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ENDPOINT CONTINUITY FOR MULTILINEAR LITTLEWOOD-PALEY OPERATORS

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In this paper, we prove the endpoint continuity for multilinear LITTLEWOOD-PALEY operators.

1. INTRODUCTION AND RESULTS

Throughout this paper, $M(f)$ will denote the HARDY-LITTLEWOOD maximal function of f, Q will denote a cube of R^n. For a cube Q, denote that $f_Q = |Q|^{-1}\int_Q f(x)\,\mathrm{d}x$ and $f^{\#}(x) = \sup\limits_{x\in Q}|Q|^{-1}\int_Q |f(y)-f_Q|\,\mathrm{d}y$.

Fix $\delta > 0$. Let ψ be a function on R^n which satisfies the following properties:

(1) $\int \psi(x)\,\mathrm{d}x = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y)-\psi(x)| \le C|y|(1+|x|)^{-(n+2-\delta)}$ when $2|y| < |x|$;

Let m be a positive integer and A be a function on R^n. The multilinear LITTLEWOOD-PALEY operator is defined by

$$g_\mu^A(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{1/2}, \quad \mu > 1,$$

where

$$F_t^A(f)(x,y) = \int_{R^n} \frac{f(z)\psi_t(y-z)}{|x-z|^m} R_{m+1}(A;x,z)\,\mathrm{d}z,$$

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

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and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. We denote that $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\mu(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu} \left| F_t(f)(y) \right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \right)^{1/2},$$

which is the LITTLEWOOD-PALEY operator (see [**17**]).

Let H be the HILBERT space $H = \Big\{ h : ||h|| = \big(\iint_{R_+^{n+1}} |h(t)|^2 \,\mathrm{d}y\mathrm{d}t/t^{n+1} \big)^{1/2} < \infty \Big\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0, +\infty)$ to H, and it is clear that

$$g_\mu^A(f)(x) = \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^A(f)(x,y) \right\|$$

and

$$g_\mu(f)(x) = \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t(f)(y) \right\|.$$

We also consider the variant of g_μ^A, which is defined by

$$\widetilde{g}_\mu^A(f)(x) = \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu} \left| \widetilde{F}_t^A(f)(x,y) \right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \right)^{1/2}, \quad \mu > 1,$$

where

$$\widetilde{F}_t^A(f)(x,y) = \int_{R^n} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} \psi_t(y-z) f(z) \,\mathrm{d}z$$

and

$$Q_{m+1}(A;x,z) = R_m(A;x,z) - \sum_{|\alpha|=m} \frac{1}{\alpha!} D^\alpha A(x)(x-z)^\alpha.$$

Note that when $m = 0$ and $\delta = 0$, g_μ^A is just the commutator of LITTLEWOOD-PALEY operator (see [**1**], [**14**], [**15**]). It is well known that multilinear operators, as the extension of commutators, are of great interest in harmonic analysis and have been widely studied by many authors (see [**3–6**], [**8**], [**9**]). In [**2**], [**7**], the $L^p (p > 1)$ boundedness of commutators generated by the CALDERON-ZYGMUND operator or fractional integral operator and BMO functions are obtained, in[**11**], [**16**], the endpoint boundedness of commutators generated by the CALDERON-ZYGMUND operator and BMO functions are obtained. The main purpose of this paper is to discuss the boundedness properties of the multilinear LITTLEWOOD-PALEY operators for the extreme cases of p.

We shall prove the following theorems in Section 3.

Theorem 1. *Let $0 \le \delta < n$ and $D^\alpha A \in BMO(R^n)$ for $|\alpha| = m$. Then g_μ^A maps $L^{n/\delta}(R^n)$ continuously into $BMO(R^n)$.*

Theorem 2. *Let* $0 \le \delta < n$ *and* $D^\alpha A \in BMO(R^n)$ *for* $|\alpha| = m$. *Then* g_μ^A *maps* $H^1(R^n)$ *continuously into weak* $L^{n/(n-\delta)}(R^n)$.

