Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 17 (2006), 18–30.
Available electronically at **http: //matematika.etf.bg.ac.yu**

A SIMPLE ALGORITHM FOR PROVING A CLASS OF INEQUALITIES

Slaviša B. Prešić

Let $E(x_1,\dots,x_n)$ be an expression of the form

$$\mathcal{A}_1|a_{11}x_1+\cdots+a_{1n}x_n+b_1|+\cdots+\mathcal{A}_m|a_{m1}x_1+\cdots+a_{m1}x_n+b_m| \\ +\mathcal{B}_1x_1+\cdots+\mathcal{B}_nx_n+\mathcal{C},$$

where $\mathcal{A}_i, a_{ij}, b_i, \mathcal{B}_j, \mathcal{C}$ are any real numbers. In this paper we introduce an algorithm *Elim*, by which one can establish whether for all $x_1,\dots,x_n\in\mathbb{R}$ the inequality $E(x_1,\dots,x_n)\rho 0$ holds, where ρ can be $>$ or $\geq$. Such an example is the following inequality

$$|x_1|+|x_2|+|x_3|-|x_1+x_2|-|x_1+x_3|-|x_2+x_3|+|x_1+x_2+x_3|\geq 0,$$

which originated from H. Hornich [**2**]. All results can be transfered to any ordered field.

Let $E(x)$ be an expression of the form

$$A_1|x-a_1|+\cdots+A_k|x-a_k|+Bx+C, \tag{1}$$

where A_i, a_i, B, C are any real numbers. The so-called *determiners* of expression $E(x)$ are

$$-\infty,\ a_1,\ \dots,\ a_k,\ +\infty.$$

Temporarily suppose that this chain of inequalities

$$a_1\leq a_2\leq\cdots\leq a_k \tag{$*1$}$$

holds. Then the expression $E(x)$ has the following property:

(2) In each interval $(-\infty,a_1]$, $[a_1,a_2],\dots,[a_{k-1},a_k]$, $[a_k,+\infty)$ $E(x)$ reduces to a certain linear expression like $ax+b$, with some real numbers a,b.

2000 Mathematics Subject Classification: 26A99
Keywords and Phrases: General inequality, field of reals, ordered field.

These linear expressions will be denoted by

$$Lin_{(-\infty,a_1]}(x),\ Lin_{[a_1,a_2]}(x),\ldots,Lin_{[a_{k-1},a_k]}(x),\ Lin_{[a_k,+\infty)}(x).$$

For instance, if $E(x) = |x-1| - 5|x-2| + 3x - 2$, then $E-$determiners are $-\infty,\ 1,\ 2,\ +\infty$ and the corresponding linear expressions are

$$Lin_{(-\infty,1]}(x) = 7x - 11,\ Lin_{[1,2]}(x) = 9x - 13,\ Lin_{[2,+\infty)}(x) = -x + 7.$$

Indeed, if $x \leq 1$ then $|x-1| = 1-x$, $|x-2| = 2-x$ therefore $E(x) = 1 - x - 5(2 - x) + 3x - 2$, i.e. $Lin_{(-\infty,1]}(x) = 7x - 11$. Similarly one can derive the equalities $Lin_{[1,2]}(x) = 9x - 13$, $Lin_{[2,+\infty)}(x) = -x + 7$. In general, one can derive these equalities

$$(*2)\qquad \begin{aligned} Lin_{(-\infty,a_1]}(x) &= x(B - A_1 - \cdots - A_k) + (C + A_1a_1 + \cdots + A_ka_k),\\ Lin_{[a_1,a_2]}(x) &= x(B + A_1 - A_2 - \cdots - A_k)\\ &\quad + (C - A_1a_1 + A_2a_2 + \cdots + A_ka_k),\\ Lin_{[a_2,a_3]}(x) &= x(B + A_1 + A_2 - A_3a_3 - \cdots - A_k)\\ &\quad + (C - A_1a_1 - A_2a_2 + A_3a_3 + \cdots + A_ka_k),\\ &\ \vdots\\ Lin_{[a_{k-1},a_k]}(x) &= x(B + A_1 + A_2 + \cdots + A_{k-1} - A_k)\\ &\quad + C - A_1a_1 - A_2a_2 - \cdots - A_{k-1}a_{k-1} + A_ka_k),\\ Lin_{[a_k,+\infty)}(x) &= x(B + A_1 + A_2 + \cdots + A_k)\\ &\quad + (C - A_1a_1 - A_2a_2 - \cdots - A_ka_k). \end{aligned}$$

Expressions $(*2)$ satisfy the following equalities

$$(3)\qquad \begin{aligned} &Lin_{(-\infty,a_1]}(a_1) = Lin_{[a_1,a_2]}(a_1),\ Lin_{[a_1,a_2]}(a_2) = Lin_{[a_2,a_3]}(a_2),\\ &\qquad\ldots,\ Lin_{[a_{k-1},a_k]}(a_k) = Lin_{[a_k,\infty)}(a_k). \end{aligned}$$

Related to (3) we shall also say: *neighbouring linear expressions are connected.*

