Univ. Beograd. Publ. Elektrotehn. Fak.
Available electronically at **http: //matematika.etf.bg.ac.yu**
Ser. Mat. 17 (2006), 1–7.

# OSCILLATIONS OF WEAKLY NONLINEAR SYSTEMS WITH NO FREQUENCIES DUE TO NONLINEARITY

*Alexandr K. Demenchuk*

We consider a nonlinear differential system of the form $\dot{x} = Ax + f(t) + \mu F(t,x)$, where $f$ and $F$ are almost periodic in $t$ and $\mu$ is a small parameter. Suppose that the frequency moduli $\mathrm{Mod}\,(f)$ and $\mathrm{Mod}\,(F)$ have a zero intersection. Our aim is to give the existence conditions and the construction procedure for the solution $x$ of this system such that $\mathrm{Mod}\,(x) \cap \mathrm{Mod}\,(F) = \{0\}$.

## 1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{R}^{n\times m}$ be the space of real $n \times m$-matrices with the norm $|\cdot|$, $\mathbb{R}^{n\times 1} = \mathbb{R}^n$, and $D$ be a compact subset of $\mathbb{R}^n$. Following A. Fink [**7**] we denote the space of Borh almost periodic functions $h : \mathbb{R} \to \mathbb{R}^{n\times m}$ by $AP(\mathbb{R}^{n\times m})$. By $AP(D, \mathbb{R}^{n\times m})$ we denote the space of functions $H : \mathbb{R} \times D \to \mathbb{R}^{n\times m}$ such that each $H \in AP(D, \mathbb{R}^{n\times m})$ is continuous on $\mathbb{R} \times D$ and almost periodic in $t \in \mathbb{R}$ uniformly for $x \in D$. The space $AP(\mathbb{R}^{n\times m})$ endowed with the norm $\|h\| = \sup\{|h(t)| : \ t \in \mathbb{R}\}$ becomes a Banach space. For $H \in AP(D, \mathbb{R}^{n\times m})$ we put

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T H(s,x)\,\mathrm{d}s = \widehat{H}(x), \quad H(t,x) - \widehat{H}(x) = H_*(t,x).$$

**Definition 1.** *The frequency module* $\mathrm{Mod}\,(h_1, \dots, h_k, H_1, \dots, H_l)$ *of* $h_i \in AP(\mathbb{R}^n)$ $(i = \overline{1,k})$ *and* $H_j \in AP(D, \mathbb{R}^n)$ $(j = \overline{1,l};\ k + l \geq 1)$ *is the smallest additive group of real numbers that contains all Fourier exponents of this functions.*

Almost periodic solutions of the general system

$$\dot{x} = g(t,x), \quad t \in \mathbb{R}, \tag{1}$$

2000 Mathematics Subject Classification: 34C27
Keywords and Phrases: Almost periodic, differential systems, solutions.

with $g \in AP(D, \mathbb{R}^n)$ were studied in a number of papers, see e.g. [**3**], [**7**], [**9**], [**14**] and the bibliography therein. However, a little inquiry into the matter shows that the most of this investigations are implemented under presupposition that $\mathrm{Mod}\,(x) \subseteq \mathrm{Mod}\,(g)$, where $x(t)$ is the required solution.

The analogous presupposition is frequently used in studying of periodic differential systems and their periodic solutions, though H. Massera [**13**], J. Kurzweil and O. Veivoda [**11**], N. Erugin [**6**], E. Grudo [**8**] and others have shown that some $\omega$-periodic differential systems possess $\Omega$-periodic solutions $x$ such that $\omega$ and $\Omega$ are incommensurable, i.e. $\mathrm{Mod}\,(x) \cap \mathrm{Mod}\,(g) = \{0\}$.

It should be noted that these solutions should not treated as some pathological object. On the contrary, one can easily discover that such oscillations are very natural for the mechanical system of two pendulums connected by a resilient coupling with periodically varying stiffness. These oscillations can be also discovered in electrical systems, for example in the system consisting of two oscillatory circuits connected by the condense with periodically varying capacity.

