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HAMILTONIANICITY OF THE TOWERS OF HANOI PROBLEM

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In this paper we analyze a variant of the n-disk Towers of Hanoi problem with an arbitrary starting and ending configuration using transition graphs representing valid configurations. In particular, we show that starting with any configuration, there is a sequence of moves that goes through each valid configuration exactly once and back to the starting configuration. Also, we show how the original Towers of Hanoi problem can be solved in any number of moves between $2^n - 1$ and $3^n - 1$ inclusive.

1. INTRODUCTION

The traditional Towers of Hanoi problem is a simple puzzle in which initially n disks of mutually different sizes are placed on one of three poles in decreasing order from the bottom. The disks are all to be moved one at a time to another pole using the third one as a spare. In addition, only the top disk of one pole may be moved to the top of other pole, and no disk may be placed on top of a smaller one.

The puzzle has become a classic paradigm for recursive programming, divide-and-conquer problem solving paradigm, and complexity analysis of algorithms. It is also related to the Gray code in an interesting way, since it is possible to use the puzzle's solution to generate Gray codes (Gardner [**8**]).

Because of its great popularity, the puzzle has been extensively studied in the literature with respect to the required number of moves, optimal solution, and similar considerations. Also, numerous variations to the original game have been proposed, such as Towers of Hanoi with four or more poles (Wood [**12**]), cyclic Towers of Hanoi (Atkinson [**1**]), color Towers of Hanoi (Er [**6**]), Towers of Hanoi with arbitrary starting configuration (Scarioni and Sperenza [**10**]), Towers of Hanoi with arbitrary starting and ending configuration (Staples [**11**]), and so on.

In this paper we analyze the last mentioned variant of the puzzle with the aid of transition graphs representing valid configurations. In particular, we show that

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starting with any configuration, there is a sequence of moves that goes through each valid configuration exactly once and back to the starting configuration. Also, we show how the original Towers of Hanoi problem can be solved in any number of moves between $2^n - 1$ and $3^n - 1$ inclusive.

2. THE PROBLEM AND ITS TRANSITION GRAPH

The original Towers of Hanoi problem consists of three poles A, B, C and n disks $d_1, d_2, \cdots, d_n$ with different sizes $d_1 < d_2 < \cdots < d_n$. Initially, all the disks are stacked on the pole A in decreasing order, i.e., the largest disk d_n is at the bottom and the smallest disk d_1 is on the top. The goal of the game is to move all the disks to the pole C subject to the following two rules:

1. only the top disk may be removed from one pole and placed onto the top of another pole, and

2. no larger disk is above a smaller one at any time.

Note that these rules imply that only one disk can be moved at a time, and all disks at all times must rest on the poles.

A *configuration* is an arbitrary distribution of the n disks among the three poles with the proviso that no disk is on top of a smaller one. That is, a configuration is any valid arrangement of all disks in the course of the problem solution. Clearly, there are 3^n distinct configurations corresponding to the 3^n strings of the form $p_1 p_2 \ldots p_n$, where each $p_i \in \{A, B, C\}$. The obvious interpretation of such a string is that p_i is the pole on which the disk d_i resides.

Without loss of generality, we further assume that the poles are arranged in the corners of a triangle as shown in Figure 1.

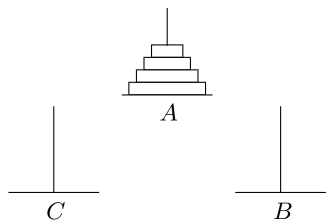

Figure 1: Triangle arrangement of the poles

This way we can define the clockwise and counterclockwise order of the poles, and for a pole $p \in \{A, B, C\}$ we denote by $S(p)$ and $P(p)$ the successor and predecessor pole of the pole p in this order. Moreover, we can distinguish a clockwise and counterclockwise move, meaning that a disk is moved in the direction $A \curvearrowright B \curvearrowright C \curvearrowright A$ and $A \curvearrowleft B \curvearrowleft C \curvearrowleft A$, respectively.

Since a configuration can be changed only by removing the top disk from a pole, it is obvious that the only legal moves are

- a clockwise move of the smallest disk, denoted by α,

- a counterclockwise move of the smallest disk, denoted by β, and

- the only legal move not involving the smallest disk, denoted by γ.