Theorem 3. *Let* $0 \le \delta < n$ *and* $D^\alpha A \in BMO(R^n)$ *for* $|\alpha| = m$. *Then* $\widetilde{g}_\mu^A$ *maps* $H^1(R^n)$ *continuously into* $L^{n/(n-\delta)}(R^n)$.

Theorem 4. *Let* $0 \le \delta < n$ *and* $D^\alpha A \in BMO(R^n)$ *for* $|\alpha| = m$.

(i) *If for any* H^1*-atom* a *supported on certain cube* Q *and* $u \in 3Q \setminus 2Q$, *there is*

$$\int_{(4Q)^c} \left\| \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \times \times \sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-u)^\alpha}{|x-u|^m} \int_Q \psi_t(y-z) D^\alpha A(z) a(z)\,\mathrm{d}z \right\|^{n/(n-\delta)} \mathrm{d}x \le C,$$

then g_μ^A *maps* $H^1(R^n)$ *continuously into* $L^{n/(n-\delta)}(R^n)$;

(ii) *If for any cube* Q *and* $u \in 3Q \setminus 2Q$, *there is*

$$\frac{1}{|Q|} \int_Q \left\| \left(\frac{t}{t+|x-y|}\right)^{n\lambda/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} |D^\alpha A(x) -(D^\alpha A)_Q| \int_{(4Q)^c} \frac{(u-z)^\alpha}{|u-z|^m} \psi_t(y-z) f(z)\,\mathrm{d}z \right\| \mathrm{d}x \le C||f||_{L^{n/\delta}},$$

then $\widetilde{g}_\mu^A$ *maps* $L^{n/\delta}(R^n)$ *continuously into* $BMO(R^n)$.

Remark. In general, g_μ^A is not continuous from H^1 to $L^{n/(n-\delta)}$.

2. SOME LEMMAS

We begin with two preliminary lemmas.

Lemma 1. (see [**6**]) *Let* A *be a function on* R^n *and* $D^\alpha A \in L^q(R^n)$ *for* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(A; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\widetilde{Q}(x,y)|} \int_{\widetilde{Q}(x,y)} |D^\alpha A(z)|^q\,\mathrm{d}z \right)^{1/q},$$

where $\widetilde{Q}(x,y)$ *is the cube centered at* x *and having side length* $5\sqrt{n}\,|x-y|$.

Lemma 2. *Let* $0 \le \delta < n$, $1 < p < n/\delta$ *and* $D^\alpha A \in BMO(R^n)$ *for* $|\alpha| = m$, $1 < r \le \infty$, $1/q = 1/p + 1/r - \delta/n$. *Then* g_μ^A *is bound from* $L^p(R^n)$ *to* $L^q(R^n)$, *that is*

$$||g_\mu^A(f)\|_{L^q} \le C \sum_{|\alpha|=m} \|D^\alpha A||_{BMO} \|f\|_{L^p}.$$

Proof. By MINKOWSKI inequality and the condition of ψ, we have

$$
\begin{aligned}
g_\mu^A(f)(x) \le & \int_{R^n} \frac{|f(z)|\,|R_{m+1}(A;x,z)|}{|x-z|^m} \\
& \left(\int_{R_+^{n+1}} |\psi_t(y-z)|^2 \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}}\right)^{1/2} \mathrm{d}z \\
\le & \, C\int_{R^n} \frac{|f(z)|\,|R_{m+1}(A;x,z)|}{|x-z|^m} \\
& \left(\int_0^\infty \int_{R^n} \frac{t^{-2n+2\delta}}{(1+|y-z|/t)^{2n+2-2\delta}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{\mathrm{d}y\mathrm{d}t}{t^{1+n}}\right)^{1/2} \mathrm{d}z \\
\le & \quad C\int_{R^n} \frac{|f(z)|\,|R_{m+1}(A;x,z)|}{|x-z|^m} \\
& \left(\int_0^\infty \left(t^{-n}\int_{R^n} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{\mathrm{d}y}{(t+|y-z|)^{2n+2-2\delta}}\right) t\,\mathrm{d}t\right)^{1/2} \mathrm{d}z,
\end{aligned}
$$

