Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 16 (2005), 77–87.
Available electronically at **http: //pefmath.etf.bg.ac.yu**

RELATIONSHIPS BETWEEN HOMOGENEITY, SUBADDITIVITY AND CONVEXITY PROPERTIES

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By using the reasonings of M. Kuczma, R. A. Rosenbaum, Z. Gajda and Z. Kominek, we establish some intimate connections between various homogeneity, subadditivity and convexity properties of a real-valued function of a vector space.

INTRODUCTION

Throughout this paper, p will denote a real-valued function of a vector space X over $\mathbb{R}$. Though, most of the forthcoming definitions and theorems could be naturally extended to some more general situations.

For some $\Lambda \subset \mathbb{R}$, the function p will be called Λ–subhomogeneous if

$$p(\lambda x) \leq \lambda p(x)$$

for all $\lambda \in \Lambda$ and $x \in X$. Moreover, p will be called Λ–convex if

$$p\big(\lambda x + (1-\lambda)y\big) \leq \lambda p(x) + (1-\lambda)p(y)$$

for all $\lambda \in \Lambda$ and $x, y \in X$. The Λ–superhomogeneous and Λ–concave functions are to be defined by the reversed inequalities.

In particular, a $\{\lambda\}$–subhomogeneous function will be simply called λ–subhomogeneous. Moreover, a -1–subhomogeneous function will be called subodd. The λ–superhomogeneous and superodd functions are to be defined analogously.

2000 Mathematics Subject Classification: 39B72
Keywords and Phrases: Subhomogeneous, subadditive and convex functions.

For instance, we shall show that if p is subadditive in the sense that

$$p(x+y) \leq p(x) + p(y)$$

for all $x, y \in X$, then p is superodd, $\mathbb{N}$–subhomogeneous and $\{0\} \cup \mathbb{N}^{-1}$–superhomogeneous. While if p subodd and subadditive, then p is already additive, and thus $\mathbb{Q}$–homogeneous.

Moreover, by using the reasonings of Rosenbaum [**12**] and Gajda and Kominek [**5**], we prove that if p is 2–subhomogeneous and 2^{-1}–convex, then p is subadditive. While, if p is 2–superhomogeneous and subadditive, then p is 2^{-1}–convex and $\{0\} \cup \mathbb{Q}_+$–homogeneous.

In this respect, it is also worth mentioning that if in particular p is $\mathbb{N}$–subhomogeneous and convex, then p is $[1, +\infty[$–subhomogeneous and $]0, 1]$–superhomogeneous. Moreover, $p(\lambda x) \leq -\lambda p(-x)$ for all $\lambda \in]-\infty, -1]$ and $x \in X$.

1. Λ–SUBHOMOGENEOUS FUNCTIONS

As a straightforward extension of the usual notion of homogeneous functions, we may naturally introduce the following

Definition 1.1. *The function p will be called Λ–subhomogeneous for some $\Lambda \subset \mathbb{R}$ if*

$$p(\lambda x) \leq \lambda(x)$$

for all $\lambda \in \Lambda$ and $x \in X$. If the inequality is reversed, then p will be called Λ–superhomogeneous.

Remark 1.2. Note that if p is $\{\lambda\}$–subhomogeneous for some $\lambda \in \mathbb{R}$, then p may be simply called λ–subhomogeneous.

Moreover, the function p may be naturally called absolutely Λ–subhomogeneous if $p(\lambda x) \leq |\lambda|\, p(x)$ for all $\lambda \in \Lambda$ and $x \in X$.

Remark 1.3. Note also that the function p may be called Λ–homogeneous if it is both Λ–subhomogeneous and Λ–superhomogeneous.

Moreover, p is Λ–superhomogeneous if and only if $-p$ is Λ–subhomogeneous. Therefore, Λ–superhomogeneous functions need not be studied separately.

Concerning Λ–subhomogeneous functions, we can easily establish the following two theorems.

Theorem 1.4. *If p is Λ–subhomogeneous for some $\Lambda \subset \mathbb{R}$, then $p(\lambda x) \leq -\lambda p(-x)$ for all $\lambda \in -\Lambda$ and $x \in X$.*

Proof. If $\lambda \in -\Lambda$, then $-\lambda \in \Lambda$. Therefore, $p(\lambda x) = p\big((-\lambda)(-x)\big) \leq (-\lambda)p(-x)$ $= -\lambda p(-x)$ for all $x \in X$.

