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ON EXTENSIONS OF TWO MAPPINGS ASSOCIATED WITH HERMITE–HADAMARD'S INEQUALITIES FOR CONVEX FUNCTIONS

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In this paper, we introduce two new mappings closely connected with Hermite-Hadamard's inequalities for convex functions and study their main properties.

1. INTRODUCTION

Let f be a given continuous function defined on a interval $[a,b]$, $a<b$. For any $x, y \in [a,b]$ and $t \in (0,1)$, we write

$$C\big(t;x,y;f(s),s\big) = \frac{t}{(1-t)(y-x)} \int_x^{tx+(1-t)y} f(s)\,\mathrm{d}s + \frac{1-t}{t(y-x)} \int_{tx+(1-t)y}^{y} f(s)\,\mathrm{d}s,$$

where, $x \neq y$. When $x = y$, $C\big(t;x,x;f(s),s\big) = f(x)$.

When f is a continuous convex function on $[a,b]$, the author of this paper showed in [**1**] and [**2**] that the following inequalities hold true:

$$f\big(ta+(1-t)b\big) \le C\big(t;a,b;f(s),s\big) \le tf(a)+(1-t)f(b). \tag{1.1}$$

We define two mappings H and h by $H : (0,1)\times[a,b]\times[a,b] \to \mathbb{R}$, if

$$\begin{aligned} H(t;x,y) &= (y-x)\big(tf(x)+(1-t)f(y)\big) - \frac{t}{1-t}\int_x^{tx+(1-t)y} f(s)\,\mathrm{d}s \\ &\quad - \frac{1-t}{t}\int_{tx+(1-t)y}^{y} f(s)\,\mathrm{d}s \\ &= (y-x)\Big(tf(x)+(1-t)f(y) - C\big(t;x,y;f(s),s\big)\Big) \end{aligned}$$

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and $h : (0,1) \times [a,b] \times [a,b] \to \mathbb{R}$, if

$$\begin{aligned} h(t;x,y) = \frac{t}{1-t}\int_x^{tx+(1-t)y} f(s)\,\mathrm{d}s + \frac{1-t}{t}\int_{tx+(1-t)y}^{y} f(s)\,\mathrm{d}s \\ -(y-x)f\big(tx+(1-t)y\big) \\ = (y-x)\Big(C\big(t;x,y;f(s),s\big) - f\big(tx+(1-t)y\big)\Big), \end{aligned}$$

they are differences generated by the inequalities (1.1).

If $t = 1/2$, then inequalities (1.1), $H(t;x,y)$ and $h(t;x,y)$ reduce to

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s \le \frac{f(a)+f(b)}{2}, \tag{1.2}$$

$$\widetilde{H}(x,y) = (y-x)\big(f(x)+f(y)\big) - 2\int_x^y f(s)\,\mathrm{d}s$$

and

$$\widetilde{h}(x,y) = \int_x^y f(s)\,\mathrm{d}s - (y-x)f\left(\frac{x+y}{2}\right),$$

respectively.

The (1.1) are called HERMITE-HADAMARD's inequalities (see [**3**] and [**4**]). $\widetilde{H}(x,y)$ and $\widetilde{h}(x,y)$ are differences generated by the inequalities (1.2).

In [**5**], S. S. DRAGOMIR and R. P. AGARWAL gave some properties of $\widetilde{H}(a,y)$ and $\widetilde{h}(a,y)$ with $y \in [a,b]$; in [**6**], the author of this paper showed some properties of $\widetilde{H}(x,b)$ and $\widetilde{h}(x,b)$ with $x \in [a,b]$ and obtained some refinements of (1.2).

The aim of this paper is to study the main properties of $H(t;x,y)$ and $h(t;x,y)$, and then obtain some refinements of (1.1).