The conclusion (3) is based on the assumption $(*1)$. In general case instead of $(*1)$ we have some chain of inequalities of the form

$$(*3)\qquad a'_1 \leq a'_2 \leq\ \cdots\ \leq a'_k,$$

where $a'_1a'_2\ \cdots\ a'_k$ is certain permutation of $a_1a_2\ \cdots\ a_k$. Notice that if we each a_i replace with a'_i then from (2), (3) we obtain new true assertions. Also, by such substitution from formulas $(*2)$ we obtain new valid formulas. It is interesting that in the formulas

$$(*4)\qquad Lin_{(-\infty,a'_1]}(x) = x(B - A_1 - \cdots - A_k) + (C + A_1a_1 + \cdots + A_ka_k),$$

$$Lin_{[a'_k,+\infty)}(x) = x(B + A_1 + A_2 + \cdots + A_k) + (C - A_1a_1 - A_2a_2 - \cdots - A_ka_k),$$

the right-hand sides do not depend on the permutation $a_1' a_2' \cdots a_k'$. Next, we introduce the following notations:

$$
\begin{array}{lll}
& E(+\infty) > 0 \ \textit{stands for}: & (\exists x_0)(\forall x \geq x_0)\ E(x) > 0 \text{ i.e. starting with some} \\
& & x_0 \text{ for all } x \geq x_0 \text{ the inequality } E(x) > 0 \text{ holds,} \\
(4) & E(+\infty) \geq 0 \ \textit{stands for}: & (\exists x_0)(\forall x \geq x_0)\ E(x) \geq 0, \\
& E(-\infty) > 0 \ \textit{stands for}: & (\exists x_0)(\forall x \leq x_0)\ E(x) > 0, \\
& E(-\infty) \geq 0 \ \textit{stands for}: & (\exists x_0)(\forall x \leq x_0)\ E(x) \geq 0.
\end{array}
$$

Bearing in mind (∗4) one can substitute definitions (4) by the following:

$$
(5)\quad
\begin{array}{ll}
E(+\infty) > 0 \ \textit{stands for}: & A_1 + \cdots + A_k + B > 0 \\
& \vee (A_1 + \cdots + A_k + B = 0,\ C - A_1 a_1 - \cdots - A_k a_k > 0), \\
E(+\infty) \geq 0 \ \textit{stands for}: & A_1 + \cdots + A_k + B > 0 \\
& \vee (A_1 + \cdots + A_k + B = 0,\ C - A_1 a_1 - \cdots - A_k a_k \geq 0), \\
E(-\infty) > 0 \ \textit{stands for}: & A_1 + \cdots + A_k - B > 0 \\
& \vee (A_1 + \cdots + A_k - B = 0,\ C + A_1 a_1 + \cdots + A_k a_k > 0), \\
E(-\infty) \geq 0 \ \textit{stands for}: & A_1 + \cdots + A_k - B > 0 \\
& \vee (A_1 + \cdots + A_k - B = 0,\ C + A_1 a_1 + \cdots + A_k a_k \geq 0).
\end{array}
$$

Each *Lin*−function of $E(x)$, being linear, has the following property:

(6) *Lin* has a fixed sign σ in its interval if and only if it has this sign σ on the ends of the interval.

For instance:

$$Lin_{[a_1', a_2']}(x) > 0 \text{ for all } x \in [a_1', a_2'] \text{ if and only if } E(a_1') > 0,\ E(a_2') > 0.$$

Lemma 1. *Let* $E(x)$ *be an expression of the form* (1)*. Then the following equivalences hold*:

$$
\begin{array}{l}
(\forall x \in \mathbb{R}) E(x) > 0 \ \Leftrightarrow\ E(-\infty) > 0,\ E(a_1) > 0,\ \ldots,\ E(a_k) > 0,\ E(+\infty) > 0, \\
(\forall x \in \mathbb{R}) E(x) \geq 0 \ \Leftrightarrow\ E(-\infty) \geq 0,\ E(a_1) \geq 0,\ \ldots,\ E(a_k) \geq 0,\ E(+\infty) \geq 0.
\end{array}
$$

Proof. We shall prove the first equivalence; the second equivalence can be proved in a similar way. The proof of $\langle if \rangle$ part is immediate. Indeed, if the inequality $E(x) > 0$ holds for all $x \in \mathbb{R}$ then it holds in "points" $a_1, \ldots, a_k$. Bearing in mind (∗4) and (5) we see that the conditions $E(-\infty) > 0$, $E(+\infty) > 0$ are satisfied also.

To prove $\langle only\ if \rangle$ part suppose that conditions

$$E(-\infty) > 0,\ E(a_1) > 0,\ \ldots,\ E(a_k) > 0,\ E(+\infty) > 0$$

hold. These conditions can be expressed in this way

$$E(-\infty) > 0,\ E(a_1') > 0,\ \ldots,\ E(a_k') > 0,\ E(+\infty) > 0.$$

Let $x \in \mathbb{R}$ be any real number. If $x \in [a'_i, a'_{i+1}]$ $(1 \le i \le k-1)$, then by (6) it follows that $E(x) > 0$ holds. Next, if $E(-\infty) > 0$ then from the facts:

1° Starting with some x_0 for all $x \le x_0$ the inequality $E(x) > 0$ holds,

2° $E(x)$ reduces to linear expression for all $x \le a'_1$,

we derive that $E(x) > 0$ for all $x \le a'_1$. In a similar way, from the assumptions $E(+\infty) > 0$ and $E(a_k) > 0$ we conclude that $E(x) > 0$ for all $x \ge a_k$.