Theorem 1.5. *If p is Λ–subhomogeneous for some $\Lambda \subset]0, +\infty[$, then p is Λ^{-1}–superhomogeneous.*

Proof. If $\lambda \in \Lambda^{-1}$, then $\lambda^{-1} \in \Lambda$. Therefore, $p(x) = p\big(\lambda^{-1}\lambda x\big) \leq \lambda^{-1}p(\lambda x)$, and hence $\lambda p(x) \leq p(\lambda x)$ for all $x \in X$.

From the latter theorem and its dual, it is clear that in particular we have

Corollary 1.6. *If* $\Lambda \subset]0, +\infty[$ *such that* $\Lambda = \Lambda^{-1}$, *then the following assertions are equivalent*:

(1) *p is* Λ*–homogeneous*;

(2) *p is* Λ*–subhomogeneous*;

(3) *p is* Λ*–superhomogeneous.*

Analogously to Theorem 1.5, we can also prove the following

Theorem 1.7. *If p is* Λ*–subhomogeneous for some* $\Lambda \subset]-\infty, 0[$, *then p is* Λ^{-1}*–subhomogeneous.*

Moreover, it is also worth proving the following

Theorem 1.8. *If p is* $\Lambda_1 \cup \Lambda_2$*–subhomogeneous for some* $\Lambda_1 \subset [0, +\infty[$ *and* $\Lambda_2 \subset \mathbb{R}$, *then p is* $\Lambda_1\Lambda_2$*–subhomogeneous.*

Proof. If $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$, then $p(\lambda_2 x) \leq \lambda_2 p(x)$, and thus $p(\lambda_1\lambda_2 x) \leq \lambda_1 p(\lambda_2 x) \leq \lambda_1\lambda_2 p(x)$ for all $x \in X$.

From the above theorem, by induction, we can immediately get

Corollary 1.9. *If p is* Λ*–subhomogeneous for some* $\Lambda \subset [0, +\infty[$, *then p is* Λ^n*–subhomogeneous for all* $n \in \mathbb{N}$.

2. FURTHER PROPERTIES OF SUBHOMOGENEOUS FUNCTIONS

In the sequel, we shall also need the next obvious

Theorem 2.1. *The following assertions are equivalent*:

(1) $p(0) \leq 0$; (2) *p is* 0*–subhomogeneous.*

Proof. Clearly, $p(0) \leq 0$ if and only if $p(0x) \leq 0\,p(x)$ for all $x \in X$. Therefore, (1) and (2) are equivalent.

By using this theorem and its dual, we can also easily prove the following two theorems.

Theorem 2.2. *If p is* λ*–subhomogeneous for some* $\lambda < 1$, *then p is* 0*–subhomogeneous.*

Theorem 2.3. *If p is* λ*–subhomogeneous for some* $\lambda > 1$, *then p is* 0*–superhomogeneous.*

Proof. In particular, $p(0) = p(\lambda 0) \leq \lambda p(0)$. Therefore, $0 \leq (\lambda - 1)\,p(0)$, and hence $0 \leq p(0)$. Thus, by the dual of Theorem 2.1, p is 0–superhomogeneous.

Since $\mathbb{Q}_+ = \mathbb{Q}_+^{-1}$, by Corollary 1.6 and Theorem 2.3, it is clear that we also have the following

Theorem 2.4. *The following assertions are equivalent*:

(1) *p is* $\mathbb{Q}_+$*–subhomogeneous*;

(2) *p is* $\mathbb{Q}_+$*–superhomogeneous*;

(3) *p is $\{0\} \cup \mathbb{Q}_+$–homogeneous.*

Hint. If (2) holds, then by Corollary 1.6 it is clear that p is actually $\mathbb{Q}_+$–homogeneous. Moreover, from Theorem 2.3 and its dual, we can see that p is, in addition, 0–homogeneous. Therefore, (3) also holds.