In [**4**], [**5**], we give the following definitions.

**Definition 2.** *An almost periodic solution $x(t)$ of system* (1) *is called irregular if* $\mathrm{Mod}\,(x) \cap \mathrm{Mod}\,(g) = \{0\}$.

**Definition 3.** *Let* $\mathrm{Mod}\,(g)$ *is splitted into direct sum of two submoduli $L_1$, $L_2$, i.e.* $\mathrm{Mod}\,(g) = L_1 \oplus L_2$.

a) *An almost periodic solution $x(t)$ of the system* (1) *is called irregular with respect to $L_1$* (*or partially irregular*) *if* $(\mathrm{Mod}\,(x) + L_2) \cap L_1 = \{0\}$.

b) *An irregular with respect to submodule $L_1$ almost periodic solution $x(t)$ of the system* (1) *is called weakly $L_1$-irregular* (*or weakly irregular*) *if* $\mathrm{Mod}\,(x) \subseteq L_2$.

In the present paper we consider a weakly nonlinear system

$$\dot{x} = Ax + f(t) + \mu F(t, x), \quad t \in \mathbb{R}, \quad x \in D \subset \mathbb{R}^n, \tag{2}$$

where $A \in \mathbb{R}^{n \times n}$, $f \in AP(\mathbb{R}^n)$, $F \in AP(D, \mathbb{R}^n)$, and $\mu$ is a small real parameter. Regular solutions of (2) were studied in [**1**], [**2**], [**3**], [**7**], [**9**], [**15**].

Let us suppose that

$$\mathrm{Mod}\,(f) \cap \mathrm{Mod}\,(F) = \{0\}. \tag{3}$$

The aim of this paper is to obtain the existence conditions and some construction procedure for irregular with respect to $\mathrm{Mod}\,(F)$ almost periodic solutions of system (2) under condition (3). By Definition 3 we have $(\mathrm{Mod}\,(x) + \mathrm{Mod}\,(f)) \cap \mathrm{Mod}\,(F) = \{0\}$ for any such solution. Now it follows from (3) that $\mathrm{Mod}\,(x) \cap \mathrm{Mod}\,(F) = \{0\}$. This means that the frequencies of nonlinear part of the system do not influence $\mathrm{Mod}\,(x)$.

## 2. THE CASE $\widehat{F}(x) \equiv 0$

Let $\lambda_1(A), \ldots, \lambda_n(A)$ be the eigenvalues of $A$. Denote Green's function of the system $\dot{x} = Ax$ by $G_A(t)$.

**Theorem 1.** *Suppose that system* (2) *satisfies condition* (3) *and*
i) $\operatorname{Re}\lambda_j(A) \neq 0 \quad (j = \overline{1,n})$, ii) $\widehat{F}(x) \equiv 0$.

*Then the following assertions are true.*

1) *If system* (2) *has an irregular with respect to* $\operatorname{Mod}(F)$ *almost periodic solutions* $x(t)$*, then* $x(t)$ *is weakly irregular and*

$$x(t) \equiv x^{(0)}(t), \quad x^{(0)}(t) = \int_{-\infty}^{+\infty} G_A(t-s) f(s)\, \mathrm{d}s. \tag{4}$$

2) *System* (2) *has solution* (4) *iff*

$$F(t, x^{(0)}(t)) \equiv 0. \tag{5}$$

**Proof.** Let (3), i), ii) hold and let $x(t)$ be a partially irregular almost periodic solution of system (2). It follows from [**4**] that $x(t)$ satisfies the system

$$\dot{x} = Ax + f(t) + \mu\widehat{F}(x). \tag{6}$$

Then by i), ii), and [**9**, p. 157], system (6) has a unique almost periodic solution $x^{(0)}(t)$. Therefore, we have (4). It follows from [**7**, p. 91] that $\operatorname{Mod}(x) = \operatorname{Mod}(f)$. Since $\operatorname{Mod}(f) \cap \operatorname{Mod}(F) = \{0\}$ and $\operatorname{Mod}(x) = \operatorname{Mod}(f)$, we see that $x(t)$ is weakly irregular.