According to Lemma 1, if we want to prove certain inequality $E(x) > 0$ (for all $x \in \mathbb{R}$), then it suffices to prove the following *conjunction*

$$E(-\infty) > 0 \ \wedge \ E(a_1) > 0 \ \wedge \cdots \ \wedge \ E(a_k) > 0 \ \wedge \ E(+\infty) > 0$$

of inequalities. A similar fact holds for inequality $E(x) \ge 0$. Notice that on the left-hand side of both equivalences in Lemma 1 stands one formula of the form $(\forall x \in \mathbb{R})E(x)\rho 0$, where ρ is $>$ or $\ge$, while on the right-hand side stands the formula in which quantifier $(\forall x \in \mathbb{R})$ does not appear. In other words, Lemma 1 is an assertion of *elimination of the quantifier* $(\forall x \in \mathbb{R})$. The right-hand sides are conjunctions, whose components we shall call *successors* of the formula on the left-hand side, i.e. of formula $(\forall x \in \mathbb{R})E(x)\rho 0$. Also, this formula shall be called *parent* (of its successors).

Mainly based on Lemma 1 and Lemma 2 below we shall gradually define an algorithm *Elim*. Briefly said, *Elim* "calculates" the logical value of given formula, the result can be either $\top$ ("true") or $\bot$ ("false"). In the sequel for *Elim* we shall use a functional denotation. Namely, if ϕ is a given formula, then by $\mathit{Elim}(\phi)$ is denoted its logical value (obtained by *Elim*−algorithm). *Elim* shall be defined by three definition-equalities $(El_1), (El_2), (El_3)$ below.

Elim deals with some formulas, belonging to the so called *Elim–class*. For instance, formulas $(\forall x \in \mathbb{R})E(x) > 0, (\forall x \in \mathbb{R})E(x) \ge 0$ from Lemma 1 are elements of *Elim–class*. Let $E(x_1, \ldots, x_n)$ be any expression with variables $x_1, \ldots, x_n$ only. *Elim–class* is determined by

Definition 1. *A formula* $(\forall x_1, \ldots, x_n \in \mathbb{R})E(x_1, \ldots, x_n)\rho 0)$, *where* ρ *is* $>$ *or* $\ge$ *belongs to Elim–class if and only Elim−algorithm can calculate*

$$\mathit{Elim}\big((\forall x_1, \ldots, x_n \in \mathbb{R})E(x_1, \ldots, x_n)\rho 0\big),$$

i.e. by Elim one can prove or disprove that the inequality $E(x_1, x_2, \ldots, x_n)\rho 0$ *holds for any real numbers* $x_1, \ldots, x_n$.

According to Lemma 1 we first introduce the following definition-equality, which is a particular case of (El_1) below.

$$(El'_1) \quad \mathit{Elim}\big((\forall x \in \mathbb{R})E(x)\rho 0\big) = \mathit{Elim}\big(E(-\infty)\rho 0\big) \wedge \mathit{Elim}\big(E(a_1)\rho 0\big) \wedge \cdots \wedge \mathit{Elim}(E(a_k)\rho 0) \wedge \mathit{Elim}(E(+\infty)\rho 0) \quad (\rho \text{ is } > \text{ or } \ge).$$

The meaning of (El'_1) is the following. Suppose that we now the $v_0, v_1, \dots, v_k, v_{k+1}$ which are the logical values for

$$Elim\big(E(-\infty)\rho 0\big),\ Elim\big(E(a_1)\rho 0\big),\ \dots,\ Elim\big(E(a_k)\rho 0\big),\ Elim\big(E(+\infty)\rho 0\big)$$

respectively. Then the value of $Elim\big((\forall x \in \mathbb{R})E(x)\rho 0\big)$ is $v_0 \wedge v_1 \wedge \cdots \wedge v_k \wedge v_{k+1}$, where the use of truth value table for logical connective $\wedge$ is supposed. Related to (El'_1) we can also say in this way: *calculation of Elim*(ϕ) *for a given formula* ϕ *is transfered to calculation of Elim-values for its successors.*

An equality $A\rho B$ (ρ is $>$ or $\geq$), where A, B are some given real numbers, will be called *a constant-inequality.* For instance, $3 > 5$ is such an inequality. Every constant-inequality is either true or false, i.e. has exactly one logical value $\top$ or $\bot$. The next compoment of *Elim*−algorithm is the following definition-equality:

(El_2) $\quad Elim(A\rho B) = v \quad (\rho$ is $>$ or $\geq)$ $\quad A,\ B$ are some real numbers and v is the logical value of the inequality $A\rho B$.

For instance, $Elim(3 > 5)$ is $\bot$, $Elim(5 \geq 3)$ is $\top$. Next, we introduce the folowing components of *Elim*−algorithm:

(El_3) $\quad Elim(\phi \wedge \psi) = Elim(\phi) \wedge Elim(\psi), \qquad Elim(\phi \vee \psi) = Elim(\phi) \vee Elim(\psi),$

where on the rigt-hand sides are supposed the corresponding truth value tables for $\wedge$ and $\vee$.

To illustrate the given definition-equalities we state one simple example. Let $E(x)$ be expression $2|x|+|x-1|+3x+1$. This is an expression of type (1). The $-\infty$ is an $E-$determiner. By (5) for the inequality $E(-\infty) \geq 0$ we have the following logical formula

$$2+1-3>0\ \vee\ (2+1-3=0\ \wedge\ 2\geq 0)$$

by which we can easily calculate the logical value of $E(-\infty) \geq 0$. Employing *Elim*−algoritm we have the following chain of equalities

$(*5)$ $\quad Elim\big(E(-\infty) \geq 0\big)$

$$\begin{aligned}
&= Elim\big(2+1-3>0\ \vee\ (2+1-3=0\ \wedge\ 2\geq 0)\big) \quad \text{(by (5))}\\
&= Elim(2+1-3>0)\ \vee\ Elim\big((2+1-3=0\ \wedge\ 2\geq 0)\big) \quad \text{(by } (El_3))\\
&= Elim(2+1-3>0) \vee \big(Elim(2+1-3=0) \wedge Elim(2\geq 0)\big) \text{ (by } (El_3))\\
&= \bot\ \vee\ (\top\ \wedge\ \top) \quad \text{(by } (El_2))\\
&= \top \quad \text{(by truth value tables for } \wedge \text{ and } \vee).
\end{aligned}$$

Now we shall see how *Elim* works in two examples.