Now, by using our former observations, we can also prove the following

Theorem 2.5. *If p is $\mathbb{N}$–homogeneous, then p is $\{0\} \cup \mathbb{Q}_+$–homogeneous.*

Proof. In this case, p is both $\mathbb{N}$–subhomogeneous and $\mathbb{N}$–superhomogeneous. Therefore, by the dual of Theorem 1.5, p is N^{-1}–subhomogeneous. Hence, by Theorem 1.8 and the fact that $\mathbb{Q}_+ = \mathbb{N}\,\mathbb{N}^{-1}$, it is clear that p is $\mathbb{Q}_+$–subhomogeneous. Therefore, by Theorem 2.4, p is $\{0\} \cup \mathbb{Q}_+$–homogeneous.

Moreover, by Theorem 2.4, it is clear that we also have the following

Theorem 2.6. *The following assertions are equivalent:*

(1) *p is $\mathbb{Q}$–homogeneous;*

(2) *p is $\mathbb{Q}_+$–subhomogeneous and $\mathbb{Q}_-$–homogeneous;*

(3) *p is $\mathbb{Q}_+$–superhomogeneous and $\mathbb{Q}_-$–homogeneous.*

Remark 2.7. Note that, because of $\mathbb{R}_+ = \mathbb{R}_+^{-1}$, in Theorems 2.4 and 2.6 we may write $\mathbb{R}$ in place of $\mathbb{Q}$.

Moreover, it is also worth mentioning that p is absolutely homogeneous if and only if it is absolutely $\mathbb{R} \setminus \{0\}$–subhomogeneous ($\mathbb{R} \setminus \{0\}$–superhomogeneous).

3. HOMOGENEITY PROPERTIES OF SUBODD FUNCTIONS

By using Definition 1.1, we may easily formulate the following

Definition 3.1. *The function p will be called subodd if it is -1–subhomogeneous. Moreover, p is called superodd if it is -1–superhomogeneous.*

Remark 3.2. Note that thus p is subodd if and only if $p(-x) \leq -p(x)$ for all $x \in X$.

Moreover, p is superodd if and only if $-p$ is subodd. That is, the reversed inequality holds.

Remark 3.3. Note also that, in contrast to Definition 3.1, the subeven and supereven functions need not be introduced.

Namely, we can at once seen that the function p is even if and only if it is absolutely -1–subhomogeneous (-1–superhomogeneous).

By Theorem 2.2, we evidently have the following

Theorem 3.4. *If p is subodd, then p is 0–subhomogeneous.*

Moreover, by using Theorem 1.4, we can easily establish the following

Theorem 3.5. *If p is subodd and Λ–subhomogeneous for some $\Lambda \subset [0, +\infty[$, then p is $-\Lambda$–subhomogeneous.*

Proof. If $\lambda \in -\Lambda$, then by Theorem 1.4 and the inequalities $p(-x) \leq -p(x)$ and $0 \leq -\lambda$, it is clear that $p(\lambda x) \leq -\lambda p(-x) \leq -\lambda\big(-p(x)\big) = \lambda p(x)$ for all $x \in X$. Therefore, p is $-\Lambda$–subhomogeneous.

Now, as a useful consequence of Theorems 3.4 and 3.5, can also state

Theorem 3.6. *If p is subodd and $\Lambda \subset \mathbb{R}$ such that $\Lambda = -\Lambda$, then the following assertions are equivalent*:

(1) *p is Λ–subhomogeneous*; (2) *p is Λ_+–subhomogeneous.*

Proof. By Theorem 3.4, p is 0–subhomogeneous. Moreover, we can easily see that

$$\Lambda \setminus \{0\} = \Lambda_- \cup \Lambda_+ = -\Lambda_+ \cup \Lambda_+.$$

Hence, by Theorem 3.5, it is clear that p is $\Lambda \setminus \{0\}$–subhomogeneous if and only if (2) holds. Therefore, (1) and (2) are also equivalent.

From the latter theorem, it is clear that in particular we also have

Corollary 3.7. *If $\Lambda \subset \mathbb{R}$ such that $1 \in \Lambda$ and $\Lambda = -\Lambda$, then the following assertions are equivalent*:

(1) *p is Λ–subhomogeneous*;

(2) *p is subodd and Λ_+–subhomogeneous.*

Moreover, from Theorems 2.5 and 3.6, we can immediately get the following

Theorem 3.8. *If p is subodd and $\mathbb{N}$–homogeneous, then p is $\mathbb{Q}$–subhomogeneous.*

Proof. By Theorem 2.5, p is $\mathbb{Q}_+$–subhomogeneous. Hence, by Theorem 3.6, p is $\mathbb{Q}$–subhomogeneous.