As shown in Boardman et al. [**2**] and Boardman and Robson [**3**], the configurations and transitions among them for the n-disk problem can be represented by the vertices and edges of a graph T_n defined inductively as follows (see Figure 2).

T_1 is a triangle graph on three vertices with labels A, B, and C. For $n \geq 2$, T_n consists of three copies X, Y, Z of T_{n-1} such that

- the vertices for X (and Y and Z) are labeled the same as those for T_{n-1}, plus one additional A (respectively B or C) appended to the right, and
- X and Y are connected with the edge whose endpoints are the vertices $C^{n-1}A$ and $C^{n-1}B$, X and Z are connected with the edge $(B^{n-1}A,\, B^{n-1}C)$, and Y and Z with the edge $(A^{n-1}B,\, A^{n-1}C)$, where p^k stands for the string of k copies of $p \in \{A, B, C\}$.

Clearly, each side of the triangle T_n represents a solution to the original n-disk problem. It is easy to see that the sides have length $2^n - 1$ and that any other path connecting any two corners of the triangle T_n has length greater than $2^n - 1$. Thus, each side corresponds to the unique optimal solution. It is also easy to show that the distance between any two vertices in T_n is at most $3^n - 1$.

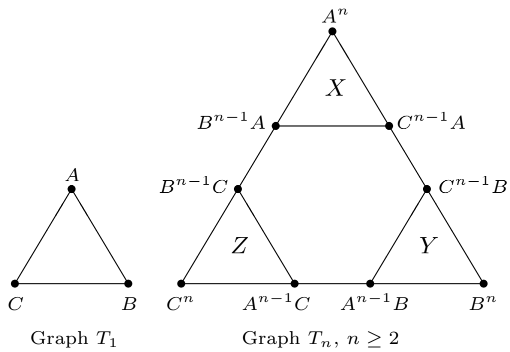

Figure 2: Transition graphs

If we consider the problem with an arbitrary starting and ending configuration, finding its optimal solution in this setting means finding a shortest path between any two vertices in T_n. When they belong to the same subgraph X, Y, or Z, the problem reduces to the one in T_{n-1}. When they belong to different subgraphs, it is tempting to go through the edge that directly connects the two subgraphs involved. However, it is pointed out in Staples [**11**] that this intuitive solution does not always give a shortest path. For example, the shortest path between two vertices $B^{n-1}A$ and $A^{n-1}B$ of T_n in Figure 2 that goes through the subgraphs X and Y, goes through $C^{n-1}A$ and $C^{n-1}B$ and has length $2^n - 1$. On the other hand, the one going through the vertices $B^{n-1}C$ and $A^{n-1}C$ of the subgraph Z has length $2^{n-1} + 1$.

3. THE PROBLEM AND ITS HAMILTONIANICITY

In this section we show that the transition graph T_n is Hamiltonian, i.e., from any configuration of n disks we can perform a sequence of 3^n moves going through every other configuration exactly once and back to the starting configuration. Also, we show that is possible to solve the original Towers of Hanoi problem for n disks in any number of m moves such that $2^n - 1 \le m \le 3^n - 1$ and no configuration is repeated.

To this end, first denote by μ^k concatenation of k copies of the move(s) $\mu \in \{\alpha, \beta, \gamma\}^*$.

Theorem 1. *For any $n \ge 1$ and $p \in \{A, B, C\}$, the sequence of moves*

$$(\beta\beta\gamma\alpha\alpha\gamma)^{(3^{n-1}-1)/2}\beta\beta$$

is a Hamiltonian path in T_n from p^n to $S(p)^n$, and the sequence of moves

$$(\alpha\alpha\gamma\beta\beta\gamma)^{(3^{n-1}-1)/2}\alpha\alpha$$

is a Hamiltonian path in T_n from p^n to $P(p)^n$.