noting that

$$
\begin{aligned}
t^{-n}\int_{R^n} \left(\frac{t}{t+|x-y|}\right)^{n\mu} \frac{\mathrm{d}y}{(t+|y-z|)^{2n+2-2\delta}} &\le CM\left(\frac{1}{(t+|x-z|)^{2n+2-2\delta}}\right) \\
&\le C\,\frac{1}{(t+|x-z|)^{2n+2-2\delta}}
\end{aligned}
$$

and

$$
\int_0^\infty \frac{t\,\mathrm{d}t}{(t+|x-z|)^{2n+2-2\delta}} = C|x-z|^{-2n+2\delta},
$$

we obtain

$$
\begin{aligned}
g_\mu^A(f)(x) &\le C\int_{R^n} \frac{|f(z)|}{|x-z|^m} |R_{m+1}(A;x,z)| \left(\int_0^\infty \frac{t\,\mathrm{d}t}{(t+|x-z|)^{2n+2-2\delta}}\right)^{1/2} \mathrm{d}z \\
&= C\int_{R^n} \frac{|f(z)|\,|R_{m+1}(A;x,z)|}{|x-z|^{m+n-\delta}}\,\mathrm{d}z,
\end{aligned}
$$

thus, the lemma follows from [**8**], [**9**].

3. PROOF OF THEOREMS

Proof of Theorem 1. It is only to prove that there exists a constant C_Q such that

$$
\frac{1}{|Q|}\int_Q |g_\mu^A(f)(x) - C_Q|\,\mathrm{d}x \le C||f||_{L^{n/\delta}}
$$

holds for any cube Q. Fix a cube $Q = Q(x_0, l)$. Let $\widetilde{Q} = 5\sqrt{n}\,Q$ and $\widetilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\widetilde{Q}} x^\alpha$, then $R_m(A;x,y) = R_m(\widetilde{A};x,y)$ and $D^\alpha \widetilde{A} = D^\alpha A - (D^\alpha A)_{\widetilde{Q}}$

for $|\alpha|=m$. We write $F_t^A(f)=F_t^A(f_1)+F_t^A(f_2)$ for $f_1=f\chi_{\widetilde{Q}}$ and $f_2=f\chi_{R^n\setminus\widetilde{Q}}$, then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \left|g_\mu^A(f)(x)-g_\mu^A(f_2)(x_0)\right|\mathrm{d}x\\
&=\frac{1}{|Q|}\int_Q\Bigg|\Big\|\Big(\frac{t}{t+|x-y|}\Big)^{n\mu/2}F_t^A(f)(x,y)\Big\|\\
&\qquad\qquad-\Big\|\Big(\frac{t}{t+|x_0-y|}\Big)^{n\mu/2}F_t^A(f_2)(x_0,y)\Big\|\Bigg|\mathrm{d}x\\
&\le\frac{1}{|Q|}\int_Q g_\mu^A(f_1)(x)\,\mathrm{d}x+\frac{1}{|Q|}\int_Q\Big\|\Big(\frac{t}{t+|x-y|}\Big)^{n\mu/2}F_t^A(f_2)(x,y)\\
&\qquad\qquad-\Big(\frac{t}{t+|x_0-y|}\Big)^{n\mu/2}F_t^A(f_2)(x_0,y)\Big\|\mathrm{d}x\\
&:=I(x)+II(x).
\end{aligned}$$

Now, let us estimate I and II . First, taking $p>1$ and $q>1$ such that $1/q=1/p-\delta/n$, by the (L^p,L^q) boundedness of g_μ^A (Lemma 2), we gain

$$I\le\left(\frac{1}{|Q|}\int_Q\left(g_\mu^A(f_1)(x)\right)^q\mathrm{d}x\right)^{1/q}\le C|Q|^{-1/q}\|f_1\|_{L^p}\le C\|f\|_{L^{n/\delta}}.$$