Now, as an immediate consequence of the latter theorem and its dual we can also state

Corollary 3.9. *If p is odd and $\mathbb{N}$–homogeneous, then p is $\mathbb{Q}$–homogeneous.*

Analogously to Theorem 3.5, we can also prove the following

Theorem 3.10. *If p is superodd and Λ–subhomogeneous for some $\Lambda \subset]-\infty, 0]$, then p is $-\Lambda$–subhomogeneous.*

Remark 3.11. Hence, as a counterpart of Theorem 3.6, we can only state that if p is superodd and $\Lambda \subset \mathbb{R}$ such that $\Lambda = -\Lambda$, then p is $\Lambda \setminus \{0\}$–subhomogeneous if and only if it is Λ_-–subhomogeneous.

Finally, we note that by Theorem 1.4 we also have the following

Theorem 3.12. *If p is a even and Λ–subhomogeneous for some $\Lambda \subset \mathbb{R}$, then $p(\lambda x) \leq -\lambda p(x)$ for all $\lambda \in -\Lambda$ and $x \in X$.*

Remark 3.13. Hence, as a counterpart of Theorem 3.6, we can only state that if p is even and $\Lambda \subset \mathbb{R}$ such that $\Lambda = -\Lambda$, then p is absolutely $\Lambda \setminus \{0\}$–subhomogeneous if and only if it is Λ_+–subhomogeneous.

Moreover, from the above theorem, we can also get the following

Corollary 3.14. *If p is even and subodd, then $p(x) \leq 0$ for all $x \in X$.*

Proof. By Definition 3.1, p is -1–subhomogeneous. Therefore, by Theorem 3.12, for any $x \in X$ we have $p(x) \leq -p(x)$. Hence, $2p(x) \leq 0$, and thus $p(x) \leq 0$.

4. HOMOGENEITY PROPERTIES OF SUBADDIVITY FUNCTIONS

According to Hille [**6**, p. 131] Rosenbaum [**12**, p. 227], we may also have the following

Definition 4.1. *The function p will be called subadditive if*

$$p(x+y) \le p(x) + p(y)$$

for all $x, y \in X$. If the inequality is reversed, then p will be called superadditive.

Remark 4.2. Note that thus p is additive if and only if it is both subadditive and superadditive.

Moreover, p is superadditive if and only if $-p$ is subadditive. Therefore, superadditive functions need not be studied separately.

By using the reasonings of Kuczma [**7**, pp. 401–402] and our former observations, we can easily prove the following two theorems.

Theorem 4.3. *If p is subadditive, then*

(1) *p is superodd*;
(2) *p is $\mathbb{N}$–subhomogeneous*;
(3) *p is $\{0\} \cup \mathbb{N}^{-1}$–superhomogeneous.*

Proof. Clearly, $p(0) = p(0+0) \le p(0) + p(0)$, and thus $0 \le p(0)$. Therefore, by Theorem 2.1, p is 0–superhomogeneous.

Moreover, if $x \in X$, then we have $0 \le p(0) = p(-x+x) \le p(-x) + p(x)$. Therefore, $-p(x) \le p(-x)$, and thus (1) also holds.

On the other hand, if $p(nx) \le np(x)$ for some $n \in \mathbb{N}$, then we also have

$$p\big((n+1)x\big) = p(nx+x) \le p(nx) + p(x) \le np(x) + p(x) = (n+1)p(x).$$

Hence, by the induction principle, it is clear (2) also holds.

Finally, to complete the proof, we note that from (2), by Theorem 1.5, it follows that p is, in addition, $\mathbb{N}^{-1}$–superhomogeneous. Therefore, (3) is also true.

Theorem 4.4. *The following assertions are equivalent*:

(1) *p is additive*;
(2) *p is odd and subadditive*;
(3) *p is subodd and subadditive.*

Proof. From Theorem 4.3 (1) and its dual, we can see that an additive function is odd. Hence, it is clear that the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are true.