By [**4**] $x(t)$ satisfies

$$F_*(t, x) = 0 \tag{7}$$

as well. From ii), (4), and (7) we can deduce (5).

Conversely, assume that the conditions of Theorem 1 hold. Putting $\widehat{F}(x) = 0$ in (6), we obtain

$$\dot{x} = Ax + f(t). \tag{8}$$

By i) system (8) has a unique almost periodic solution (4) and $\operatorname{Mod}(x) = \operatorname{Mod}(f)$. Since $x(t) \equiv x^{(0)}(t)$ satisfies (5) and (8), we see that $x(t)$ is a solution to (2). It follows from $\operatorname{Mod}(x) = \operatorname{Mod}(f)$ and (3) that this solution is weakly irregular. □

It should be noted that Theorem 1 is not valid in critical case when some of $\operatorname{Re}\lambda_j(A)$ are zero. However, some critical cases can be treated similarly. Let $\mathrm{M}\{a_1, \dots, a_p\}$ be a module formed by some real numbers $a_1, \dots, a_p$.

Suppose that

$$\operatorname{Re}\lambda_j(A) = 0 \ \ (j = \overline{1,p}), \quad \operatorname{Re}\lambda_s(A) \neq 0, \ \ (s = \overline{p+1,n}),$$

$$\operatorname{Im}\lambda_k(A) \neq \operatorname{Im}\lambda_m(A) \ \text{ for all } k \neq m, \quad k, m = \{1, \dots, p\}$$

$$(\mathrm{M}\{\operatorname{Im}\lambda_1(A), \dots, \operatorname{Im}\lambda_p(A)\} + \operatorname{Mod}(f)) \cap \operatorname{Mod}(F) = \{0\}. \tag{9}$$

Then the system

$$\dot{y} = -A^{\mathsf{T}} y \tag{10}$$

has $p$ linearly independent quasiperiodic solutions

$$y^{(1)}, \dots, y^{(p)}. \tag{11}$$

**Theorem 2.** *Suppose that* (3), ii), *and* (9) *hold. System* (2) *has a partially irregular almost periodic solution iff*

$$\left\| \int_{t_0}^{t} \sum_{k=1}^{n} y_k^{(j)}(s) f_k(s)\,\mathrm{d}s \right\| < +\infty \quad (j = \overline{1,p}) \tag{12}$$

*and*

$$F\big(t, x(t)\big) \equiv 0 \tag{13}$$

*hold, where* $x(t)$ *is an almost periodic solution of linear approximation system for* (2).

**Proof.** Let $x(t)$ be a partially irregular almost periodic solution of system (2). Then by [**4**] and ii) $x(t)$ satisfies (8). It follows from [**12**] that estimate (12) holds. Note that $x(t)$ satisfies (7) as well. Then, since $\widehat{F}\big(x(t)\big) \equiv 0$, we obtain (13).

Let us show that the opposite assertion also holds. Since $A$ has eigenvalues (9), we see that (11) satisfies (10). By [**12**], (9), and (12), the linear approximation system for (2) has an almost periodic solution $x(t)$ and $\big(\mathrm{Mod}\,(x) + \mathrm{Mod}\,(f)\big) \cap \mathrm{Mod}\,(F) = 0$, i.e. $x(t)$ is irregular with respect to $\mathrm{Mod}\,(F)$. It follows from (13) that $x(t)$ satisfies (2). Hence, system (2) has a partially irregular almost periodic solution $x(t)$. □

## 3. THE CASE $\widehat{F}(x) \not\equiv 0$

Consider the system

$$\dot{x} = Ax + f(t) + \mu \widehat{F}(x), \quad \mathrm{Re}\,\lambda_j(A) \neq 0,\ (j = \overline{1,n}), \quad \widehat{F}(x) \not\equiv 0. \tag{14}$$