To estimate II, we write

$$\begin{aligned}
&\Big(\frac{t}{t+|x-y|}\Big)^{n\mu/2}F_t^A(f_2)(x,y)-\Big(\frac{t}{t+|x_0-y|}\Big)^{n\mu/2}F_t^A(f_2)(x_0,y)\\
&=\int\Big(\frac{1}{|x-z|^m}-\frac{1}{|x_0-z|^m}\Big)\Big(\frac{t}{t+|x-y|}\Big)^{n\mu/2}\psi_t(y-z)R_m(\widetilde{A};x,z)f_2(z)\,\mathrm{d}z\\
&\quad+\int\frac{\psi_t(y-z)f_2(z)}{|x_0-z|^m}\Big(\frac{t}{t+|x-y|}\Big)^{n\mu/2}\big(R_m(\widetilde{A};x,z)-R_m(\widetilde{A};x_0,z)\big)\,\mathrm{d}z\\
&\quad+\int\Big(\Big(\frac{t}{t+|x-y|}\Big)^{n\mu/2}-\Big(\frac{t}{t+|x_0-y|}\Big)^{n\mu/2}\Big)\frac{\psi_t(y-z)R_m(\widetilde{A};x_0,z)f_2(z)}{|x_0-z|^m}\,\mathrm{d}z\\
&\qquad-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int\Big(\Big(\frac{t}{t+|x-y|}\Big)^{n\mu/2}\frac{(x-z)^\alpha}{|x-z|^m}\\
&\qquad-\Big(\frac{t}{t+|x_0-y|}\Big)^{n\mu/2}\frac{(x_0-z)^\alpha}{|x_0-z|^m}\Big)\psi_t(y-z)D^\alpha\widetilde{A}(z)f_2(z)\,\mathrm{d}z\\
&:=II_1^t(x)+II_2^t(x)+II_3^t(x)+II_4^t(x).
\end{aligned}$$

We choose $r>1$ such that $1/r+\delta/n=1$, note that $|x-z|\sim|x_0-z|$ for $x\in\widetilde{Q}$

and $z \in R^n \setminus \widetilde{Q}$, similar to the proof of Lemma 2 and by Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \|II_1^t(x)\|\,\mathrm{d}x\\
&\le \frac{C}{|Q|}\int_Q\left(\int_{R^n\setminus\widetilde{Q}} \frac{|x-x_0|\,|f(z)|}{|x-z|^{n+m+1-\delta}}|R_m(\widetilde{A};x,z)|\,\mathrm{d}z\right)\mathrm{d}x\\
&\le \frac{C}{|Q|}\int_Q\left(\sum_{k=0}^{\infty}\int_{2^{k+1}\widetilde{Q}\setminus 2^k\widetilde{Q}} \frac{|x-x_0|\,|f(z)|}{|x-z|^{n+m+1-\delta}}\left|R_m(\widetilde{A};x,z)\right|\mathrm{d}z\right)\mathrm{d}x\\
&\le C\sum_{k=1}^{\infty}\frac{kl(2^kl)^m}{(2^kl)^{n+m+1-\delta}}\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\left(\int_{2^k\widetilde{Q}}|f(z)|\,\mathrm{d}z\right)\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^{n/\delta}}\sum_{k=1}^{\infty}k\,2^{-k}\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\,\|f\|_{L^{n/\delta}}.
\end{aligned}$$

For $II_2^t(x)$, by the formula (see [**6**]):

$$\begin{aligned}
R_m(\widetilde{A};x,z)-R_m(\widetilde{A};x_0,z)&=R_m(\widetilde{A};x,x_0)\\
&\quad+\sum_{0<|\beta|<m}\frac{1}{\beta!}\,R_{m-|\beta|}(D^\beta\widetilde{A};x_0,z)(x-x_0)^\beta
\end{aligned}$$

and Lemma 1, we get

$$\begin{aligned}
&\left|R_m(\widetilde{A};x,z)-R_m(\widetilde{A};x_0,z)\right|\\
&\quad\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\left(|x-x_0|^m+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}\right),
\end{aligned}$$

thus, for $x\in Q$,

$$\begin{aligned}
\|II_2^t(x)\| &\le C\int_{R^n}\frac{|f_2(z)|}{|x-z|^{m+n-\delta}}\left|R_m(\widetilde{A};x,z)-R_m(\widetilde{A};x_0,z)\right|\mathrm{d}z\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\\
&\qquad\int_{R^n}\frac{|x-x_0|^m+\sum_{0<|\beta|<m}|x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}}{|x_0-z|^{m+n-\delta}}\,|f_2(z)|\,\mathrm{d}z\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\sum_{k=0}^{\infty}\frac{kl^m}{(2^kl)^{m+n-\delta}}\int_{2^k\widetilde{Q}}|f(z)|\,\mathrm{d}z\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^{n/\delta}}\sum_{k=1}^{\infty}k\,2^{-km}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{BMO}\|f\|_{L^{n/\delta}}.
\end{aligned}$$