Proof. By induction on n. The base case $n = 1$ is trivial. For $n \ge 2$, observe that by the induction hypothesis, the sequence of moves $(\beta\beta\gamma\alpha\alpha\gamma)^{(3^{n-2}-1)/2}\beta\beta$ takes the starting configuration $p^{n-1}p$ to the ending configuration $S(p)^{n-1}p$, the sequence $(\alpha\alpha\gamma\beta\beta\gamma)^{(3^{n-2}-1)/2}\alpha\alpha$ takes $S(p)^{n-1}P(p)$ to $p^{n-1}P(p)$, and the sequence $(\beta\beta\gamma\alpha\alpha\gamma)^{(3^{n-2}-1)/2}\beta\beta$ takes $p^{n-1}S(p)$ to $S(p)^n$. Moreover, each path representing these sequences in T_n traverses all other vertices in the same subgraph X, Y, or Z exactly once. Hence, a Hamiltonian path of T_n from p^n to $S(p)^n$ can be obtained by combining these sequences of moves with two γ moves that take $S(p)^{n-1}p$ to $S(p)^{n-1}P(p)$ and $p^{n-1}P(p)$ to $p^{n-1}S(p)$. That is, the sequence of moves

$$\left[(\beta\beta\gamma\alpha\alpha\gamma)^{(3^{n-2}-1)/2}\beta\beta\right]\gamma\left[(\alpha\alpha\gamma\beta\beta\gamma)^{(3^{n-2}-1)/2}\alpha\alpha\right]\gamma\left[(\beta\beta\gamma\alpha\alpha\gamma)^{(3^{n-2}-1)/2}\beta\beta\right]$$

is a Hamiltonian path from p^n to $S(p)^n$. This simplifies to the sequence

$$(\beta\beta\gamma\alpha\alpha\gamma)^{(3^{n-1}-1)/2}\beta\beta.$$

The claim for the case p^n to $P(p)^n$ can be easily established by interchanging $P(p)$ with $S(p)$ and β with α in the previous argument. $\square$

Theorem 2. *For any $n \ge 1$, T_n is a Hamiltonian graph.*

Proof. The graph T_1 is obviously Hamiltonian, since α^3 and β^3 are Hamiltonian cycles starting and ending at any of its three vertices.

For $n \ge 2$ and $p \in \{A, B, C\}$, we claim that the sequence of moves

$$\left[(\beta\beta\gamma\alpha\alpha\gamma)^{(3^{n-2}-1)/2}\beta\beta\gamma\right]^3$$

is a Hamiltonian cycle of T_n starting and ending at $S(p)^{n-1}p$. Indeed, by Theorem 1, $(\beta\beta\gamma\alpha\alpha\gamma)^{(3^{n-2}-1)/2}\beta\beta$ is a Hamiltonian path of each of the three subgraphs of T_n from $S(p)^{n-1}p$ to $P(p)^{n-1}p$, from $P(p)^{n-1}S(p)$ to $p^{n-1}S(p)$, and from $p^{n-1}P(p)$ to $S(p)^{n-1}P(p)$, respectively. Combining these sequences of moves with three γ moves that connect the subgraphs gives the Hamiltonian cycle, as claimed. $\square$

We conclude with an observation about the original Towers of Hanoi problem. Consider solutions for the n-disk problem in which no configuration is repeated. It is well known that the shortest such solution has length $2^n - 1$, and we have just demonstrated in Theorem 1 that the longest such solution has $3^n - 1$ moves. Next we show that, in fact, there exists such a solution with any number of moves in-between these two extremes.

Theorem 3. *For any integers n and m such that $n \geq 1$ and $2^n - 1 \leq m \leq 3^n - 1$, there exists an m-move solution to the original n-disk Towers of Hanoi problem with no configuration repeated.*

Proof. Let $p \to q$ denote the move that removes the top disk from a pole p and puts it onto the top of a pole q. Given $n \geq 1$ and m such that $2^n - 1 \leq m \leq 3^n - 1$, and given poles s, d, and a mutually different from $\{A, B, C\}$, by $H(s, d, a, n, m)$ denote a sequence of m moves that take n topmost disks from pole s (source) to pole d (destination) using pole a as the auxiliary pole. Then, for any s, d, and a mutually different from $\{A, B, C\}$, the sequence H can be defined recursively on n and m as follows.