2. MAIN RESULTS

The main properties of $H(t;x,y)$ are given in the following two theorems:

Theorem 2.1. *Let f be a continuous convex function defined on $[a,b]$. For any $t \in (0,1)$, then we have the following*:

(1) *$H(t;a,y)$ is nonnegative and monotonically increasing with y on $[a,b]$, $H(t;x,b)$ is nonnegative and monotonically decreasing with x on $[a,b]$;*

(2) *For any $x \in (a,b)$, we have the following three refinements of the right side in* (1.1) :

$$\begin{aligned} & C\big(t;a,b;f(s),s\big) \\ & \le \frac{x-a}{b-a}\Big(tf(a)+(1-t)f(x) - C\big(t;a,x;f(s),s\big)\Big) + C\big(t;a,b;f(s),s\big) \\ & \le tf(a)+(1-t)f(b), \end{aligned} \tag{2.1}$$

$$
\begin{aligned}
(2.2)\quad & C\big(t;a,b;f(s),s\big)\\
&\le \frac{b-x}{b-a}\Big(tf(x)+(1-t)f(b)-C\big(t;x,b;f(s),s\big)\Big)+C\big(t;a,b;f(s),s\big)\\
&\le tf(a)+(1-t)f(b)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.3)\quad & C\big(t;a,b;f(s),s\big)\\
&\le \frac{1}{2}\bigg(\frac{x-a}{b-a}\,tf(a)+\frac{b-x}{b-a}\,(1-t)f(b)+\Big(\frac{b-x}{b-a}t+\frac{x-a}{b-a}(1-t)\Big)f(x)\\
&\quad -\frac{x-a}{b-a}\,C\big(t;a,x;f(s),s\big)-\frac{b-x}{b-a}\,C\big(t;x,b;f(s),s\big)\bigg)+C\big(t;a,b;f(s),s\big)\\
&\le tf(a)+(1-t)f(b).
\end{aligned}
$$

Theorem 2.2. *Let f be a continuous convex function defined on $[a,b]$. For any $\alpha\in(0,1)$, then we have the following*:

(1) *When $1/2\le t<1$, $H(t;a,y)$ is convex with y on $[a,b]$ and we have the following refinement of the right side in* (1.1) :

$$
\begin{aligned}
(2.4)\quad & C\big(t;a,b;f(s),s\big)\\
&\le tf(a)+(1-t)f\big(\alpha a+(1-\alpha)b\big)-C\big(t;a,\alpha a+(1-\alpha)b;f(s),s\big)\\
&\qquad\qquad +C\big(t;a,b;f(s),s\big)\\
&\le tf(a)+\frac{1-t}{(1-\alpha)(b-a)}\,C\big(\alpha;a,b;(x-a)f(x),x\big)\\
&\quad -\frac{1}{(1-\alpha)(b-a)}\,C\Big(\alpha;a,b;(x-a)C\big(t;a,x;f(s),s\big),x\Big)+C\big(t;a,b;f(s),s\big)\\
&\le tf(a)+(1-t)f(b);
\end{aligned}
$$

(2) *When $0<t\le 1/2$, $H(t;x,b)$ is convex with x on $[a,b]$ and we have the following refinement of the right side in* (1.1) :

$$
\begin{aligned}
(2.5)\quad & C\big(t;a,b;f(s),s\big)\\
&\le tf\big(\alpha a+(1-\alpha)b\big)+(1-t)f(b)-C\big(t;\alpha a+(1-\alpha)b,b;f(s),s\big)\\
&\qquad +C\big(t;a,b;f(s),s\big)\\
&\le (1-t)f(b)+\frac{t}{\alpha(b-a)}\,C\big(\alpha;a,b;(b-x)f(x),x\big)\\
&\quad -\frac{1}{\alpha(b-a)}\,C\Big(\alpha;a,b;(b-x)C\big(t;x,b;f(s),s\big),x\Big)+C\big(t;a,b;f(s),s\big)\\
&\le tf(a)+(1-t)f(b).
\end{aligned}
$$

Remark 1. The conditions "$0<t<1/2$" and "$1/2<t<1$" do not imply convexity of $H(t;a,y)$ and $h(t;x,b)$ on $[a,b]$, respectively. Indeed, we have the following simple counterexample:

Example. Let $k = 1 + 1 \times 10^{-11}$, $t_1 = 0.002523$ and $t_2 = 1 - t_1$. Then $0 < t_1 < 1/2$, $1/2 < t_2 < 1$, $\big((k+1)t_1 - t_1^2(1-t_1)^{k-1} - 1 + (1-t_1)^{k+1}\big) = \big((k+1)(1-t_2) -(1-t_2)^2 t_2^{k-1} - 1 + t_2^{k+1}\big) < 0$ and $f(s) = |s|^k$ is convex on $[-100, 100]$. Hence,

$$\begin{aligned} H(t_1; 0, y) &= (y-0)\big(t_1 0^k + (1-t_1)y^k\big) - \frac{t_1}{1-t_1}\int_0^{(1-t_1)y} s^k\,\mathrm{d}s - \frac{1-t_1}{t_1}\int_{(1-t_1)y}^{y} s^k\,\mathrm{d}s \\ &= \frac{1-t_1}{t_1(k+1)}\left((k+1)t_1 - t_1^2(1-t_1)^{k-1} - 1 + (1-t_1)^{k+1}\right) y^{k+1} \end{aligned}$$

is concave with y on $[0, 100]$ and

$$\begin{aligned} H(t_2; x, 0) &= (0-x)\big(t_2(-x)^k + (1-t_2)0^k\big) - \frac{t_2}{1-t_2}\int_x^{t_2 x} (-s)^k\,\mathrm{d}s - \frac{1-t_2}{t_2}\int_{t_2 x}^{0} (-s)^k\,\mathrm{d}s \\ &= \frac{t_2}{(1-t_2)(k+1)}\big((k+1)(1-t_2) - (1-t_2)^2 t_2^{k-1} - 1 + t_2^{k+1}\big)(-x)^{k+1} \end{aligned}$$

is concave with x on $[-100, 0]$.

The main properties of $h(t; x, y)$ are embodied in the following theorem:

Theorem 2.3. *Let f be a continuous convex function defined on $[a, b]$. For any $t \in (0, 1)$, we have the following*:

(1) *$h(t; a, y)$ is nonnegative and monotonically increasing with y on $[a, b]$, $h(t; x, b)$ is nonnegative and monotonically decreasing with x on $[a, b]$.*

(2) *We have the inequality*:

$$h(t; x, y) \le H(t; x, y), \qquad a \le x < y \le b. \tag{2.6}$$

(3) *For any $x \in (a, b)$, we have the following three refinements of the left side in* (1.1) :

$$\begin{aligned} & f\big(ta + (1-t)b\big) \\ & \le \frac{x-a}{b-a}\Big(C\big(t; a, x; f(s), s\big) - f\big(ta + (1-t)x\big)\Big) + f\big(ta + (1-t)b\big) \\ & \le C\big(t; a, b; f(s), s\big), \end{aligned} \tag{2.7}$$

$$\begin{aligned} & f\big(ta + (1-t)b\big) \\ & \le \frac{b-x}{b-a}\Big(C\big(t; x, b; f(s), s\big) - f\big(tx + (1-t)b\big)\Big) + f\big(ta + (1-t)b\big) \\ & \le C\big(t; a, b; f(s), s\big) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned}(2.9)\quad & f\big(ta+(1-t)b\big)\\ &\le \frac{1}{2(b-a)}\bigg((x-a)\Big(C\big(t;a,x;f(s),s\big)-f\big(ta+(1-t)x\big)\Big)\\ &\qquad +(b-x)\Big(C\big(t;x,b;f(s),s\big)-f\big(tx+(1-t)b\big)\Big)\bigg)+f\big(ta+(1-t)b\big)\\ &\le C\big(t;a,b;f(s),s\big).\end{aligned}$$

Remark 2. When we choose $t = 1/2$, (2.1) and (2.7) reduce to (2) and (10) in [**5**], (2.2)-(2.3) and (2.8)-(2.9) reduce to (12)-(13) and (15)-(16) in [**6**], respectively.