For $II_3^t(x)$, by the inequality: $a^{1/2} - b^{1/2} \leq (a-b)^{1/2}$ for $a \geq b > 0$, we obtain, similar to the estimate of Lemma 2 and II_1,

$$
\begin{aligned}
&|II_3^t(x)\| \\
&\leq C \int_{R^n} \left(\int_{R_+^{n+1}} \left(\frac{t^{n\mu/2}|x-x_0|^{1/2}|\psi_t(y-z)|\,|f_2(z)|}{(t+|x-y|)^{(n\mu+1)/2}|x_0-z|^m} \left|R_m(A;x_0,z)\right| \right)^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \right)^{1/2} \mathrm{d}z \\
&\leq C \int_{R^n} \frac{|f_2(z)|\,|x-x_0|^{1/2}|R_m(A;x_0,z)|}{|x_0-z|^m} \\
&\qquad\qquad \left(\iint_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu+1} \frac{t^{-n}\mathrm{d}y\mathrm{d}t}{(t+|y-z|)^{2n+2-2\delta}} \right)^{1/2} \mathrm{d}z \\
&\leq C \int_{R^n} \frac{|f_2(z)|\,|x-x_0|^{1/2}|R_m(A;x_0,z)|}{|x_0-z|^m} \left(\int_0^\infty \frac{\mathrm{d}t}{(t+|x-z|)^{2n+2-2\delta}} \right)^{1/2} \mathrm{d}z \\
&\leq C \int_{R^n} \frac{|f_2(z)|\,|x-x_0|^{1/2}|R_m(\widetilde{A};x_0,z)|}{|x_0-z|^{m+n+1/2-\delta}} \mathrm{d}z \\
&\leq C \sum_{k=1}^\infty \frac{kl^{1/2}(2^k l)^m}{(2^k l)^{n+m+1/2-\delta}} \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \left(\int_{2^k\widetilde{Q}} |f(z)|\,\mathrm{d}z \right) \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}} \sum_{k=0}^\infty k\,2^{-k/2} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}}.
\end{aligned}
$$

For $II_4^t(x)$, similar to the estimates of $II_1^t(x)$ and $II_3^t(x)$, we have

$$
\begin{aligned}
\|II_4^t(x)\| &\leq C \int_{R^n\setminus\widetilde{Q}} \left(\frac{|x-x_0|}{|x-z|^{n+1-\delta}} + \frac{|x-x_0|^{1/2}}{|x-z|^{n+1/2-\delta}} \right) \sum_{|\alpha|=m} |D^\alpha \widetilde{A}(z)|\,|f(z)|\,\mathrm{d}z \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}} \sum_{k=0}^\infty k\,2^{-k/2} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \|f\|_{L^{n/\delta}}.
\end{aligned}
$$

Combining these estimates , we complete the proof of Theorem 1.

Prof of Theorem 2. It suffices to show that there exists a constant $C > 0$ such that for every H^1-atom a(that is a satisfying: supp $a \subset Q = Q(x_0, r)$, $\|a\|_{L^\infty} \leq |Q|^{-1}$ and $\int a(y)\,\mathrm{d}y = 0$(see[**10**])), we have

$$\|\widetilde{g}_\mu^A(a)\|_{L^{n/(n-\delta)}} \leq C.$$

We write

$$\int_{R^n} \big(\widetilde{g}_\mu^A(a)(x)\big)^{n/(n-\delta)} \mathrm{d}x$$
$$= \Big(\int_{|x-x_0|\le 2r} + \int_{|x-x_0|>2r}\Big)\big(\widetilde{g}_\mu^A(a)(x)\big)^{n/(n-\delta)} \mathrm{d}x := J + JJ.$$