Moreover, if (3) holds, then we can see that

$$p(x) = p(x+y-y) \le p(x+y) + p(-y) \le p(x+y) - p(y),$$

and hence $p(x) + p(y) \le p(x+y)$. Therefore, (1) also holds.

From the above two theorems, by using Corollary 3.9, we can easily derive the following

Corollary 4.5. *If p is subodd and subadditive, then p is $\mathbb{Q}$–homogeneous.*

Proof. By Theorem 4.4, p is odd and additive. Moreover, by Theorem 4.3 (2) and its dual p is $\mathbb{N}$–homogeneous. Thus, Corollary 3.9 can be applied.

In addition to the above theorems, it is also worth proving the following

Theorem 4.6. *If p is subadditive and $\Lambda_1 \cup \Lambda_2$–subhomogeneous for some $\Lambda_1, \Lambda_2 \subset \mathbb{R}$, then p is $\Lambda_1 + \Lambda_2$–subhomogeneous.*

Proof. If $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$, then by the hypotheses of the theorem we evidently have

$$\begin{aligned} p\big((\lambda_1 + \lambda_2)x\big) &= p(\lambda_1 x + \lambda_2 x) \leq p(\lambda_1 x) + p(\lambda_2 x) \leq \lambda_1 p(x) + \lambda_2 p(x) \\ &= (\lambda_1 + \lambda_2) p(x) \end{aligned}$$

for all $x \in X$.

5. HOMOGENEITY PROPERTIES OF 0–SUBHOMOHENEOUS CONVEX FUNCTIONS

Analogously to Definition 1.1, we may also naturally introduced the following

Definition 5.1. *The function p will be called Λ–convex for some $\Lambda \subset \mathbb{R}$ if*

$$p\big(\lambda x + (1-\lambda)y\big) \leq \lambda p(x) + (1-\lambda)p(y)$$

for all $\lambda \in \Lambda$ and $x, y \in X$. If the inequality is reversed, then p will be called Λ–concave.

Remark 5.2. Note that thus p may be called Λ–affine if it is both Λ–convex and Λ–concave.

Moreover, p is Λ–concave if and only if $-p$ is Λ–convex. Therefore, Λ–concave functions need not be studied separately.

Concerning Λ–convex functions, we can easily establish the following

Theorem 5.3. *If p is 0–subhomogeneous and Λ–convex for some $\Lambda \subset]-\infty, 1]$, then p is Λ–subhomogeneous.*

Proof. By Theorem 2.1, $p(0) \leq 0$. Hence, by the Λ–convexity of p, it is clear that

$$p(\lambda x) = p\big(\lambda x + (1-\lambda)\big) \leq \lambda p(x) + (1-\lambda)p(0) \leq \lambda p(x)$$

for all $\lambda \in \Lambda$ $x \in X$.

Now, by using Theorems 5.3, 1.4 and 1.5, we can also prove the following

Theorem 5.4. *If p is 0–subhomogeneous and convex, then*

(1) *p is $[0, 1]$–subhomogeneous;*

(2) *p is $[1,+\infty[$–superhomogeneous;*
(3) *$p(\lambda x) \le -\lambda p(-x)$ for all $\lambda \in [-1,0]$ and $x \in X$.*

Proof. In this case, p is $[0,1]$–convex. Therefore, (1) is a particular case of Theorem 5.3.

From (1), since $-[0,1] = [-1,0]$ and $]0,1]^{-1} = [1,+\infty[$, by Theorems 1.4 and 1.5 it is clear that (3) and (2) are also true.

From the above theorem, by using Theorems 3.4 and 3.5, we can get the following two corollaries.

Corollary 5.5. *If p is subodd and convex, then p is $[-1,1]$–subhomogeneous.*

Proof. By Theorem 3.4, p is 0–subhomogeneous. Thus, by Theorem 5.4 (1), p is $[0,1]$–subhomogeneous. Hence, by Theorem 3.5, p is $[-1,0]$–subhomogeneous. Therefore, the required assertion is also true.