Towards proving these theorems we shall need the following lemma:

Lemma 2.4. *Let g be a continuous function defined on $[a, b]$. Then we have the following*:

(1) *Let g'_- and g'_+ exist on (a, b). When $g'_- \ge 0$ and $g'_+ \ge 0$, g is monotonically increasing on $[a, b]$. When $g'_- \le 0$ and $g'_+ \le 0$, g is monotonically decreasing on $[a, b]$* (see [**7**]).

(2) *If g'_+ exist and it is monotonically increasing on (a, b), then g is convex on $[a, b]$* (see [**6**–**7**]).

3. PROOFS OF THEOREMS

Proof of Theorem 2.1. (1) The fact that $H(t; a, y)$ and $H(t; x, b)$ are nonnegative follows from (1.1).

By the continuity of f, $H(t; a, y)$ with y and $H(t; x, b)$ with x are continuous on $[a, b]$.

For any $x, y \in (a, b)$ and $t \in (0, 1)$, the right derivative of $H(t; a, y)$ with y and $H(t; x, b)$ with x are:

$$\begin{aligned}(3.1)\quad & H'_+(t;a,y)=tf(a)+(1-t)f(y)+(y-a)(1-t)f'_+(y)\\ &-tf\big(ta+(1-t)y\big)-\frac{1-t}{t}\Big(f(y)-(1-t)f\big(ta+(1-t)y\big)\Big)\\ &=\frac{1}{t}\Big(t(1-t)(y-a)f'_+(y)+t^2f(a)-(1-t)^2f(y)-(2t-1)f\big(ta+(1-t)y\big)\Big)\end{aligned}$$

and

$$\begin{aligned}(3.2)\quad & H'_+(t;x,b)=-\big(tf(x)+(1-t)f(b)\big)+(b-x)tf'_+(x)\\ &-\frac{t}{1-t}\Big(tf\big(tx+(1-t)b\big)-f(x)\Big)+(1-t)f\big(tx+(1-t)b\big)\\ &=\frac{1}{1-t}\Big(t(1-t)(b-x)f'_+(x)-(1-t)^2f(b)+t^2f(x)\\ &\qquad +(1-2t)f\big(tx+(1-t)b\big)\Big),\end{aligned}$$

respectively.

Using (3.1) and convexity of f, we get

$$
\begin{aligned}
H'_+(t;a,y) &= \frac{1}{t}\Big(t^2\Big(f(a)-f\big(ta+(1-t)y\big)\Big) \\
&\quad +(1-t)^2\Big(f\big(ta+(1-t)y\big)-f(y)\Big)+t(1-t)(y-a)f'_+(y)\Big) \\
&\geq \frac{1}{t}\Big(t^2(1-t)(a-y)f'_+\big(ta+(1-t)y\big)+t(1-t)^2(a-y)f'_+(y) \\
&\qquad\qquad +t(1-t)(y-a)f'_+(y)\Big) \\
&= t(1-t)(y-a)\Big(f'_+(y)-f'_+\big(ta+(1-t)y\big)\Big)\geq 0.
\end{aligned}
\tag{3.3}
$$

By the same method as in the proof of (3.3), we can show that the left derivative of $H(t;a,y)$ with y satisfies

$$H'_-(t;a,y)\geq 0, \qquad y\in(a,b). \tag{3.4}$$

From (3.3)–(3.4) and (1) in the Lemma 2.4, we get that $H(t;a,y)$ is monotonically increasing with y on $[a,b]$.