For J, by the following equality

$$Q_{m+1}(A;x,y) = R_{m+1}(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}\,(x-y)^\alpha \big(D^\alpha A(x) - D^\alpha A(y)\big),$$

we have, similar to the proof of Lemma 2,

$$\widetilde{g}_\mu^A(a)(x) \le g_\mu^A(a)(x) + C \sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^{n-\delta}}\,|a(y)|\,\mathrm{d}y,$$

thus, $\widetilde{g}_\mu^A$ is (L^p, L^q)-bounded by Lemma 2 and [**1**],[**2**], where $1/q = 1/p - \delta/n$. We see that

$$J \le C\|\widetilde{g}_\mu^A(a)\|_{L^q}^{n/((n-\delta)q)} |2Q|^{1-n/((n-\delta)q)} \le C\|a\|_{L^p}^{n/(n-\delta)} |Q|^{1-n/((n-\delta)q)} \le C.$$

To obtain the estimate of JJ, we denote that $\widetilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}\,(D^\alpha A)_{2B} x^\alpha$. Then $Q_m(A;x,y) = Q_m(\widetilde{A};x,y)$. We write, by the vanishing moment of a and $Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}\,(x-y)^\alpha D^\alpha A(x)$, for $x \in (2Q)^c$,

$$\begin{aligned}
\widetilde{F}_t^A(a)(x,y) &= \int \frac{\psi_t(y-z) R_m(\widetilde{A};x,z)}{|x-z|^m}\, a(z)\,\mathrm{d}z \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \frac{\psi_t(y-z) D^\alpha \widetilde{A}(x)(x-z)^\alpha}{|x-z|^m}\, a(z)\,\mathrm{d}z \\
&= \int \left(\frac{\psi_t(y-z) R_m(\widetilde{A};x,z)}{|x-z|^m} - \frac{\psi_t(y-x_0) R_m(\widetilde{A};x,x_0)}{|x-x_0|^m} \right) a(z)\,\mathrm{d}z \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \left(\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(y-x_0)(x-x_0)^\alpha}{|x-x_0|^m} \right) D^\alpha \widetilde{A}(x) a(z)\,\mathrm{d}z,
\end{aligned}$$

thus, similar to the proof of II in Theorem 1, we obtain

$$\begin{aligned}
\|\widetilde{F}_t^A(a)(x,y)\| \le C\Big(&\sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} |Q|^{1/n} |x-x_0|^{-n-1+\delta} \\
&+ |Q|^{1/n} |x-x_0|^{-n-1+\delta} |D^\alpha \widetilde{A}(x)| \Big),
\end{aligned}$$

so that,

$$JJ \le C\Big(\sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}\Big)^{n/(n-\delta)} \sum_{k=1}^{\infty} k\, 2^{-kn/(n-\delta)} \le C,$$

which together with the estimate for J yields the desired result. This finishes the proof of Theorem 2.

Proof of Theorem 3. By the equality

$$R_{m+1}(A;x,y) = Q_{m+1}(A;x,y) + \sum_{|\alpha|=m} \frac{1}{\alpha!}\,(x-y)^\alpha \big(D^\alpha A(x) - D^\alpha A(y)\big)$$

and similar to the proof of Lemma 2, we get

$$g_\mu^A(f)(x) \le \widetilde{g}_\mu^A(f)(x) + C\sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^{n-\delta}}\,|f(y)|\,\mathrm{d}y,$$

by Theorem 1 and 2 with [**1**], [**2**], we obtain

$$\begin{aligned}\big|\{x\in R^n : g_\mu^A(f)(x) > \lambda\}\big| &\le \big|\{x\in R^n : \widetilde{g}_\mu^A(f)(x) > \lambda/2\}\big| \\ &\quad + \Big|\Big\{x\in R^n : \sum_{|\alpha|=m}\int \frac{\big|D^\alpha A(x) - D^\alpha A(y)\big|}{|x-y|^{n-\delta}}\,\big|f(y)\big|\,\mathrm{d}y > C\lambda\Big\}\Big| \\ &\le C\big(\|f\|_{H^1}/\lambda\big)^{n/(n-\delta)}.\end{aligned}$$

This completes the proof of Theorem 3.