By [**9**, p. 157] for $\mu = 0$ system (14) has a unique almost periodic solution

$$x^{(0)}(t) = \int_{-\infty}^{+\infty} G_A(t-s) f(s)\,\mathrm{d}s, \qquad \int_{-\infty}^{+\infty} \|G_A(s)\|\,\mathrm{d}s \le c, \quad \|x^{(0)}\| \le M_0.$$

Assume that $\widehat{F}(x)$ satisfies the Lipscitz condition

$$|\widehat{F}(x') - \widehat{F}(x'')| \le L|x'' - x'|, \quad x', x'' \in D_\rho = \{x \in \mathbb{R}^n :\ |x| \le \rho,\ \rho > 2M_0\}, \tag{15}$$

By [**15**, p. 281] (14) is equivalent to integral equation

$$x(t) = Px(t), \quad Px(t) = x^{(0)}(t) + \mu \int_{-\infty}^{+\infty} G_A(t-s) \widehat{F}\big(x(s)\big)\,\mathrm{d}s. \tag{16}$$

We take $\mathrm{AP}_\rho(\mathbb{R}^n) = \{h \in \mathrm{AP}(\mathbb{R}^n),\ \|x\| \le \rho\}$. It follows from [**3**, p. 426], i), and (15) that there is a $\mu^* > 0$ such that for $|\mu| < \mu^*$ and $x \in \mathrm{AP}_\rho(\mathbb{R}^n)$ we have $Px \subset \mathrm{AP}_\rho(\mathbb{R}^n)$, moreover $P$ is a contraction operator on $\mathrm{AP}_\rho(\mathbb{R}^n)$. Then by [**10**, p. 75] there is a unique fixed point of $P$. Consequently, (16) has a unique almost periodic solution $x(t,\mu) \in \mathrm{AP}_\rho(\mathbb{R}^n)$.

Let us show that $\mathrm{Mod}(x) \subseteq \mathrm{Mod}(f)$. From [**7**, p. 27] we have $\widehat{F}(t, x(t,\mu)) \in \mathrm{AP}(\mathbb{R}^n)$. Let $\mathrm{T}(f,\varepsilon)$ be an $\varepsilon$-translation set of $f$ and let $\tau \in \mathrm{T}(f,\varepsilon)$. Then

$$\begin{aligned}\|x(t+\tau,\mu) - x(t,\mu)\| &\le \|x^{(0)}(t+\tau,\mu) - x^{(0)}(t,\mu)\| \\ &\qquad + |\mu|\,\|\widehat{F}(x(t+\tau,\mu)) - \widehat{F}(x(t,\mu))\| \int_{-\infty}^{+\infty} \|G_A(t-s)\|\,\mathrm{d}s \\ &\le c\varepsilon + |\mu| cL \|x(t+\tau,\mu) - x(t,\mu)\|.\end{aligned}$$

It follows that

$$\|x(t+\tau,\mu) - x(t,\mu)\| < \varepsilon_1, \quad \varepsilon_1 = \varepsilon(1 - |\mu| cL)^{-1}, \quad |\mu| < (cL)^{-1}.$$

Since $\varepsilon_1$ is sufficiently small, we see that $\tau$ is an $\varepsilon_1$-almost period of $x(t,\mu)$. Therefore, for every $\varepsilon > 0$ there is an $\varepsilon_1 > 0$ such that $\mathrm{T}(f,\varepsilon) \subseteq \mathrm{T}(x,\varepsilon_1)$. Then by [**7**, p. 61] we have $\mathrm{Mod}(x) \subseteq \mathrm{Mod}(f)$.

Consequently, for $|\mu| \le \mu^{**} = \min\{\mu^*, (cL)^{-1}\}$, we obtain

$$\dot{x}(t,\mu) \equiv Ax(t,\mu) + f(t) + \mu\widehat{F}\big(x(t,\mu)\big), \quad \mathrm{Mod}(x) \subseteq \mathrm{Mod}(f). \tag{17}$$

