AN OSTROWSKI TYPE INEQUALITY FOR CONVEX FUNCTIONS

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An Ostrowski type integral inequality for convex functions and applications for quadrature rules and integral means are given. A refinement and a counterpart result for Hermite–Hadamard inequalities are obtained and some inequalities for pdf’s and \((HH)−\)divergence measure are also mentioned.

1. INTRODUCTION

The following result is known in the literature as Ostrowski’s inequality [1].

**Theorem 1.** Let \(f : [a, b] \rightarrow \mathbb{R}\) be a differentiable mapping on \((a, b)\) with the property that \(|f'(t)| \leq M\) for all \(t \in (a, b)\). Then

\[
|f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt| \leq \left( \frac{1}{4} + \frac{(x - a + b)^2}{(b - a)^2} \right) (b - a) \, M
\]

for all \(x \in [a, b]\).

The constant \(1/4\) is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

\[
f(x) = \frac{1}{b - a} \int_a^b f(t) \, dt + \frac{1}{b - a} \int_a^b p(x, t)f'(t) \, dt, \quad x \in [a, b],
\]

where

\[
p(x, t) := \begin{cases} 
  t - a & \text{if } a \leq t \leq x \\
  t - b & \text{if } x < t \leq b
\end{cases}
\]

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which holds for absolutely continuous functions $f : [a, b] \to \mathbb{R}$.

The following Ostrowski type result holds (see [2], [3] and [4]).

**Theorem 2.** Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. Then, for all $x \in [a, b]$, we have:

\[
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \begin{cases}
\frac{1}{4} + \left( \frac{x-a+b}{b-a} \right)^2 (b-a) \|f'\|_{\infty} & \text{if } f' \in L_{\infty}[a, b] ; \\
\frac{1}{(p+1)^{1/p}} \left( \left( \frac{x-a}{b-a} \right)^{p+1} + \left( \frac{b-x}{b-a} \right)^{p+1} \right)^{1/p} (b-a)^{1/p} \|f'\|_q & \text{if } f' \in L_q[a, b] , \frac{1}{p} + \frac{1}{q} = 1 , p > 1 ; \\
\left( \frac{1}{2} + \frac{x-a+b}{b-a} \right) \|f'\|_1 ;
\end{cases}
\]

where $\|\cdot\|_r$ ($r \in [1, \infty]$) are the usual Lebesgue norms on $L_r[a, b]$, i.e.,

\[
\|g\|_{\infty} := \text{ess sup}_{t \in [a, b]} |g(t)|
\]

and

\[
\|g\|_r := \left( \int_a^b |g(t)|^r \, dt \right)^{1/r} , \quad r \in [1, \infty).
\]

The constants $1/4, 1/(p+1)^{1/p}$ and $1/2$ respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from Fink’s result in [5] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that $f$ is Hölder continuous, then one may state the result (see [6]):

**Theorem 3.** Let $f : [a, b] \to \mathbb{R}$ be of $r$–Hölder type, i.e.,

\[
|f(x) - f(y)| \leq H |x-y|^r , \text{ for all } x, y \in [a, b] ,
\]

where $r \in (0, 1]$ and $H > 0$ are fixed. Then for all $x \in [a, b]$ we have the inequality:

\[
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \frac{H}{r+1} \left( \left( \frac{b-x}{b-a} \right)^{r+1} + \left( \frac{x-a}{b-a} \right)^{r+1} \right) (b-a)^r .
\]
The constant $1/(r+1)$ is also sharp in the above sense.

Note that if $r = 1$, i.e., $f$ is LIPSCHITZ continuous, then we get the following version of OSTROWSKI’s inequality for Lipschitzian functions (with $L$ instead of $H$) (see [7])

$$
(1.6) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left( \frac{1}{4} + \left( \frac{x-a+b}{b-a} \right)^2 \right) (b-a) L.
$$

Here the constant $1/4$ is also best.