Proof of Theorem 4. (**i**) It suffices to show that there exists a constant $C>0$ such that for every H^1-atom a with supp $a\subset Q = Q(x_0,d)$, there is

$$\|g_\mu^A(a)\|_{L^{n/(n-\delta)}} \le C.$$

Let $\widetilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}\,(D^\alpha A)_{\widetilde{Q}} x^\alpha$. We write, by the vanishing moment of a and for $u\in 3Q\setminus 2Q$,

$$\begin{aligned}F_t^A(a)(x,y) &= \chi_{4Q}(x) F_t^A(a)(x,y) \\ &\quad + \chi_{(4Q)^c}(x)\int_{R^n}\left(\frac{R_m(\widetilde{A};x,z)\psi_t(y-z)}{|x-z|^m} - \frac{R_m(\widetilde{A};x,u)\psi_t(y-u)}{|x-u|^m}\right) a(z)\,\mathrm{d}z \\ &\quad - \chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left(\frac{(x-z)^\alpha}{|x-z|^m} - \frac{(x-u)^\alpha}{|x-u|^m}\right)\psi_t(y-z) D^\alpha A(z) a(z)\,\mathrm{d}z \\ &\quad - \chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\,\psi_t(y-z) D^\alpha A(z) a(z)\,\mathrm{d}z,\end{aligned}$$

then

$$
\begin{aligned}
g_\mu^A(a)(x) &= \left\| \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(a)(x,y) \right\| \\
&\le \chi_{4Q}(x) \left\| \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(a)(x,y) \right\| \\
&\quad + \chi_{(4Q)^c}(x) \bigg\| \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \int_{R^n} \bigg(\frac{R_m(\widetilde{A};x,z)\psi_t(y-z)}{|x-z|^m} \\
&\quad - \frac{R_m(\widetilde{A};x,u)\psi_t(y-u)}{|x-u|^m} \bigg) a(z)\,\mathrm{d}z \bigg\| \\
&\quad + \chi_{(4Q)^c}(x) \bigg\| \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \bigg(\frac{(x-z)^\alpha}{|x-z|^m} \\
&\quad - \frac{(x-u)^\alpha}{|x-u|^m} \bigg) \psi_t(y-z) D^\alpha A(z) a(z)\,\mathrm{d}z \bigg\| \\
&+ \chi_{(4Q)^c}(x) \left\| \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x-u)^\alpha}{|x-u|^m} \psi_t(y-z) a(z)\,\mathrm{d}z \right\| \\
&= L_1(x) + L_2(x) + L_3(x,u) + L_4(x,u).
\end{aligned}
$$

Taking $n/(n-\delta) < q$ and p so that $1/q = 1/p - \delta/n$, it follows the (L^p, L^q)-boundedness of g_μ^A that

$$
\|L_1\|_{L^{n/(n-\delta)}} \le |4Q|^{(n-\delta)/n-1/q} \, \|g_\mu^A(a)\|_{L^q} \le C|Q|^{1-1/p} ||a||_{L^p} \le C.
$$

Similar to the proof of Theorem 1, we obtain

$$
\|L_2\|_{L^{n/(n-\delta)}} \le C \quad \text{and} \quad \|L_3(\cdot,u)\|_{L^{n/(n-\delta)}} \le C.
$$

Thus, using the condition of $L_4(x,u)$, we obtain $\|g_\mu^A(a)\|_{L^{n/(n-\delta)}} \le C$.

(ii). We write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $u \in 3Q \setminus 2Q$,

$$
\begin{aligned}
\widetilde{F}_t^A(f)(x,y) &= \widetilde{F}_t^A(f_1)(x,y) + \int_{R^n} \frac{R_m(\widetilde{A};x,z)}{|x-z|^m} \psi_t(y-z) f_2(z)\,\mathrm{d}z \\
&- \sum_{|\alpha|=m} \frac{1}{\alpha!} \left(D^\alpha A(x) - (D^\alpha A)_Q\right) \int_{R^n} \left(\frac{(x-z)^\alpha}{|x-z|^m} - \frac{(u-z)^\alpha}{|u-z|^m} \right) \psi_t(y-z) f_2(z)\,\mathrm{d}z \\
&- \sum_{|\alpha|=m} \frac{1}{\alpha!} \left(D^\alpha A(x) - (D^\alpha A)_Q\right) \int_{R^n} \frac{(u-z)^\alpha}{|u-z|^m} \psi_t(y-z) f_2(z)\,\mathrm{d}z,
\end{aligned}
$$