Corollary 5.6. *If p is superodd, 0–subhomogeneous and convex, then p is $\mathbb{R}\setminus]-1,0[$–superhomogeneous.*

Proof. By Theorem 5.4 (2), p is $[1,+\infty[$–superhomogeneous. Hence, by the dual of Theorem 3.5, p is $]-\infty,-1]$–superhomogeneous. Therefore, the required assertion is also true.

Moreover, as an immediate consequence of Theorem 5.4 (3), we can also state

Corollary 5.7. *If p is even, 0–subhomogeneous and convex, then $p(\lambda x) \le -\lambda p(x)$ for all $\lambda \in [-1,0]$.*

6. HOMOGENEITY PROPERTIES OF 2–HOMOGENEOUS CONVEX FUNCTIONS

By using a reasoning of Rosenbaum [**12**, p. 234], we can prove the following

Theorem 6.1. *If p is 2–subhomogeneous and 2^{-1}–convex, then p is subadditive.*

Proof. By the above hypotheses, we evidently have

$$\begin{aligned} p(x+y) &= p\Big(2\big(2^{-1}x + (1-2^{-1})y\big)\Big) \le 2p\big(2^{-1}x + (1-2^{-1})y\big) \\ &\le 2\big(2^{-1}p(x) + (1-2^{-1})p(y)\big) = p(x) + p(y) \end{aligned}$$

for all $x, y \in X$. Therefore, p is subadditive.

From the above theorem, by Theorem 4.3, it is clear that in particular we have

Corollary 6.2. *If p is 2-subhomogeneous and 2^{-1}–convex, then p is superodd, $\mathbb{N}$–subhomogeneous and $\{0\} \cup \mathbb{N}^{-1}$–superhomogeneous.*

In this respect, it is also worth mentioning that, by using another reasoning of Rosenbaum [**12**, p. 235], we can also prove the following partial converse of Theorem 6.1.

Theorem 6.3. *If p is 2–superhomogeneous and subadditive, then p is 2^{-1}–convex.*

Proof. In this case, by the dual of Theorem 1.5, p is 2^{-1}–subhomogeneos. Now, by the 2^{-1}–convexity of p, it is clear that

$$\begin{aligned} p\big(2^{-1}x+(1-2^{-1})y\big) &= p\big(2^{-1}(x+y)\big) \leq 2^{-1}p(x+y) \\ &\leq 2^{-1}\big(p(x)+p(y)\big) = 2^{-1}p(x)+(1-2^{-1})p(y) \end{aligned}$$

for all $x, y \in X$. Therefore, p is 2^{-1}–convex.

Now, as an immediate consequence of Theorems 6.1 and 6.3, we can also state

Corollary 6.4. *If p is 2–homogeneous, then p is subadditive if and only if p is 2^{-1}–convex.*

However, it is now more interesting that by using a reasoning of Gajda and Kominek [**5**, p. 27] we can also prove the following

Theorem 6.5. *If p is 2–superhomogeneous and subadditive, then p is $\{0\} \cup \mathbb{Q}_+$–homogeneous.*

Proof. To prove this, by Theorem 2.5, it is enough to show only that p is $\mathbb{N}$–homogeneous. For this, note that, by Theorem 4.3, p is $\mathbb{N}$–subhomogeneous. Moreover, by the dual of Corollary 1.9, p is 2^n–superhomogeneous for all $n \in \mathbb{N}$. Therefore, if the required assertion does not hold, then there exist $x \in X$ and $1 < k \in \mathbb{N} \setminus \{2^n\}_{n=1}^{\infty}$ such that $p(kx) < kp(x)$.

Now, by defining

$$m = \max\{n \in \mathbb{N} :\ 2^n \leq k\} \quad \text{and} \quad r = k - 2^m,$$

we can state that

$$k = 2^m + r \quad \text{and} \quad r \in \{1, 2, \dots, 2^m - 1\}.$$

Moreover, we can note that

$$2^{m+1} = 2^m + 2^m = 2^m + r + 2^m - r = k + 2^m - r \quad \text{and} \quad 2^m - r \in \mathbb{N}.$$

Hence, by our former observations, it is clear that

$$\begin{aligned} (2^m + 2^m)p(x) &= 2^{m+1}p(x) \leq p(2^{m+1}x) = p\big((k + 2^m - r)x\big) \\ &= p\big(kx + (2^m - r)x\big) \leq p(kx) + p\big((2^m - r)x\big) \\ &\leq p(kx) + \big(2^m - r\big)p(x), \end{aligned}$$

and thus $2^m p(x) \leq p(kx) - rp(x)$. Therefore, we also have

$$kp(x) = (2^m + r)p(x) = 2^m p(x) + rp(x) \leq p(kx).$$

This contradiction proves the $\mathbb{N}$–homogeneity of p.