Moreover, if one drops the continuity condition of the function, and assumes that it is of bounded variation, then the following result may be stated (see [8]).

**Theorem 4.** Assume that $f : [a, b] \to \mathbb{R}$ is of bounded variation and denote by $\sqrt[\to a]{(f)}$ its total variation. Then

$$
(1.7) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left( \frac{1}{2} + \left| \frac{x-a+b}{b-a} \right| \right) \sqrt[\to a]{(f)}
$$

for all $x \in [a, b]$.

The constant $1/2$ is the best possible.

If we assume more about $f$, i.e., $f$ is monotonically increasing, then the inequality (1.7) may be improved in the following manner [9] (see also [10]).

**Theorem 5.** Let $f : [a, b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a, b]$, we have the inequality:

$$
(1.8) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \\
\quad \leq \frac{1}{b-a} \left( (2x - (a+b)) f(x) + \int_a^b \text{sgn}(t-x) f(t) \, dt \right) \\
\quad \leq \frac{1}{b-a} \left( (x-a)(f(x) - f(a)) + (b-x)(f(b) - f(x)) \right) \\
\quad \leq \left( \frac{1}{2} + \left| \frac{x-a+b}{b-a} \right| \right) (f(b) - f(a)).
$$

All the inequalities in (1.8) are sharp and the constant $1/2$ is the best possible.

For recent generalisations of OSTROWSKI inequality in various directions and for $n$-time differentiable functions, see [14] and the research monograph [13] where further references are provided.

In this paper we establish an OSTROWSKI type inequality for convex functions. Applications for quadrature rules, for integral means, for probability distribution functions, and for $HH$–divergences in Information Theory are also considered.
2. THE RESULTS

The following theorem providing a lower bound for the OSTROWSKI difference \( \int_a^b f(t) \, dt - (b-a)f(x) \) holds.

**Theorem 6.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function on \([a, b]\). Then for any \( x \in (a, b) \) we have the inequality:

\[
\frac{1}{2} \left( (b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right) \leq \int_a^b f(t) \, dt - (b-a)f(x).
\]

The constant \( 1/2 \) in the left hand side of (2.1) is sharp in the sense that it cannot be replaced by a larger constant.

**Proof.** It is easy to see that for any locally absolutely continuous function \( f : (a, b) \to \mathbb{R} \), we have the identity

\[
\int_a^x (t-a)f'(t) \, dt + \int_x^b (t-b)f'(t) \, dt = f(x) - \int_a^b f(t) \, dt,
\]

for any \( x \in (a, b) \) where \( f' \) is the derivative of \( f \) which exists a.e. on \((a, b)\).

Since \( f \) is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any \( x \in (a, b) \), we have the inequalities

\[
f'(t) \leq f'_-(x) \text{ for a.e. } t \in [a, x]
\]

and

\[
f'(t) \geq f'_+(x) \text{ for a.e. } t \in [x, b].
\]

If we multiply (2.3) by \( t-a \geq 0, t \in [a, x] \), and integrate on \([a, x]\), we get

\[
\int_a^x (t-a)f'(t) \, dt \leq \frac{1}{2} (x-a)^2 f'_-(x)
\]

and if we multiply (2.4) by \( b-t \geq 0, t \in [x, b] \), and integrate on \([x, b]\), we also have

\[
\int_x^b (b-t)f'(t) \, dt \geq \frac{1}{2} (b-x)^2 f'_+(x).
\]

Finally, if we subtract (2.6) from (2.5) and use the representation (2.2) we deduce the desired inequality (2.1).