It follows from [**9**, p. 159] that $x(t,\mu)$ can be evaluated via successive approximations method

$$x(t,\mu) = \lim_{m\to+\infty} x^{(m)}(t,\mu),$$

$$x^{(m+1)}(t,\mu) = x^{(0)}(t) + \mu \int_{-\infty}^{+\infty} G_A(t-s)\widehat{F}\big(x^{(m)}(s,\mu)\big)\,\mathrm{d}s \quad (m = 0, 1, 2, \ldots). \tag{18}$$

Suppose that

$$F_*\big(t, x(t,\mu)\big) \equiv 0. \tag{19}$$

It follows from (17) and (19) that $x(t,\mu)$ satisfies (2). Since $\mathrm{Mod}(x) \subseteq \mathrm{Mod}(f)$ and $\mathrm{Mod}(f) \cap \mathrm{Mod}(F) = \{0\}$, we see that $x(t,\mu)$ is weakly irregular.

Thus, we have proved

**Theorem 3.** *Suppose that system* (2) *satisfies conditions* (3), i), (15), *and* (19). *Then there exists a* $\mu^{**}$ *such that for* $|\mu| \le \mu^{**}$ *system* (2) *has a unique in* $D_\rho$ *weakly irregular almost periodic solution* (18).

## 4. EXAMPLES

EXAMPLE 1. Let $A \in \mathbb{R}^{2\times 2}$ $(a_{11} > 0,\ a_{21} \neq 0,\ \operatorname{Re}\lambda_j(A) \neq 0,\ j = 1,2)$, $f_1 \in \mathrm{AP}(\mathbb{R})$, $F_j \in \mathrm{AP}(D,\mathbb{R})$ $(j = 1,2)$. Suppose that

$$\widehat{F}_j(x,y) \equiv 0,\ F_j(t,x,0) \equiv 0\ (j = 1,2),\quad \mathrm{Mod}\,(f_1) \cap \mathrm{Mod}\,(F_1,F_2) = \{0\}.$$

Consider the system

$$\dot{x} = a_{11}x + a_{12}y + f_1(t) + \mu F_1(t,x,y),\quad \dot{y} = a_{21}x + a_{22}y + f_2(t) + \mu F_2(t,x,y),$$

where $f_2(t) = a_{21}\int_t^{+\infty} \exp\big(a_{11}(t-s)\big) f_1(s)\,\mathrm{d}s$, $\mu$ is a small parameter. By Theorem 1 this system has a unique irregular with respect to $\mathrm{Mod}\,(F_1,F_2)$ almost periodic solution $x(t) = -(a_{21})^{-1} f_2(t),\ y(t) = 0$.

EXAMPLE 2. Let $a,b \in \mathbb{R}$, $p,q \in \mathrm{AP}(\mathbb{R})$, and $\mathbb{Z} \cap \mathrm{Mod}\,(p,q) = \{0\}$. Consider the system

$$\dot{x} = x + ay + \cos t - \sin t + \mu\big(x^2 + p(t)h_1(x,y)\big),\quad \dot{y} = by + \mu q(t)h_2(x,y), \tag{20}$$

where $h_j(x,y)$ is continuous on $\mathbb{R}^2$ and $h_j(x,0) \equiv 0$ $(j = 1,2)$.

For (20) the linear approximation system is $2\pi-$periodic and $F_*(t,x,y) = \mathrm{col}\,[p_*(t)h_1(x,y), q_*(t)h_2(x,y)]$. Note that $y(t) \equiv 0$ satisfies $F_*(t,x,y) \equiv 0$. Putting $y = 0$ in (20), we obtain $\dot{x} = x + \cos t - \sin t + \mu x^2,\quad y = 0$. For $|\mu| < 1/4$ this system has a unique in a neighbourhood of the origin $2\pi-$periodic solution

$$x(t,\mu) = \lim_{m\to+\infty} x^{(m)}(t,\mu),\quad y = 0,$$
$$x^{(m+1)}(t,\mu) = \sin t + \mu \int_t^{+\infty} \exp\,(t-s)\big(x^{(m)}(s,\mu)\big)^2\,\mathrm{d}s\quad (m = 0,1,2,\ldots).$$

It can easily see that $x = x(t,\mu),\ y = 0$ is $2\pi$-periodic and, therefore it is a weakly $\mathrm{Mod}\,(p,q)-$irregular almost periodic solution of system (20).