Using (3.2) and the convexity of f, we have

$$
\begin{aligned}
H'_+(t;x,b) &= \frac{1}{1-t}\Big(t^2\Big(f(x)-f\big(tx+(1-t)b\big)\Big) \\
&\quad +(1-t)^2\Big(f\big(tx+(1-t)b\big)-f(b)\Big)+t(1-t)(b-x)f'_+(x)\Big) \\
&\leq \frac{1}{1-t}\Big(t^2(1-t)(x-b)f'_+(x)+t(1-t)^2(x-b)f'_+\big(tx+(1-t)b\big) \\
&\qquad\qquad +t(1-t)(b-x)f'_+(x)\Big) \\
&= t(1-t)(b-x)\Big(f'_+(x)-f'_+\big(tx+(1-t)b\big)\Big)\leq 0.
\end{aligned}
\tag{3.5}
$$

By the same method as in the proof of (3.5), we can show that the left derivative of $H(t;x,b)$ with x satisfies

$$H'_-(t;x,b)\leq 0, \qquad x\in(a,b). \tag{3.6}$$

From (3.5)–(3.6) and (1) in the Lemma 2.4, we get that $H(t;x,b)$ is monotonically decreasing with x on $[a,b]$.

(2) For any $x\in(a,b)$, from the monotonically increasing properties of $H(t;a,y)$ with y on $[a,b]$ and the mapping $H(t;x,y)$, we have

$$
\begin{aligned}
0=H(t;a,a)\leq H(t;a,x) &= (x-a)\Big(tf(a)+(1-t)f(x)-C\big(t;a,x;f(s),s\big)\Big) \\
&\leq H(t;a,b)=(b-a)\Big(tf(a)+(1-t)f(b)-C\big(t;a,b;f(s),s\big)\Big),
\end{aligned}
$$

which implies the inequalities (2.1).

From the monotonically decreasing properties of $H(t;x,b)$ with x on $[a,b]$ and the same method as in the proof of (2.1), we can prove (2.2). Expression of (2.1) plus (2.2) and a simple manipulation yields (2.3).

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. (1) For $1/2 \le t < 1$, the continuity of $H(t;a,y)$ with y on $[a,b]$ has been proved in the proof of Theorem 2.1.

For $\forall y_1, y_2 \in (a,b), y_1 < y_2$, from (3.1), $1/2 \le t < 1$ and convexity of f, we obtain

$$\begin{aligned}
&t\big(H'_+(t;a,y_2) - H'_+(t;a,y_1)\big)\\
&\quad = t(1-t)\big((y_2-a)f'_+(y_2) - (y_1-a)f'_+(y_1)\big) + (1-t)^2\big(f(y_1)-f(y_2)\big)\\
&\qquad +(2t-1)\Big(f\big(ta+(1-t)y_1\big) - f\big(ta+(1-t)y_2\big)\Big)\\
&\quad \ge t(1-t)(y_1-a)\big(f'_+(y_2)-f'_+(y_1)\big) + t(1-t)(y_2-y_1)f'_+(y_2)\\
&\qquad +(1-t)^2(y_1-y_2)f'_+(y_2) + (2t-1)(1-t)(y_1-y_2)f'_+\big(ta+(1-t)y_2\big)\\
&\quad = t(1-t)(y_1-a)\big(f'_+(y_2)-f'_+(y_1)\big)\\
&\qquad +(2t-1)(1-t)(y_2-y_1)\Big(f'_+(y_2) - f'_+\big(ta+(1-t)y_2\big)\Big) \ge 0,
\end{aligned}$$

which implies that $H'_+(t;a,y)$ is monotonically increasing with y on (a,b). By (2) in the Lemma 2.4, we get that $H(t;a,y)$ is convex with y on $[a,b]$.