Now, assume that (2.1) holds with a constant \( C > 0 \) instead of \( 1/2 \), i.e.,

\[
C \left( (b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right) \leq \int_a^b f(t) \, dt - (b-a)f(x).
\]

Consider the convex function \( f_0(t) := k \left| t - \frac{a+b}{2} \right|, k > 0, t \in [a, b] \). Then

\[
f_0'\left( \frac{a+b}{2} \right) = k, \quad f_0'\left( \frac{a+b}{2} \right) = -k, \quad f_0\left( \frac{a+b}{2} \right) = 0
\]
and
\[ \int_{a}^{b} f_0(t) \, dt = \frac{1}{4} k(b - a)^2. \]

If in (2.7) we choose \( f_0 \) as above and \( x = (a + b)/2 \), then we get
\[ C \left( \frac{1}{4} (b - a)^2 k + \frac{1}{4} (b - a)^2 k \right) \leq \frac{1}{4} k(b - a)^2, \]
which gives \( C \leq 1/2 \), and the sharpness of the constant is proved.

Now, recall that the following inequality, which is well known in the literature as the Hermite–Hadamard inequality for convex functions, holds:
\[
(HH) \quad f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(t) \, dt \leq \frac{f(a) + f(b)}{2}.
\]

The following corollary which improves the first Hermite–Hadamard inequality (HH) holds.

**Corollary 1.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function on \( [a, b] \). Then
\[
0 \leq \frac{1}{8} \left( f' \left( \frac{a + b}{2} \right) - f' \left( \frac{a + b}{2} \right) \right) (b - a) \leq \frac{1}{b - a} \int_{a}^{b} f(t) \, dt - f \left( \frac{a + b}{2} \right).
\]
The constant \( 1/8 \) is sharp.

The proof is obvious by the above theorem. The sharpness of the constant is obtained for \( f_0(t) := k \left| t - \frac{a + b}{2} \right|, t \in [a, b], k > 0 \).

When \( x \) is a point of differentiability, we may state the following corollary as well.

**Corollary 2.** Let \( f \) be as in Theorem 6. If \( x \in (a, b) \) is a point of differentiability for \( f \), then
\[
\left( a + b \over 2 - x \right) f'(x) \leq \frac{1}{b - a} \int_{a}^{b} f(t) \, dt - f(x).
\]

**Remark 1.** If \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is convex on \( I \) and if we choose \( x \in \overset{\circ}{I} \) (\( \overset{\circ}{I} \) is the interior of \( I \)), \( b = x + h, a = x - h, h > 0 \) is such that \( a, b \in I \), then from (2.1) we may write
\[
0 \leq \frac{1}{8} h^2 \left( f'_+ (x) - f'_- (x) \right) \leq \int_{x - \frac{h}{2}}^{x + \frac{h}{2}} f(t) \, dt - hf(x),
\]
and the constant \( 1/8 \) is sharp in (2.10).
The following result providing an upper bound for the Ostrowski difference
\[
\int_a^b f(t) \, dt - (b - a)f(x)
\] also holds.

**Theorem 7.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a convex function on \([a, b]\). Then for any \( x \in [a, b] \), we have the inequality:

\[
\int_a^b f(t) \, dt - (b - a)f(x) \leq \frac{1}{2} \left( (b - x)^2 f'_-(b) - (x - a)^2 f'_+(a) \right).
\]

The constant 1/2 is sharp in the sense that it cannot be replaced by a smaller constant.

**Proof.** If either \( f'_+(a) = -\infty \) or \( f'_-(b) = +\infty \), then the inequality (2.11) evidently holds true.

Assume that \( f'_+(a) \) and \( f'_-(b) \) are finite.

Since \( f \) is convex on \([a, b]\), we have

\[
f'(t) \geq f'_+(a) \text{ for a.e. } t \in [a, x]
\]

and

\[
f'(t) \leq f'_-(b) \text{ for a.e. } t \in [x, b].
\]

If we multiply (2.12) by \( t - a \geq 0, t \in [a, x] \), and integrate on \([a, x]\), then we deduce

\[
\int_a^x (t - a)f'(t) \, dt \geq \frac{1}{2} (x - a)^2 f'_+(a)
\]

and if we multiply (2.13) by \( b - t \geq 0, t \in [x, b] \), and integrate on \([x, b]\), then we also have

\[
\int_x^b (b - t)f'(t) \, dt \leq \frac{1}{2} (b - x)^2 f'_-(b).
\]

Finally, if we subtract (2.14) from (2.15) and use the representation (2.2), we deduce the desired inequality (2.11).