Some iterations for $x(t,\mu)$ are given by

$$\begin{aligned}
x^{(0)}(t,\mu) =&\ \sin t,\\
x^{(1)}(t,\mu) =&\ \sin t - \left(\frac{1}{2} + \frac{1}{5}\sin t - \frac{1}{10}\cos 2t\right)\mu,\\
x^{(2)}(t,\mu) =&\ \sin t + \left(-\frac{1}{2} + \frac{1}{10}\cos 2t - \frac{1}{5}\sin 2t\right)\mu\\
&+ \left(\frac{9}{20}\sin t + \frac{13}{20}\cos t + \frac{1}{20}\sin 3t - \frac{1}{20}\cos 3t\right)\mu^2\\
&+ \left(-\frac{11}{40} - \frac{3}{50}\cos 2t - \frac{2}{25}\sin 2t + \frac{19}{3400}\cos 4t - \frac{1}{425}\sin 4t\right)\mu^3.
\end{aligned}$$

Here $\|x(t,\mu) - x^{(2)}(t,\mu)\| \leq 0.064$ by [**15**, p. 285].

## REFERENCES


1. Y. N. Bibikov: *On existence of quasiperiodic motions of quasilinear systems.* Prikl. Mat. Mekh., **59** (1995), 21–29. (in Russian)
2. G. I. Biryuk: *On existence theorem of almost periodic solutions for some nonlinear systems of differential equations with small parameter.* Dokl. Akad. Nauk SSSR, **96** (1954), 5–7. (in Russian)
3. B. P. Demidovich: *Lectures on Mathematical Stability Theory.* Nauka, Moskva, 1967. (in Russian)
4. A. K. Demenchuk: *Partially irregular almost periodic solutions of ordinary differential systems.* Math. Bohem., **126** (2001), 221–228.
5. A. K. Demenchuk: *Almost periodic weakly irregular solutions of ordinary nonlinear differential systems.* Differ. Uravn., **37** (2001), 730–734. (in Russian); English translation: Differ. Equ., **37** (2001), 763–767.
6. N. P. Erugin: *Linear Systems of Ordinary Differential Equations with Periodic and Quasiperiodic Coefficients.* Izdatelstvo Akad. Nauk BSSR, Minsk, 1963. (in Russian)
7. A. M. Fink: *Almost Periodic Differential Equations.* Lecture Notes in Math., **377**, Springer, Berlin, 1974.
8. E. I. Grudo : *On periodic solutions with incommensurable periods of periodic differential systems.* Differ. Uravn. **22** (1986), 1409–1506. (in Russian). English translation: Differ. Equ., **22** (1986), 1036–1041.
9. J. K. Hale: *Oscillations in Nonlinear Systems.* Mc Graw-Hill, New York, 1963.
10. A. N. Kolmogorov, S. V. Fomin: *Elements of Functional Theory and Functional Analysis.* Nauka, Moskva, 1981. (in Russian)
11. J. Kurzweil, O. Veivoda: *On periodic and almost periodic solutions of a system of ordinary differential systems.*, Czehoslovak Math. J., **5** (1955), 362–370.
12. I. G. Malkin: *On resonance in quasiharmonic systems.* Prikl. Mat. Mekh., **18** (1954), 459–463. (in Russian)
13. J. L. Massera: *Observaciones sobre les soluciones periodicas de ecuaciones differentiales.* Boletin de la Facultad de Ingenieria Montevideo, **4**, (1950) 37–45.
14. A. M. Samoilenko: *The elements of mathematical theory of multifrequency oscillations.* Nauka, Moskva, 1987.(in Russian)
15. V. I. Zubov: *Oscillation Theory.* Nauka, Moskva, 1979. (in Russian)


Department of Differential Equations,
Institute of Mathematics,
National Academy of Sciences of Belarus,
Surganova, 11, 220072, Minsk, Belarus
E–mail: demenchuk@im.bas-net.by

(Received February 2, 2004)