EXAMPLE 1. Prove or disprove the given inequality (for any $x \in \mathbb{R}$):

(i) $\quad E(x) \geq 0,\quad$ where $E(x) = 2|x| + |x-1| + 3x + 1$.

(ii) $\quad E(x) > 0,\quad$ where $E(x) = 1 + x - |x|$.

Solution. (i) The $E(x)$ is an expression of type (1). The $E-$determiners are $-\infty, 0, 1, +\infty$. By (El_1') (ρ is $\geq$) we have the following equality:

$$(*) \quad Elim\big((\forall x \in \mathbb{R})\,(2|x| + |x-1| + 3x + 1 \geq 0)\big)$$
$$= Elim(E(-\infty) \geq 0) \ \wedge \ Elim(E(0) \geq 0) \ \wedge \ Elim(E(1) \geq 0) \ \wedge \ Elim(E(\infty) \geq 0).$$

Now we calculate the $\wedge-$components. For the first one (see $(*5)$) we already have the result $Elim\big(E(-\infty)\big) \geq 0) = \top$. For the second and third one we have equivalities

$$Elim\big(E(0) \geq 0\big) = Elim(2 \geq 0) = \top, \quad Elim\big(E(1) \geq 0\big) = Elim(6 \geq 0) = \top$$

respectively. For the fourth one we have the following chain of equalities:

$$\begin{aligned} Elim(E(\infty) \geq 0) &= Elim\big(2+1+3 > 0 \vee (2+1+3 = 0 \ \wedge \ 0 \geq 0)\big) \\ &= Elim(2+1+3 > 0) \vee Elim\big(2+1+3 = 0 \ \wedge \ 0 \geq 0\big) \\ &= Elim(2+1+3 > 0) \vee \big(Elim(2+1+3 = 0) \ \wedge \ Elim(0 \geq 0)\big) \\ &= \top \vee (\bot \wedge \top) \\ &= \top. \end{aligned}$$

Notice that we can shorten this calculation. Namely, when we have calculated $Elim(2+1+3 > 0)$ and obtained $\top$, then we could conclude that the total result is $\top$. After these calculations for formula $(*)$ we have the final result $\top$. In other words inequality (i) is proved.

(ii) The $E(x)$ is an expression of type (1). The $E-$determiners are $-\infty, 0, +\infty$. By (El_1') (ρ is $>$) we have the following equality:

$$(**) \quad Elim\big((\forall x \in \mathbb{R})\,(1 + x - |x| > 0)\big)$$
$$= Elim(E(-\infty) > 0) \ \wedge \ Elim(E(0) > 0) \ \wedge \ Elim(E(+\infty) > 0).$$

Now we calculate the first $\wedge-$component. We have the following calculation:

$$Elim(E(-\infty) > 0) = Elim(-1-1 > 0 \vee (-1-1 = 0 \wedge 1 > 0) = \bot \vee (\bot \wedge \top) = \bot.$$

Since the first $\wedge-$component is $\bot$ we do not need to calculate other $\wedge-$components, the total result for $(**)$ is $\bot$, i.e. the inequality (ii) is not true for all $x \in \mathbb{R}$.

According to the solutions, stated in Example 1, we see that if $E(x)$ is an expression of the form (1) then *Elim*$-$algorithm is able to prove or disprove the inequality $E(x)\rho 0$ (ρ is $>$ or $\geq$) for any real number x. In other words formula $(\forall x \in \mathbb{R})E(x)\rho 0$ belongs to *Elim–class.*

Let now $E(x)$ be an expression of the form (1), such that $a_1, \ldots, a_k$ and C can be expressions containing some new variables, say $y_1, \ldots, y_n$. However, we suppose that $A_1, \ldots, A_k$ and B are some real numbers. Let $E(x)$ be also denoted by $E(x, y_1, \ldots, y_n)$. Suppose that we want to prove that inequality $E(x, y_1, \cdots, y_n)\rho 0$

(ρ is $>$ or $\geq$) holds for any real numbers $x, y_1, \ldots, y_n$. We can find $x-$successors of $E(x, y_1, \ldots, y_n)$, which are

$$(*)\quad E(-\infty, y_1, \ldots, y_n)\rho 0,\ E(a_1, y_1, \ldots, y_n)\rho 0, \ldots, \\ E(a_k, y_1, \ldots, y_n)\rho 0,\ E(\infty, y_1, \ldots, y_n)\rho 0.$$

Obviously the left-hand sides of the successors for $a_1, \ldots a_k$ are some expressions, consequently these successors are some $\rho-$inequalitites. But the successors for $-\infty$ and $+\infty$ are defined by (5) which yuilds a logical formula. For instance, for $E(+\infty, y_1, \ldots, y_n) > 0$ we have the logical formula of the form

$$A_1 + \cdots + A_k + B > 0 \ \lor\ (A_1 + \cdots + A_k + B = 0, C - A_1a_1 - \cdots - A_ka_k > 0).$$

Let S be denotation for the summ $A_1 + \cdots + A_k + B$. Then, if $S < 0$ then $E(+\infty, y_1, \ldots, y_n) > 0$ reduces to $\bot$, if $S > 0$ reduces to $\top$, and if $S = 0$ then $E(+\infty, y_1, \ldots, y_n) > 0$ reduces to the inequality $C - A_1a_1 - \cdots - A_ka_k > 0$ containing variables $y_1, \ldots, y_n$ only. Notice that this inequality is with $> -$sign, just as the successor $E(+\infty, y_1, \ldots, y_n) > 0$.