Now, assume that (2.11) holds with a constant \( D > 0 \) instead of 1/2, i.e.,

\[
\int_a^b f(t) \, dt - (b - a)f(x) \leq D \left( (b - x)^2 f'_-(b) - (x - a)^2 f'_+(a) \right).
\]

If we consider the convex function \( f_0 : [a, b] \rightarrow \mathbb{R}, \ f_0(t) = k \left| t - \frac{a + b}{2} \right| \), then we have \( f'_-(b) = k, \ f'_+(a) = -k \) and by (2.16) we deduce for \( x = \frac{a + b}{2} \) that

\[
\frac{1}{4} k(b - a)^2 \leq D \left( \frac{1}{4} k(b - a)^2 + \frac{1}{4} k(b - a)^2 \right),
\]

giving \( D \geq 1/2 \), and the sharpness of the constant is proved.
The following corollary related to the Hermite–Hadamard inequality is interesting as well.

**Corollary 3.** Let \( f : [a, b] \to \mathbb{R} \) be convex on \([a, b]\). Then

\[
(2.17) \quad 0 \leq \frac{1}{b - a} \int_a^b f(t) \, dt - f\left(\frac{a + b}{2}\right) \leq \frac{1}{8} \left(f'_+(b) - f'_+(a)\right)(b - a)
\]

and the constant \(1/8\) is sharp.

**Remark 2.** Denote \( B := f'_+(b) \), \( A := f'_+(a) \) and assume that \( B \neq A \), i.e., \( f \) is not constant on \((a, b)\). Then

\[
(b - x)^2B - (x - a)^2A = (B - A) \left(x - \left(\frac{bB - aA}{B - A}\right)\right)^2 - \frac{AB}{B - A} (b - a)^2
\]

and by (2.11) we get

\[
(2.18) \quad \int_a^b f(t) \, dt - (b - a)f(x) \leq \frac{1}{2} (B - A) \left(\left(x - \left(\frac{bB - aA}{B - A}\right)\right)^2 - \frac{AB}{(B - A)^2} (b - a)^2\right)
\]

for any \( x \in [a, b] \).

If \( A \geq 0 \) then \( x_0 = \frac{bB - aA}{B - A} \in [a, b] \) and by (2.18) we get, choosing \( x = \frac{bB - aA}{B - A} \), that

\[
(2.19) \quad 0 \leq \frac{1}{2} \frac{AB}{B - A} (b - a) \leq f\left(\frac{bB - aA}{B - A}\right) - \frac{1}{b - a} \int_a^b f(t) \, dt,
\]

which is an interesting inequality in itself.

**Remark 3.** If \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is convex on \( I \) and if we choose \( x \in I \), \( b = x + \frac{h}{2} \), \( a = x - \frac{h}{2} \), \( h > 0 \) is such that \( a, b \in I \), then from (2.11) we deduce:

\[
(2.20) \quad 0 \leq \int_{x - \frac{h}{2}}^{x + \frac{h}{2}} f(t) \, dt - hf(x) \leq \frac{1}{8} h^2 \left(f'_-(x + \frac{h}{2}) - f'_+(x - \frac{h}{2})\right)
\]

and the constant \(1/8\) is sharp.

### 3. THE COMPOSITE CASE

Consider the division \( I_n : a = x_0 < x_1 < \cdots < x_{n - 1} < x_n = b \) and denote \( h_i := x_{i + 1} - x_i, i = 0, n - 1 \). If \( \xi_i \in [x_i, x_{i+1}] (i = 0, n - 1) \) are intermediate points, then we will denote by

\[
(3.1) \quad R_n (f; I_n, \xi) := \sum_{i=0}^{n-1} h_i f(\xi_i)
\]

the Riemann sum associated to \( f, I_n \) and \( \xi \).
The following theorem providing upper and lower bounds for the remainder in approximating the integral \( \int_a^b f(t) \, dt \) of a convex function \( f \) in terms of a general Riemann sum holds.