Now, as an immediate consequence of Theorems 6.1 and 6.5, we can also state

Corollary 6.6. *If p is 2–homogeneous and 2^{-1}–convex, then p is $\{0\} \cup \mathbb{Q}_+$–homogeneous.*

7. HOMOGENEITY PROPERTIES OF SUBHOMOGENEOUS CONVEX FUNCTIONS

A simple application of Theorems 5.3, 1.5 and 1.4 gives the following

Theorem 7.1. *If p is $\{0\}$–subhomogeneous and convex, then*

(1) *p is $[0,1]$–subhomogeneous;*
(2) *p is $[1,+\infty[$–superhomogeneous;*
(3) *$p(\lambda x) \le -\lambda p(-x)$ for all $\lambda \in [-1,0]$ and $x \in X$.*

Proof. From Theorem 5.3, we can see that p is $[0,1]$–subhomogeneous. Moreover, from (1), by using Theorem 1.5 and 1.4, we can see that (2) and (3) are also true.

By using a reasoning of Rosenbaum [**12**, p. 234], we can also prove the following

Lemma 7.2. *If p is convex, then*

$$p(\lambda x) \le \langle\lambda\rangle\, p\big(([\lambda]+1)x\big) + \big(1-\langle\lambda\rangle\big)p\big([\lambda]x\big).$$

for all $\lambda \in \mathbb{R}$ and $x \in X$.

Proof. If $\lambda \in \mathbb{R}$, then under the notation

$$k = [\lambda] = \max\{l \in \mathbb{Z} :\ l \le \lambda\} \quad \text{and} \quad t = \langle\lambda\rangle = \lambda - [\lambda],$$

we have not only $k \in \mathbb{Z}$ and $0 \le t < 1$, but also

$$\lambda = \langle\lambda\rangle + [\lambda] = t + k = t(k+1-k) + k = t(k+1) + (1-t)k.$$

Hence, by the convexity of p, it is already clear that

$$p(\lambda x) = p\big(t(k+1)x + (1-t)kx\big) \le tp\big((k+1)x\big) + (1-t)p\big(kx\big)$$

for all $x \in X$. Therefore, the required inequality is also true.

Now, by using the above lemma, we can also prove an improvement of the "only if part" of Rosenbaum's [**12**, Theorem 1.3.6].

Theorem 7.3. *If p is $\mathbb{N}$–subhomogeneous and convex, then*

(1) *p is $[1,+\infty[$–subhomogeneous;*
(2) *p is $]0,1]$–superhomogeneous;*
(3) *$p(\lambda x) \le -\lambda p(-x)$ for all $\lambda \in]-\infty,-1]$ and $x \in X$.*

Proof. If $\lambda \in [1,+\infty[$, and moreover

$$k = [\lambda] \quad \text{and} \quad t = \langle\lambda\rangle,$$

then by Lemma 7.2 and the $\mathbb{N}$–subhomogenity of p it is clear that

$$\begin{aligned} p(\lambda x) &\le tp\big((k+1)x\big) + (1-t)p(kx) \\ &\le t(k+1)p(x) + (1-t)kp(x) = (t+k)p(x) = \lambda p(x) \end{aligned}$$

for all $x \in X$. Therefore, (1) is true.

Moreover, from (1), by using Theorem 1.5 and 1.4, we can see that (2) and (3) are also true.

Now, as an immediate consequence of Theorems 5.3 and 7.2, we can also state

Corollary 7.4. *If p is $\{0\} \cup \mathbb{N}$–subhomogeneous and convex, then*

(1) *p is $[0, +\infty[$–subhomogeneous;*
(2) *p is $]0, +\infty[$–superhomogeneous;*
(3) *$p(\lambda x) \le -\lambda p(-x)$ for all $\lambda \in] -\infty, 0]$ and $x \in X$.*

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(November 25, 2004)