Similarly, any successor for $-\infty$ or $+\infty$ reduces to $\bot$, or to $\top$ or to some inequality of the form $L(y_1, \ldots, y_n)\rho 0$, where L is some expression with variables $y_1, \ldots, y_n$ only. That fact is essential, we particularly express it by

(7) In virtue of the supposition that $A_1, \ldots, A_k,\ B$ are some real numbers the successors $E(-\infty, y_1, \ldots, y_n)\rho 0, E(+\infty, y_1, \ldots, y_n)\rho 0$ reduce to $\bot$, or to $\top$ or to some inequality of the form $L(y_1, \ldots, y_n)\rho 0$, where L is some expression with variables $y_1, \ldots, y_n$ only.

Now concerning the mentioned problem we put the following question: whether the problem *to prove inequality* $E(x, y_1, \ldots, y_n)\rho 0$ *for all* $x, y_1, \ldots, y_n \in \mathbb{R}$ can be reduced to the problem *to prove that all successor-inequalities hold for any* $y_1, \ldots, y_n \in \mathbb{R}$? The answer is positive. Related to this we have the following assertion:

Lemma 2. *Let* $E(x, y_1, \ldots, y_n)$ *be an expression of the form* (1) *with respect to* x, *allowing that* a_i, C *may be expressions containing the variables* $y_1, \ldots, y_n$ *only, while* A_i, B *must be some real numbers. Then the following equivalence holds*:

$$(i)\quad (\forall y_1, \ldots, y_n \in \mathbb{R})(\forall x \in \mathbb{R})E(x, y_1, \ldots, y_n)\rho 0 \\ \Leftrightarrow\ (\forall y_1, \ldots, y_n \in \mathbb{R})E(-\infty, y_1, \ldots, y_n)\rho 0, \\ (\forall y_1, \ldots, y_n \in \mathbb{R})\ E(a_1, y_1, \ldots, y_n)\rho 0, \\ \ldots, \\ (\forall y_1, \ldots, y_n \in \mathbb{R})E(a_k, y_1, \ldots, y_n)\rho 0, \\ (\forall y_1, \ldots, y_n \in \mathbb{R})E(\infty, y_1, \ldots, y_n)\rho 0.$$

Proof. Suppose that ρ is $>$. Let variables $y_1, \ldots, y_n$ have any values $v_1, \ldots, v_n$ from $\mathbb{R}$. Consider the formula $(\forall x \in \mathbb{R})E(x, v_1, \ldots, v_n)$.

The expression $E(x, v_1, \ldots, v_n)$ has the form (1), the corresponding subexpressions a_i, A_j, B, C are certain real numbers. Applying Lemma 1 to that expression we obtain the following equivalence:

$$(\forall x \in \mathbb{R})E(x, v_1, \ldots, v_n) > 0$$
$$\Leftrightarrow E(-\infty, v_1, \ldots, v_n) > 0, E(a_1, v_1, \ldots, v_n) > 0,$$
$$\ldots,\ E(a_k, v_1, \ldots, v_n) > 0, E(+\infty, v_1, \ldots, v_n) > 0$$

Temporarily denote this equivalence by $L(v_1, \ldots, v_n) \Leftrightarrow R(v_1, \ldots, v_n)$. Having in mind that $v_1, \ldots, v_n$ may be *any real numbers* we have the following conclusion

$$(\forall v_1, \ldots, v_n \in \mathbb{R})\big(L(v_1, \ldots, v_n) \Leftrightarrow R(v_1, \ldots, v_n)\big).$$

From this formula immediately follows the following equivalence

$$(*) \qquad (\forall v_1, \ldots, v_n \in \mathbb{R})L(v_1, \ldots, v_n) \Leftrightarrow (\forall v_1, \ldots, v_n \in \mathbb{R})R(v_1, \ldots, v_n).$$

We have used the following general property of quantifier $\forall$

$$(\forall \vec{V})(P(\vec{V}) \Leftrightarrow Q(\vec{V})) \Rightarrow (\forall \vec{V})P(\vec{V}) \Leftrightarrow (\forall \vec{V})Q(\vec{V}),$$

where $\vec{V}$ stands for $v_1, \ldots, v_n$, and P, Q are some logical formulas.

Using y_i instead of v_i from $(*)$ we obtain

$$(*') \qquad (\forall y_1, \ldots, y_n \in \mathbb{R})L(y_1, \ldots, y_n) \Leftrightarrow (\forall y_1, \ldots, y_n \in \mathbb{R})R(y_1, \ldots, y_n).$$

$R(y_1, \ldots, y_n)$ is the conjunction

$$E(-\infty, y_1, \ldots, y_n) > 0, E(a_1, y_1, \ldots, y_n) > 0,$$
$$\ldots,\ E(a_k, y_1, \ldots, y_n) > 0, E(+\infty, y_1, \ldots, y_n) > 0.$$