**Theorem 8.** Let \( f : [a, b] \to \mathbb{R} \) be a convex function and \( I_n \) and \( \xi \) be as above. Then we have:

\[
(3.2) \quad \int_a^b f(t) \, dt = R_n(f; I_n, \xi) + W_n(f; I_n, \xi),
\]

where \( R_n(f; I_n, \xi) \) is the Riemann sum defined by (3.1) and the remainder \( W_n(f; I_n, \xi) \) satisfies the estimate:

\[
(3.3) \quad \frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_+(\xi_i) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_-(\xi_i) \leq W_n(f; I_n, \xi) \leq \frac{1}{2} \left( (b - \xi_{n-1})^2 f'_+(b) + \sum_{i=1}^{n-1} (x_i - \xi_{i-1})^2 f'_-(x_i) \right.
\]

\[\left. - (\xi_i - x_i)^2 f'_+(x_i) \right) - (\xi_0 - a)^2 f'_+(a) \right).
\]

**Proof.** If we write the inequalities (2.1) and (2.11) on the interval \([x_i, x_{i+1}]\) and for the intermediate points \( \xi_i \in [x_i, x_{i+1}] \), then we have

\[
\frac{1}{2} \left( (x_{i+1} - \xi_i)^2 f'_+(\xi_i) - (\xi_i - x_i)^2 f'_-(\xi_i) \right) \leq \int_{x_i}^{x_{i+1}} f(t) \, dt - h_i f(\xi_i) \leq \frac{1}{2} \left( (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) - (\xi_i - x_i)^2 f'_+(x_i) \right).
\]

Summing the above inequalities over \( i \) from 0 to \( n - 1 \), we deduce

\[
(3.4) \quad \frac{1}{2} \sum_{i=0}^{n-1} \left( (x_{i+1} - \xi_i)^2 f'_+(\xi_i) - (\xi_i - x_i)^2 f'_-(\xi_i) \right) \leq \int_a^b f(t) \, dt - R_n(f; I_n, \xi) \leq \frac{1}{2} \left( \sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) - \sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+(x_i) \right).
\]

However,

\[
\sum_{i=0}^{n-1} (x_{i+1} - \xi_i)^2 f'_-(x_{i+1}) = (b - \xi_{n-1})^2 f'_-(b) + \sum_{i=0}^{n-2} ((x_{i+1} - \xi_i)^2 f'_-(x_{i+1})) = (b - \xi_{n-1})^2 f'_-(b) + \sum_{i=1}^{n-1} ((x_i - \xi_i-1)^2 f'_-(x_i))
\]

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and
\[
\sum_{i=0}^{n-1} (\xi_i - x_i)^2 f'_+(x_i) = \sum_{i=1}^{n-1} (\xi_i - x_i)^2 f'_+(x_i) + (\xi_0 - a)^2 f'_+(a)
\]
and then, by (3.4), we deduce the desired estimate (3.3).

The following corollary may be useful in practical applications.

**Corollary 4.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable convex function on \((a, b)\). Then we have the representation (3.1) and the remainder \( W_n(f; I_n, \xi) \) satisfies the estimate:

\[
\sum_{i=0}^{n-1} \left( \frac{x_i + x_{i+1}}{2} - \xi_i \right) h_i f'(\xi_i) \leq W_n(f; I_n, \xi_i)
\]

\[
\leq \frac{1}{2} \left( (b - \xi_{n-1})^2 f'_-(b) - (\xi_0 - a)^2 f'_+(a) \right)
\]

\[
+ \sum_{i=1}^{n-1} \left( x_i - \frac{\xi_i + \xi_{i-1}}{2} \right) (\xi_i - \xi_{i-1}) f'(x_i).
\]

We may also consider the mid-point quadrature rule:

\[
(3.6) \quad M_n(f, I_n) := \sum_{i=0}^{n-1} h_i f \left( \frac{x_i + x_{i+1}}{2} \right).
\]

Using Corollaries 1 and 2, we may state the following result as well.