For $n = 1$ and $m = 1, 2$:

$$H(s, d, a, 1, 1) = s \to d$$

and

$$H(s, d, a, 1, 2) = s \to a, a \to d.$$

For $n \geq 2$ and m such that $2^n - 1 \leq m \leq 3^n - 1$:

$$H(s,d,a,n,m) = \begin{cases} H(s,a,d,n-1,\lfloor m_1 \rfloor), s \to d, & \\ H(a,d,s,n-1,\lceil m_1 \rceil) & \text{, if } \lceil m_1 \rceil \leq 3^{n-1} - 1 \\ & \\ H(s,d,a,n-1,\lfloor m_2 \rfloor), s \to a, & \\ H(d,s,a,n-1,\lceil m_2 \rceil), a \to d, & \\ H(s,d,a,n-1,m_3) & \text{, if } \lceil m_1 \rceil > 3^{n-1} - 1 \end{cases}$$

where $m_1 = (m-1)/2$, $m_2 = (m-2)/3$, and $m_3 = m - 2 - \lfloor m_2 \rfloor - \lceil m_2 \rceil$.

Assuming the recursion makes sense, it is easy to check that the sequence H is the desired solution with m moves and non-repeating configurations. To see that the recursion is well-defined we argue by induction on n. The base case $n = 1$ is trivial. For $n \geq 2$, suppose inductively that $H(s, d, a, n-1, m)$ is well-defined for all m in the interval $2^{n-1} - 1 \leq m \leq 3^{n-1} - 1$ and for any s, d, and a mutually different from $\{A, B, C\}$. We need to show that $H(s, d, a, n, m)$ is well-defined for any m such that $2^n - 1 \leq m \leq 3^n - 1$. To this end, we consider the two cases of the above recursive definition of $H(s, d, a, n, m)$ and show that m_1, m_2, and m_3 belong to the proper range of integers for m in $H(s, d, a, n-1, m)$, i.e., $2^{n-1}-1 \le m_1, m_2, m_3 \le 3^{n-1}-1$. In other words, in the first case $\lceil (m-1)/2 \rceil \le 3^{n-1}-1$, we show that

$$2^{n-1} - 1 \le \lfloor (m-1)/2 \rfloor \le \lceil (m-1)/2 \rceil \le 3^{n-1} - 1.$$

Next, in the second case $\lceil (m-1)/2 \rceil > 3^{n-1} - 1$, we show that

$$2^{n-1} - 1 \le \lfloor (m-2)/3 \rfloor \le \lceil (m-2)/3 \rceil \le 3^{n-1} - 1$$

and

$$2^{n-1} - 1 \le m - 2 - \lfloor (m-2)/3 \rfloor - \lceil (m-2)/3 \rceil \le 3^{n-1} - 1.$$

Now, to prove this, in the first case we have

$$\lfloor \frac{m-1}{2} \rfloor \ge \lfloor \frac{2^n - 1 - 1}{2} \rfloor = \lfloor 2^{n-1} - 1 \rfloor = 2^{n-1} - 1.$$

In the second case, since $\lceil (m-1)/2 \rceil > 3^{n-1} - 1$ if and only if $m > 2 \cdot 3^{n-1} - 1$, it follows that

$$\lfloor \frac{m-2}{3} \rfloor \ge \lfloor \frac{2 \cdot 3^{n-1} - 1 - 2}{3} \rfloor \ge \lfloor 2 \cdot 3^{n-2} - 1 \rfloor \ge 2 \cdot 2^{n-2} - 1 = 2^{n-1} - 1.$$

Also,

$$\lceil \frac{m-2}{3} \rceil \le \lceil \frac{3^n - 1 - 2}{3} \rceil = \lceil 3^{n-1} - 1 \rceil = 3^{n-1} - 1.$$

Finally, observe that $m - 2 - \lfloor (m-2)/3 \rfloor - \lceil (m-2)/3 \rceil = \lfloor (m-1)/3 \rfloor$ for all $m \ge 3$. Then,

$$\lfloor \frac{m-1}{3} \rfloor \le \lfloor \frac{3^n - 1 - 1}{3} \rfloor = \lfloor 3^{n-1} - \frac{2}{3} \rfloor = 3^{n-1} - 1$$

and

$$\lfloor \frac{m-1}{3} \rfloor \ge \lfloor \frac{2 \cdot 3^{n-1} - 1 - 1}{3} \rfloor = \lfloor 2 \cdot 3^{n-2} - \frac{2}{3} \rfloor \ge \lfloor 2 \cdot 2^{n-2} - \frac{2}{3} \rfloor = 2^{n-1} - 1,$$

as required. $\square$

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