Using the general connection between the quantifier $\forall$ and $\wedge$, expressed by the equivalence $(\forall x)(P \wedge Q) \Leftrightarrow (\forall x)P \wedge (\forall x)Q$ we get the following equivalence

$$(*'') \quad (\forall y_1, \ldots, y_n \in \mathbb{R})R(y_1, \ldots, y_n)$$
$$\Leftrightarrow (\forall y_1, \ldots, y_n \in \mathbb{R})E(-\infty, y_1, \ldots, y_n) > 0,$$
$$(\forall y_1, \ldots, y_n \in \mathbb{R})E(a_1, y_1, \ldots, y_n) > 0,$$
$$\ldots,\ (\forall y_1, \ldots, y_n \in \mathbb{R})E(a_k, y_1, \ldots, y_n) > 0,$$
$$(\forall y_1, \ldots, y_n \in \mathbb{R})E(+\infty, y_1, \ldots, y_n) > 0.$$

From $(*')$ and $(*'')$ we derive the following equivalence

$$(\forall y_1, \ldots, y_n \in \mathbb{R})L(y_1, \ldots, y_n)$$
$$\Leftrightarrow (\forall y_1, \ldots, y_n \in \mathbb{R})E(-\infty, y_1, \ldots, y_n) > 0,$$
$$(\forall y_1, \ldots, y_n \in \mathbb{R})E(a_1, y_1, \ldots, y_n) > 0,$$
$$\ldots,\ (\forall y_1, \ldots, y_n \in \mathbb{R})E(a_k, y_1, \ldots, y_n) > 0,$$
$$(\forall y_1, \ldots, y_n \in \mathbb{R})E(\infty, y_1, \ldots, y_n) > 0.$$

As a matter of fact, in case ρ is $>$ we have obtained the equivalence (i). In a similar way one can prove (i) in case ρ is $\geq$.

In connection with Lemma 2 for *Elim* we have the last definition-equality:

$$(El_1)\quad Elim\big((\forall y_1,\dots,y_n\in\mathbb{R})(\forall x\in\mathbb{R})E(x,y_1,\dots,y_n)\rho 0\big)$$
$$= Elim\big((\forall y_1,\dots,y_n\in\mathbb{R})E(-\infty,y_1,\dots,y_n)\rho 0\big),$$
$$Elim\big((\forall y_1,\dots,y_n\in\mathbb{R})\ E(a_1,y_1,\dots,y_n)\rho 0\big),$$
$$\dots,$$
$$Elim\big((\forall y_1,\dots,y_n\in\mathbb{R})E(a_k,y_1,\dots,y_n)\rho 0\big),$$
$$Elim\big((\forall y_1,\dots,y_n\in\mathbb{R})E(\infty,y_1,\dots,y_n)\rho 0\big).$$

Notice that (El_1) is techically a bit complex. Therefore when we use them we shall make two steps:

(8) 1° For inequality $E(x,y_1,\dots,y_n)\rho 0$ we make $x-$succerors, which are:

$$E(-\infty,y_1,\dots,y_n)\rho 0,\ E(a_1,y_1,\dots,y_n)\rho 0,\dots,$$
$$E(a_k,y_1,\dots,y_n)\rho 0,\ E(+\infty,y_1,\dots,y_n)\rho 0.$$

2° After that we apply (El_1).

As we have already said, the *Elim*−algorithm is defined by equalities (El_1), (El_2), (El_3). By convention, each separate use of these equalities will be called *a step of Elim−algortihm.* During this algorithm at each step appears certain conjunction of the form

$$Elim(S_1)\wedge Elim(S_2)\wedge\cdots\wedge Elim(S_r)\quad (r\ge 1).$$

It can happen that some $Elim(S_i)$ is equal to $\bot$. In such a case *Elim*−algorithm halts, and the total *Elim*−result is $\bot$.

Now we state some examples in which *Elim*−algoritm is applied.

Example 2. Prove the inequality

$$(*1)\qquad |a|+|b|-|a+b|\ge 0,$$

where $a,\ b$ are any real numbers.

Proof. In other words we should prove the formula[1]

$$(\forall b\in\mathbb{R})(\forall a\in\mathbb{R})|a|+|b|-|a+b|\ge 0.$$

We shall apply *Elim*−algorithm, i.e. we shall calculate

$$(*)\qquad Elim\big((\forall b\in\mathbb{R})(\forall a\in\mathbb{R})|a|+|b|-|a+b|\ge 0\big).$$

According to (8) we shall first consider expression $|a|+|b|-|a+b|$ as $a-$expression, denoted by $E(a)$. It has the form (1). The $E-$determiners are $-\infty,0,-b,+\infty$. The coresponding $a-$successors of inequality $|a|+|b|-|a+b|\ge 0$ are

$$|b|+b\ge 0,\quad |b|-|b|\ge 0,\quad |-b|+|b|\ge 0,\quad |b|-b\ge 0.$$

Now applying (El_1) to $(*)$ we obtain the equality

[1]Instead of this formula we can use the formula $(\forall a\in\mathbb{R})(\forall b\in\mathbb{R})|a|+|b|-|a+b|\ge 0$ too.

$$\begin{aligned}
&Elim\big((\forall b\in\mathbb{R})(\forall a\in\mathbb{R})|a|+|b|-|a+b|\geq 0\big)\\
&\qquad = Elim\big((\forall b\in\mathbb{R})|b|+b\geq 0\big)\wedge Elim\big((\forall b\in\mathbb{R})|b|-|b|\geq 0\big)\\
&\qquad\qquad\qquad \wedge Elim\big((\forall b\in\mathbb{R})|-b|+|b|\geq 0\big)\wedge Elim\big((\forall b\in\mathbb{R})|b|-b\geq 0\big).
\end{aligned}$$

Now for each $\wedge-$component we apply (El_1). After a simple '*Elim*-calculation' for each ot them we obtain $\top$. Consequently the *Elim*-result is $\top$.