**Corollary 5.** Assume that \( f : [a, b] \to \mathbb{R} \) is a convex function on \([a, b]\) and \( I_n \) is a division as above. Then we have the representation:

\[
(3.7) \quad \int_a^b f(x) \, dx = M_n(f, I_n) + S_n(f, I_n),
\]

where \( M_n(f, I_n) \) is the mid-point quadrature rule given in (3.6) and the remainder \( S_n(f, I_n) \) satisfies the estimates:

\[
(3.8) \quad 0 \leq \frac{1}{8} \sum_{i=0}^{n-1} \left( f'_+ \left( \frac{x_i + x_{i+1}}{2} \right) - f'_- \left( \frac{x_i + x_{i+1}}{2} \right) \right) h_i^2
\]

\[
\leq S_n(f, I_n) \leq \frac{1}{8} \sum_{i=0}^{n-1} \left( f'_-(x_{i+1}) - f'_+(x_i) \right) h_i^2.
\]

The constant \( 1/8 \) is sharp in both inequalities.
4. INEQUALITIES FOR INTEGRAL MEANS

We may prove the following result in comparing two integral means.

**Theorem 9.** Let $f : [a, b] \to \mathbb{R}$ be a convex function and $c, d \in [a, b]$ with $c < d$. Then we have the inequalities

$$
\frac{a + b}{2} \cdot \frac{f(d) - f(c)}{d - c} - \frac{df(d) - cf(c)}{d - c} + \frac{1}{d - c} \int_c^d f(x) \, dx
$$

$$
\leq \frac{1}{b - a} \int_a^b f(t) \, dt - \frac{1}{d - c} \int_c^d f(x) \, dx
$$

$$
\leq \frac{f'(b)((b - d)^2 + (b - d)(b - c) + (b - c)^2)}{6(b - a)}
$$

$$
- \frac{f'(a)((d - a)^2 + (d - a)(c - a) + (c - a)^2)}{6(b - a)}.
$$

**Proof.** Since $f$ is convex, then for a.e. $x \in [a, b]$, we have (by (2.9)) that

$$
\left(\frac{a + b}{2} - x\right) f'(x) \leq \frac{1}{b - a} \int_a^b f(t) \, dt - f(x).
$$

Integrating (4.2) on $[c, d]$ we deduce

$$
\frac{1}{d - c} \int_c^d \left(\frac{a + b}{2} - x\right) f'(x) \, dx \leq \frac{1}{b - a} \int_a^b f(t) \, dt - \frac{1}{d - c} \int_c^d f(x) \, dx.
$$

Since

$$
\frac{1}{d - c} \int_c^d \left(\frac{a + b}{2} - x\right) f'(x) \, dx
$$

$$
= \frac{1}{d - c} \left(\left(\frac{a + b}{2} - d\right) f(d) - \left(\frac{a + b}{2} - c\right) f(c) + \int_c^d f(x) \, dx\right)
$$

then by (4.3) we deduce the first part of (4.1).

Using (2.11), we may write for any $x \in [a, b]$ that

$$
\frac{1}{b - a} \int_a^b f(t) \, dt - f(x) \leq \frac{1}{2(b - a)} \left((b - x)^2 f'(b) - (x - a)^2 f'(a)\right).
$$

Integrating (4.4) on $[c, d]$, we deduce
\[(4.5)\]
\[
\frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{d-c} \int_c^d f(x) \, dx 
\leq \frac{1}{2(b-a)} \left( f'(b) \frac{1}{d-c} \int_c^d (b-x)^2 \, dx - f'_-(a) \frac{1}{d-c} \int_c^d (x-a)^2 \, dx \right).
\]

Since
\[
\frac{1}{d-c} \int_c^d (b-x)^2 \, dx = \frac{(b-d)^2 + (b-d)(b-c) + (b-c)^2}{3}
\]
and
\[
\frac{1}{d-c} \int_c^d (x-a)^2 \, dx = \frac{(d-a)^2 + (d-a)(c-a) + (c-a)^2}{3},
\]
then by (4.5) we deduce the second part of (4.1).