For any $\alpha \in (0,1)$, using the nonnegativity and convexity properties of $H(t;a,y)$ and (1.1), we obtain

$$\begin{aligned}
(3.7)\qquad 0 &\le \frac{1}{(1-\alpha)(b-a)}\, H\big(t;a,\alpha a+(1-\alpha)b\big)\\
&\le \frac{1}{(1-\alpha)(b-a)}\, C\big(\alpha;a,b;H(t;a,y),y\big)\\
&\le \frac{1}{(1-\alpha)(b-a)}\, \big(\alpha H(t;a,a) + (1-\alpha)H(t;a,b)\big).
\end{aligned}$$

From mapping $H(t;x,y)$, we get

$$\begin{aligned}
(3.8)\qquad &\frac{1}{(1-\alpha)(b-a)}\, H\big(t;a,\alpha a+(1-\alpha)b\big)\\
&\quad = tf(a) + (1-t)f\big(\alpha a+(1-\alpha)b\big) - C\big(t;a,\alpha a+(1-\alpha)b;f(s),s\big),
\end{aligned}$$

$$(3.9)\qquad \frac{1}{(1-\alpha)(b-a)}\, C\big(\alpha;a,b;H(t;a,y),y\big)$$

$$= \frac{1}{(1-\alpha)(b-a)} \Big(tf(a)C\big(\alpha; a, b; (y-a), y\big)$$
$$+(1-t)C\big(\alpha; a, b; (y-a)f(y), y\big) - C\Big(\alpha; a, b; (y-a)C\big(t; a, y; f(s), s\big), y\Big)\Big)$$
$$= tf(a) + \frac{1-t}{(1-\alpha)(b-a)}\, C\big(\alpha; a, b; (x-a)f(x), x\big)$$
$$- \frac{1}{(1-\alpha)(b-a)}\, C\Big(\alpha; a, b; (x-a)C\big(t; a, x; f(s), s\big), x\Big)$$

and

$$\frac{1}{(1-\alpha)(b-a)} \big(\alpha H(t; a, a) + (1-\alpha)H(t; a, b)\big) \tag{3.10}$$
$$= \frac{1}{b-a}\, H(t; a, b) = tf(a) + (1-t)f(b) - C\big(t; a, b; f(s), s\big).$$

Combining (3.7)–(3.10), a simple manipulation yields (2.4).

(2) For $0 < t \le 1/2$, the continuity of $H(t; x, b)$ with x on $[a, b]$ has been proved in the proof of Theorem 2.1.

For $\forall x_1, x_2 \in (a, b), x_1 < x_2$, from (3.2), $0 < t \le 1/2$ and convexity of f, we obtain

$$\begin{aligned}
&(1-t)\big(H'_+(t; x_2, b) - H'_+(t; x_1, b)\big)\\
&\quad = t(1-t)(b-x_2)\big(f'_+(x_2) - f'_+(x_1)\big) - t(1-t)(x_2-x_1)f'_+(x_1)\\
&\qquad\qquad + t^2\big(f(x_2) - f(x_1)\big)\\
&\qquad + (1-2t)\Big(f\big(tx_2 + (1-t)b\big) - f\big(tx_1 + (1-t)b\big)\Big)\\
&\quad \ge t(1-t)(b-x_2)\big(f'_+(x_2) - f'_+(x_1)\big) - t(1-t)(x_2-x_1)f'_+(x_1)\\
&\qquad + t^2(x_2-x_1)f'_+(x_1) + (1-2t)t(x_2-x_1)f'_+\big(tx_1 + (1-t)b\big)\\
&\quad = t(1-t)(b-x_2)\big(f'_+(x_2) - f'_+(x_1)\big)\\
&\qquad + (1-2t)t(x_2-x_1)\Big(f'_+\big(tx_1 + (1-t)b\big) - f'_+(x_1)\Big) \ge 0,
\end{aligned}$$

which implies that $H'_+(t; x, b)$ is monotonically increasing with x on (a, b). By (2) in the Lemma 2.4, we get that $H(t; x, b)$ is convex with x on $[a, b]$.

Using the nonnegativity and convexity properties of $H(t; x, b)$ and the same method as in the proof of (2.4), we can prove (2.5).

This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. (1) The fact that $h(t; a, y)$ and $h(t; x, b)$ are nonnegative follows from (1.1).

By the continuity of f, $h(t; a, y)$ with y and $h(t; x, b)$ with x are continuous on $[a, b]$.