Example 3. The inequality (See [**2**])

$$(*1)\qquad |a|+|b|+|c|-|a+b|-|a+c|-|b+c|+|a+b+c|\geq 0$$

holds for any $a,b,c\in\mathbb{R}$.

Proof. In other words we should prove the formula

$$(\forall c\in\mathbb{R})(\forall b\in\mathbb{R})(\forall a\in\mathbb{R})|a|+|b|+|c|-|a+b|-|a+c|-|b+c|+|a+b+c|\geq 0.$$

We shall apply *Elim*$-$algorithm, i.e. we shall calculate

$$\begin{aligned}
(*2)\quad &Elim\big((\forall c\in\mathbb{R})(\forall b\in\mathbb{R})(\forall a\in\mathbb{R})\\
&\qquad\qquad |a|+|b|+|c|-|a+b|-|a+c|-|b+c|+|a+b+c|\geq 0\big).
\end{aligned}$$

According to (8) we shall first consider the expression

$$|a|+|b|+|c|-|a+b|-|a+c|-|b+c|+|a+b+c|$$

as $a-$expression, denoted by $E(a)$. It has the form (1). The $E-$determiners are

$$-\infty,\ 0,\ -b,\ -c,\ -b-c,\ +\infty$$

The coresponding $a-$successors of inequality

$$|a|+|b|+|c|-|a+b|-|a+c|-|b+c|+|a+b+c|\geq 0$$

are

$$\begin{aligned}
(*3)\quad &|b|+|c|-|b+c|\geq 0,\quad 0\geq 0,\quad |b|+|c|-|c-b|+|b|+|c|-|b+c|\geq 0\\
&\quad |b|+|c|-|b-c|+|b|+|c|-|b+c|\geq 0,\quad 0\geq 0,\quad |b|+|c|-|b+c|\geq 0.
\end{aligned}$$

By (El_1) the formula $(*2)$ is equal to

$$\begin{aligned}
&Elim\big((\forall c\in\mathbb{R})(\forall b\in\mathbb{R})|b|+|c|-|b+c|\geq 0\big)\\
&\qquad\wedge\ Elim\big((\forall c\in\mathbb{R})(\forall b\in\mathbb{R})|b|+|c|-|c-b|+|b|+|c|-|b+c|\geq 0\big)\\
&\qquad\qquad\wedge\ Elim\big((\forall c\in\mathbb{R})(\forall b\in\mathbb{R})|b|+|c|-|b-c|+|b|+|c|-|b+c|\geq 0\big)\\
&\qquad\qquad\qquad\wedge\ Elim\big((\forall c\in\mathbb{R})(\forall b\in\mathbb{R})|b|+|c|-|b+c|\geq 0\big).
\end{aligned}$$

Components $0\geq 0$ are omitted, because their *Elim*$-$values are $\top$. The remaining *Elim*$-$components are similar to that in Example 2, they can be easily calculated, each of them is equal to $\top$. Consequently the result for $(*2)$ is $\top$, i.e. the inequality $(*)$ is proved.

Now we shall generalize the inequalities occuring in Example 2 and Example 3. These inequalities, written with quantifiers in front, are elements of *Elim–class.* Consider expression $E(x_1,\dots,x_n)$ of the following form

$$(9)\quad \mathcal{A}_1|a_{11}x_1+\cdots+a_{1n}x_n+b_1|+\cdots+\mathcal{A}_m|a_{m1}x_1+\cdots+a_{m1}x_n+b_m| \\ +\mathcal{B}_1x_1+\cdots+\mathcal{B}_nx_n+\mathcal{C},$$

where $\mathcal{A}_i, a_{ij}, b_i, \mathcal{B}_j, \mathcal{C}$ are any real numbers. We shall, by induction on n, prove that formulas $(\forall x_1,\dots,x_n\in\mathbb{R})E(x_1,\dots,x_n)\rho 0$, where ρ is $>$ or $\geq$ belong to *Elim–class.*

If $n=1$ then obviously $E(x_1)$ can be transformed to form (1). By (El_1'), (El_2), (El_3) one can calculate *Elim*$((\forall x_1\in\mathbb{R})E(x_1)\rho 0)$, consequently in case $n=1$ proof complets.

Let $n>1$. Consider expression $E(x_1,\dots,x_n)$ as an x_1-expression, denoted temporarily by $E(x_1)$ also. Easily one can transform $E(x_1)$ into the form (1), where instead of x stands x_1. The number k is the number of non-zero products $\mathcal{A}_ia_{i1}$ where $1\leq i\leq m$. It is important that the coefficients $A_1,\dots,A_r$ and B are some real numbers. However $a_1,\dots,a_k,\mathcal{C}$ can contain the variables $x_2,\dots,x_n$ (but not x_1). Then $E-$determiners are $-\infty, a_1,\dots,a_k,+\infty$. Consider the cooresponding successors

$$E(-\infty,x_2,\dots,x_n)\rho 0,\ E(a_1,x_2,\dots,x_n)\rho 0,\ \dots, \\ E(a_k,x_2,\dots,x_n)\rho 0, E(+\infty,x_2,\dots,x_n)\rho 0.$$

To complete inductive proof it suffices to prove that the left-hand sides of all successors are expressions of the form (9), where instead of n stands $n-1$. This is obvious for the successors

$$E(a_1,x_2,\dots,x_n)\rho 0,\ \dots,\ E(a_k,x_2,\dots,x_n)\rho 0.$$

Having in mind (7) we see that this is valid for the successors with respect to $-\infty$, $+\infty$.