then

$$\begin{aligned}
&\left|\widetilde{g}_\mu^A(f)(x) - g_\mu\left(\frac{R_m(\widetilde{A};x_0,\cdot)}{|x_0-\cdot|^m} f_2\right)(x_0)\right| = \Bigg|\Bigg\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\widetilde{F}_t^A(f)(x,y)\Bigg\| \\
&\qquad -\Bigg\|\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} F_t\left(\frac{R_m(\widetilde{A};x_0,\cdot)}{|x_0-\cdot|^m} f_2\right)(x_0)\Bigg\|\Bigg| \\
&\quad \le \Bigg\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\widetilde{F}_t^A(f)(x,y) \\
&\qquad -\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} F_t\left(\frac{R_m(\widetilde{A};x_0,\cdot)}{|x_0-\cdot|^m} f_2\right)(x_0)\Bigg\| \\
&\quad \le \Bigg\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\widetilde{F}_t^A(f_1)(x,y)\Bigg\| \\
&\qquad +\Bigg\|\int_{R^n}\Bigg(\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\frac{R_m(\widetilde{A};x,z)}{|x-z|^m}\psi_t(y-z) \\
&\qquad -\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}\frac{R_m(\widetilde{A};x_0,z)}{|x_0-z|^m}\psi_t(x_0-z)\Bigg)f_2(z)\,\mathrm{d}z\Bigg\| \\
&\qquad +\Bigg\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\sum_{|\alpha|=m}\frac{1}{\alpha!}\big(D^\alpha A(x) \\
&\qquad -(D^\alpha A)_Q\big)\int_{R^n}\left(\frac{(y-z)^\alpha}{|y-z|^m}-\frac{(u-z)^\alpha}{|u-z|^m}\right)\psi_t(y-z)f_2(z)dz\Bigg\| \\
&\qquad +\Bigg\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\sum_{|\alpha|=m}\frac{1}{\alpha!}\big(D^\alpha A(x) \\
&\qquad -(D^\alpha A)_Q\big)\int_{R^n}\frac{(u-z)^\alpha}{|u-z|^m}\psi_t(y-z)f_2(z)\,\mathrm{d}z\Bigg\| \\
&\quad = M_1(x)+M_2(x)+M_3(x,u)+M_4(x,u).
\end{aligned}$$

By the the $(L^p(R^n),\ L^q(R^n))$-boundedness of $\widetilde{g}_\mu^A$ for $1<p<n/\delta$ with $1/q = 1/p-\delta/n$, we get

$$\frac{1}{|Q|}\int_Q M_1(x)\,\mathrm{d}x \le C\left(\frac{1}{|Q|}\int_Q |\widetilde{g}_\lambda^A(f_1)(x)|^q\,\mathrm{d}x\right)^{1/q} \le C|Q|^{-1/q}||f_1||_{L^p} \le C||f||_{L^{n/\delta}}.$$

Similar to the proof of Theorem 1, we obtain

$$\frac{1}{|Q|}\int_Q M_2(x)\,\mathrm{d}x \le C||f||_{L^{n/\delta}} \quad\text{and}\quad \frac{1}{|Q|}\int_Q M_3(x,u)\,\mathrm{d}x \le C||f||_{L^{n/\delta}}.$$

Thus, using the estimates of $M_4(x,u)$, we obtain

$$\frac{1}{|Q|}\int_Q\left|\widetilde{g}_\mu^A(x) - g_\mu\left(\frac{R_m(\widetilde{A};x_0,\cdot)}{|x_0-\cdot|^m} f_2\right)(x_0)\right|\mathrm{d}x \le C||f||_{L^{n/\delta}}.$$

This completes the proof of Theorem 4.

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