $p_1=p_2$ and $q_1=q_2$. Further, since $\rho=p\pi$ we get $\pi_1=\pi_2$ and since $\varphi=q\omega$ we get $\omega_1=\omega_2$. Next, since $(\alpha-1,\omega)=m$ and $\alpha-1=\nu m$ and $\omega=mt$ we easily find that $m_1=m_2$, $\nu_1=\nu_2$ and $t_1=t_2$. Since $(\nu,\pi)=n$, $\nu=kn$ and $\pi=ns$ we get $n_1=n_2$, $k_1=k_2$ and $s_1=s_2$. Finely, since $\gamma=kr$ we obtain that $r_1=r_2$. □

Remark 3. If (x_0,y_0) is obtained by using the EUCLID algorithm then a fixed μ-integral graph $\overline{\alpha K_{xa,yb}}$ also uniquely determines the parameters x_0, y_0, z.

Table 1 contains the set of all μ-integral graphs from the class $\overline{\alpha K_{xa,yb}}$, whose order '$o$' does not exceed 50. In this table a μ-integral graph is described [2] by the parameters α, x, a, y, b and ones presented in the class of integral graphs in Theorem 2. In Table 1 identification numbers 6, 7, 14, 25, 51, 54 and 56 are related to the integral graphs whose complementary graphs are also integral. Identification numbers 3, 18, 22, 35, 40 and 50 are related to the μ-integral graphs with the largest eigenvalue $\overline{\mu}_1=(ab-1)$. Graphs whose order does not exceed 50 with the largest eigenvalue $\overline{\mu}_1<(ab-1)$ have the identification numbers 10, 19, 27, 36, 43, 44, 45, 52 and 55.

[2] In this work the data given in Table 1 are obtained in two different ways: (i) they are generated by using relation (8) and (ii) by varying the parameters α, x, a, y, b in all possible ways in equation (4).

i	x_0	y_0	z	o	α	x	a	y	b	k	m	n	p	q	r	s	t	τ	z^+	μ_1	μ_2
1	0	1	−1	14	2	1	3	4	1	1	1	1	1	1	2	4	1	3	1	8	1
2	0	1	−1	14	2	1	4	3	1	1	1	1	2	1	3	3	1	5	1	9	1
3	0	1	−1	22	2	1	5	2	3	1	1	1	1	1	2	2	1	1	1	14	3
4	0	1	−1	22	2	1	8	1	3	1	1	1	4	1	5	1	1	3	1	17	3
5	0	1	−1	24	2	1	5	7	1	1	1	1	1	1	2	7	1	3	1	14	2
6	0	1	−1	24	2	1	6	6	1	1	1	1	3	1	5	3	1	8	1	15	2
7	0	1	−1	24	3	1	6	2	1	1	1	2	2	1	5	2	1	7	1	20	1
8	1	−3	13	26	2	2	3	7	1	1	1	1	1	1	2	7	2	5	1	14	1
9	0	1	−2	26	2	2	4	5	1	1	1	1	1	1	3	5	1	4	1	15	1
10	0	1	−1	26	2	1	8	1	5	1	1	1	2	1	3	1	1	1	1	19	5
11	0	1	−1	28	2	1	4	10	1	1	1	1	1	1	3	5	1	4	1	15	2
12	0	1	−1	28	2	1	6	4	2	1	1	1	1	1	2	4	1	3	2	17	3
13	0	1	−1	28	2	1	8	3	2	1	1	1	2	1	3	3	1	5	2	19	3
14	0	1	−1	28	2	1	9	5	1	1	1	1	3	1	4	5	1	7	1	20	2
15	0	1	−1	28	4	1	5	2	1	1	1	3	1	1	4	2	1	5	1	24	1
16	0	1	−2	32	2	2	5	3	2	1	1	1	1	1	3	3	1	2	1	19	2
17	0	1	−2	32	2	2	6	4	1	1	1	1	2	1	5	4	1	7	1	20	1
18	0	1	−1	32	2	1	7	3	3	1	1	1	1	1	2	3	1	1	1	20	4
19	0	1	−1	32	2	1	10	2	3	1	1	1	5	1	7	1	1	4	1	23	4
20	0	1	−1	33	3	1	5	6	1	1	1	2	1	1	4	3	1	5	1	24	2
21	1	−7	12	33	3	1	6	5	1	1	2	1	3	1	5	5	1	13	1	25	2
22	0	1	−1	33	3	1	7	1	4	1	1	2	1	1	3	1	1	1	1	27	4
23	0	1	−1	33	3	1	9	1	2	1	1	2	3	1	7	1	1	5	1	29	2
24	0	1	−1	34	2	1	5	6	2	1	1	1	1	1	3	3	1	2	1	19	3
25	0	1	−1	34	2	1	7	10	1	1	1	1	1	1	2	10	1	3	1	20	3
26	0	1	−1	34	2	1	8	9	1	1	1	1	4	1	7	3	1	11	1	21	3
27	0	1	−1	34	2	1	9	2	4	1	1	1	3	1	5	1	1	2	1	23	5
28	0	1	−1	36	2	1	6	12	1	1	1	1	2	1	5	4	1	7	1	20	3
29	0	1	−1	36	2	1	10	8	1	1	1	1	2	1	3	8	1	5	1	24	3
30	1	−3	19	38	2	3	3	10	1	1	1	1	1	1	2	10	3	7	1	20	1
31	2	−9	41	38	2	3	4	7	1	1	1	1	2	1	3	7	3	11	1	21	1
32	1	−4	13	39	3	1	4	9	1	1	2	1	1	1	3	9	1	7	1	27	2
33	0	1	−1	39	3	1	9	4	1	1	1	2	3	1	8	2	1	11	1	32	2
34	0	1	−3	42	2	3	5	6	1	1	1	1	1	1	4	6	1	5	1	24	1
35	0	1	−1	42	2	1	9	4	3	1	1	1	1	1	2	4	1	3	3	26	5
36	0	1	−1	42	2	1	12	3	3	1	1	1	2	1	3	3	1	5	3	29	5
37	0	1	−2	44	2	2	5	12	1	1	1	1	1	1	4	6	1	5	1	24	2
38	1	−7	24	44	2	2	6	10	1	1	1	1	3	1	5	5	2	13	1	25	2
39	0	1	−1	44	2	1	9	13	1	1	1	1	1	1	2	13	1	3	1	26	4
40	0	1	−2	44	2	2	7	2	4	1	1	1	1	1	3	2	1	1	1	27	4
41	0	1	−1	44	2	1	10	12	1	1	1	1	5	1	9	3	1	14	1	27	4
42	0	1	−2	44	2	2	9	2	2	1	1	1	3	1	7	2	1	5	1	29	2