Now we shall estimate the number of steps for *Elim* when it is employed to calculate formula $(\forall x_1,\dots,x_n\in\mathbb{R})E(x_1,\dots,x_n)\rho 0$, where $E-$expression is defined by (9). Denote desired number by K. K is the number of all successors which appeared during *Elim*$-$algorithm. In the first step the number of all successors was $k+2$. For k we have the inequality $k\leq m$, where m is the number of $|\ |$-subexpessions in (9). We shall call this m also $m-$*number* of the expression of the form (9). One can easily conclude this fact:

> During *Elim*$-$algorithm $m-$number of any successor is less of $m-$number of its parent.

According to this we can roughly caclulate K in the following way:

> At the first step we have at most $m+2$ successors. Each of them has at most $(m+2)-1$ successors, and so on. Therefore, for K we have the following inequality

$$K \leq (m+2)(m+2-1)\cdots(m+2-n),$$

where n is the number of initial varibles $x_1,\dots,x_n$.

In such a way we obtained an estimate of K. Notice also that we count as *one step* a calculation by which using (5) we find the corresponding expression of successors related to $-\infty$ and $+\infty$.

In the sequell we state some generalizations of the results obtained until now, including generalizations concerning *Elim*–algorithm. These generalizations will be denoted by *Gen*1, *Gen*2, ...

*Gen*1. Notice that variables $x_1,\dots,x_n$ in all inequalities, considered until now, run from $-\infty$ to $+\infty$. For the variables we can use any segment $[p,q]$, where p,q are given real numbers with $p < q$. Then, all previous assertions including *Elim*–algorithm can be modified to the assumption $x_1,\dots,x_n \in [p,q]$. For instance, Lemma 1 transfers to the following one:

Lemma 1*. *Let $E(x)$ be an expression od the form (1). Then the following equivalence*

$$(\forall x \in \mathbb{R})E(x)\rho 0 \;\Leftrightarrow\; E(p)\rho 0,\, E(v_1)\rho 0,\dots,\, E(v_r)\rho 0, E(q)\rho 0 \quad (\rho \text{ is } > \text{ or } \leq)$$

holds, where $v_1,\dots,v_r$ are all those a_i which belong to the interval $[p,q]$.
End of *Gen*1.

*Gen*2. *Elim*–algorithm can be generalized to class of some inequalities which contain some *unknowns*, say $a,b,\dots \in \mathbb{R}$. In such a case we use the following extension of (El_2) :

$$Elim(A\rho B) = A\rho B \quad (\rho \text{ is } > \text{ or } \geq) \qquad A,\ B \text{ are some expressions.}$$

To illustrate this we state the following example.

Example 4. Find all values for a,b,c such that the inequality $a|x| + bx + c \geq 0$ holds for any real number x.

Solution. Temporarily denote $a|x| + bx + c$ by $E(x)$. We want to find all values $a,b,c \in \mathbb{R}$ such that $Elim(\forall x \in \mathbb{R})E(x) \geq 0$ is $\top$. The expression $a|x| + bx + c$ has the form (1). The $E-$determiners are $-\infty$, 0, $-\infty$. The corresponding successors are $E(-\infty) \geq 0$, $E(0) \geq 0$, $E(+\infty) \geq 0$. By (El_1) we have the following equality:

$$Elim\big((\forall x \in \mathbb{R})\, E(x) \geq 0\big) = Elim\big(E(-\infty) \geq 0\big) \wedge Elim\big((E(0) \geq 0\big) \wedge Elim\big(E(+\infty) \geq 0\big).$$

By (El_3) and (El_2^*) we have equalities:

$$Elim\big(E(-\infty) \geq 0\big) = a - b > 0 \wedge (a - b = 0 \wedge c \geq 0) \qquad \text{(by (5))}$$

$$Elim\big(E(0) \geq 0\big) = c \geq 0$$

$$Elim\big(E(+\infty) \geq 0\big) = a + b > 0 \wedge (a + b = 0 \wedge c \geq 0) \qquad \text{(by (5))}$$

So, the problem reduces to finding a,b,c satisfying the following conditions:

$$\big(a-b>0 \lor (a-b=0 \land c\geq 0)\big)$$
$$\land\, c\geq 0 \land \big(a+b>0 \lor (a+b=0 \land c\geq 0)\big).$$

These conditions can be easily reduced to the following conditions

$$c\geq 0 \lor a\geq |b|.$$

Obviously by the obtained conditions one can easily describe all desired values for a, b, c. End of *Gen*2.

*Gen*3. One can consider expressions built up from *max*, *min*, *sgn*, $\|, \ldots$ and may seek those expressions which have properties like (2) and (3). By argumentation similar to that applied until now one can obtain various new assertions. End of *Gen*3.

*Gen*4. Real numbers are elements of *a complete ordered field.* In all assertions and in all argumentations we have only used a part of axioms of such a field, more precisely we have used only the axioms of an ordered field. Consequently, all results in this article can be transfered to the case of ordered fields. End of *Gen*4.

REFERENCES

1. D. S. Mitrinović, P. M. Vasić: *Analytic inequalities.* Springer-Verlag, 1970.
2. H. Hornich: *Eine Ungleichung für Vektorlaugen.* Math. Zeitschrift, **48** (1942), 268–274.

Matematički fakultet,
Studentski trg 16,
Beograd, Serbia

(Received October 25, 2004)