For any $x \in (a,b)$, using convexity of f, the left derivative of $h(t;x,b)$ with x holds the following

$$
\begin{aligned}
(3.11)\quad h'_-(t;x,b) &= \frac{t}{1-t}\Big(tf\big(tx+(1-t)b\big)-f(x)\Big)-(1-t)f\big(tx+(1-t)b\big)\\
&\quad +f\big(tx+(1-t)b\big)-(b-x)tf'_-\big(tx+(1-t)b\big)\\
&= \frac{t}{1-t}\Big(f\big(tx+(1-t)b\big)-f(x)\Big)-(b-x)tf'_-\big(tx+(1-t)b\big)\\
&\le \frac{t}{1-t}\Big(-(1-t)(x-b)f'_-\big(tx+(1-t)b\big)\Big)-(b-x)tf'_-\big(tx+(1-t)b\big)=0.
\end{aligned}
$$

By the same method as in the proof of (3.11), we can show that the right derivative of $h(t;x,b)$ with x in (a,b) satisfies

$$
(3.12)\qquad h'_+(t;x,b) \le 0, \qquad x \in (a,b).
$$

From (3.11)–(3.12) and (1) in the Lemma 2.4, we get that $h(t;x,b)$ is monotonically decreasing with x on $[a,b]$.

By the same method as in the proof of (3.11), we can prove

$$
(3.13)\qquad h'_+(t;a,y) \ge 0, \quad h'_-(t;a,y) \ge 0, \qquad y \in (a,b).
$$

Using (3.13) and (1) in the Lemma 2.4, we get that $h(t;a,y)$ is monotonically increasing with y on $[a,b]$.

(2) By (1.2) and $x<y$, we have

$$
\begin{aligned}
(3.14)\qquad & \frac{t}{1-t}\int_x^{tx+(1-t)y} f(s)\,\mathrm{d}s\\
&\le \frac{t}{1-t}\big(tx+(1-t)y-x\big)\,\frac{f\big(tx+(1-t)y\big)+f(x)}{2}\\
&= t(y-x)\,\frac{f\big(tx+(1-t)y\big)+f(x)}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.15)\qquad & \frac{1-t}{t}\int_{tx+(1-t)y}^{y} f(s)\,\mathrm{d}s\\
&\le \frac{1-t}{t}\Big(y-\big(tx+(1-t)y\big)\Big)\,\frac{f\big(tx+(1-t)y\big)+f(y)}{2}\\
&= (1-t)(y-x)\,\frac{f\big(tx+(1-t)y\big)+f(y)}{2}.
\end{aligned}
$$

Expression of (3.14) plus (3.15) and a simple manipulation we obtain

$$
\begin{aligned}
&2\Big(\frac{t}{1-t}\int_x^{tx+(1-t)y} f(s)\,\mathrm{d}s+\frac{1-t}{t}\int_{tx+(1-t)y}^{y} f(s)\,\mathrm{d}s\Big)\\
&\qquad \le (y-x)\Big(tf(x)+(1-t)f(y)+f\big(tx+(1-t)y\big)\Big),
\end{aligned}
$$

which implies the inequality (2.6).

(3) For any $x \in (a,b)$, from the monotonically decreasing property of $h(t;x,b)$ with x on $[a,b]$ and the definition of the mapping h, we have

$$\begin{aligned} 0 = h(t;b,b) \le h(t;x,b) &= (b-x)\Big(C\big(t;x,b;f(s),s\big) - f\big(tx+(1-t)b\big)\Big) \\ \le h(t;a,b) &= (b-a)\Big(C\big(t;a,b;f(s),s\big) - f\big(ta+(1-t)b\big)\Big), \end{aligned}$$

which implies the inequalities (2.8).

From the monotonically increasing property of $h(t;a,y)$ with y on $[a,b]$ and the same method as in the proof of (2.8), we can prove (2.7). Expression of (2.7) plus (2.8) and a simple manipulation yields (2.9).

This completes the proof of Theorem 2.3.

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