Table 1. (continued)

i	x_0	y_0	z	o	α	x	a	y	b	k	m	n	p	q	r	s	t	τ	z^+	μ_1	μ_2
43	0	1	−1	44	2	1	10	2	6	1	1	1	1	1	2	2	1	1	2	29	7
44	0	1	−1	44	2	1	15	1	7	1	1	1	3	1	4	1	1	1	1	34	8
45	0	1	−1	44	2	1	16	1	6	1	1	1	4	1	5	1	1	3	2	35	7
46	0	1	−1	45	5	1	7	1	2	1	1	4	1	1	5	1	1	3	1	41	2
47	0	1	−1	46	2	1	5	18	1	1	1	1	1	1	4	6	1	5	1	24	3
48	0	1	−2	46	2	2	7	9	1	1	1	1	1	1	3	9	1	4	1	27	2
49	1	−3	8	46	2	1	8	5	3	1	1	1	2	3	5	5	1	7	1	27	5
50	0	1	−1	46	2	1	7	4	4	1	1	1	1	1	3	2	1	1	1	27	5
51	0	1	−1	46	2	1	16	7	1	1	1	1	4	1	5	7	1	9	1	35	3
52	1	−2	3	46	2	1	15	1	8	1	1	1	5	2	7	1	1	3	1	35	9
53	0	1	−1	48	2	1	10	7	2	1	1	1	1	1	2	7	1	3	2	29	5
54	0	1	−1	48	2	1	12	6	2	1	1	1	3	1	5	3	1	8	2	31	5
55	0	1	−1	48	2	1	14	2	5	1	1	1	2	1	3	2	1	1	1	34	7
56	0	1	−1	48	3	1	12	2	2	1	1	2	2	1	5	2	1	7	2	41	3
57	1	−3	25	50	2	4	3	13	1	1	1	1	1	1	2	13	4	9	1	26	1
58	1	−4	25	50	2	4	4	9	1	1	1	1	1	1	3	9	2	7	1	27	1
59	0	1	−1	50	2	1	7	18	1	1	1	1	1	1	3	9	1	4	1	27	4
60	0	1	−1	50	2	1	9	8	2	1	1	1	3	1	7	2	1	5	1	29	5
61	0	1	−1	50	2	1	15	10	1	1	1	1	5	1	7	5	1	12	1	35	4

Table 1.

Theorem 3. *The most general positive integral solution of the Diophantine equation* (4) *is in the form*:

- $\alpha = kmn + 1$;
- $a = \left[\dfrac{knprs + pq}{\tau}\right] z^+$ and $b = \left[\dfrac{kmqrt + pq}{\tau}\right] z^+$;
- $x = \big(rst\big)x_0 - \big(mqt\big)z$ and $y = \big(rst\big)y_0 + \big(nps\big)z$;
- $\delta = 2\left[\dfrac{\big(knrs + q\big)\big(kmrt + p\big)}{\tau}\right] z^+ - \big(a + b\big) - kmn(ax + by)$,

with the same conditions (i), (ii) *and* (iii) *as given in Theorem* (2).

Proof. According to Theorem 2 it suffices to derive the last relation of the Theorem 3. We note first if $(\alpha, x, y, a, b, \delta)$ is a solution of the equation (4) then $(\alpha, x, y, az^+, bz^+, \delta z^+)$ also represents a solution of (4) for any $z^+ \in \mathbb{N}$. Consequently, without loss of generality we may assume that $(a, b) = 1$.

Using (3) we have $\overline{\mu}_1 + \overline{\mu}_2 = (xa + yb) - (x-1)a - (y-1)b - 2$. Since $(\overline{\mu}_1 + 1) = \theta ab$ we get $\overline{\mu}_1 = \dfrac{(knrs + q)(kmrt + p)}{\tau} - 1$, which provides the proof using that $\delta = \overline{\mu}_1 - \overline{\mu}_2$.

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