**Remark 4.** If we choose \(f(x) = x^p, \ p \in (-\infty, 0) \cup [1, \infty)\ \{ -1 \}\) or \(f(x) = 1/x\) or even \(f(x) = -\ln x, x \in [a, b] \subset (0, \infty)\), in the above inequalities, then a great number of interesting results for \(p\)-logarithmic, logarithmic and identric means may be obtained. We leave this as an exercise to the interested reader.

### 5. APPLICATIONS FOR P.D.F.S

Let \(X\) be a random variable with the probability density function \(f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+\) and with cumulative distribution function \(F(x) = \Pr(X \leq x)\).

The following theorem holds.

**Theorem 10.** If \(f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+\) is monotonically increasing on \([a, b]\), then we have the inequality:

\[(5.1)\]
\[
\frac{1}{2} \left( (b-x)^2 f_+(x) - (x-a)^2 f_-(x) \right) 
\leq b - E(X) - (b-a)F(x) 
\leq \frac{1}{2} \left( (b-x)^2 f_-(b) - (x-a)^2 f_+(a) \right)
\]
for any \(x \in (a, b)\), where \(f_-(\alpha)\) means the left limit in \(\alpha\) while \(f_+(\alpha)\) means the right limit in \(\alpha\) and \(E(X)\) is the expectation of \(X\).

The constant \(1/2\) is sharp in both inequalities.

The second inequality also holds for \(x = a\) or \(x = b\).

**Proof.** Follows by Theorem 6 and 7 applied for the convex cdf function \(F(x) = \int_a^x f(t) \, dt, \ x \in [a, b]\) and taking into account that
\[
\int_a^b F(x) \, dx = b - E(X).
\]

Finally, we may state the following corollary in estimating the probability \(\Pr \left( X \leq \frac{a+b}{2} \right) \).
Corollary 6. With the above assumptions, we have

\[
\begin{align*}
(5.2) \quad b - E(X) & - \frac{1}{8} (b-a)^2 (f_-(b) - f_+(a)) \\
& \leq \Pr \left( X \leq \frac{a+b}{2} \right) \\
& \leq b - E(X) - \frac{1}{8} (b-a)^2 \left( f_+ \left( \frac{a+b}{2} \right) - f_- \left( \frac{a+b}{2} \right) \right).
\end{align*}
\]

6. APPLICATIONS FOR HH–DIVERGENCE

Assume that a set \( \chi \) and the \( \sigma \)–finite measure \( \mu \) are given. Consider the set of all probability densities on \( \mu \) to be

(6.1) \[ \Omega := \left\{ p \mid p : \Omega \to \mathbb{R}, \, p(x) \geq 0, \, \int_{\chi} p(x) \, d\mu(x) = 1 \right\}. \]

Csiszár’s \( f \)–divergence is defined as follows [11]

(6.2) \[ D_f(p, q) := \int_{\chi} p(x) f \left( \frac{q(x)}{p(x)} \right) \, d\mu(x), \, p, q \in \Omega, \]

where \( f \) is convex on \((0, \infty)\). It is assumed that \( f(u) \) is zero and strictly convex at \( u = 1 \). By appropriately defining this convex function, various divergences are derived.

In [12], Shioya and Da-te introduced the generalised Lin-Wong \( f \)–divergence \( D_f \left( p, \frac{1}{2} p + \frac{1}{2} q \right) \) and the Hermite-Hadamard (HH) divergence

(6.3) \[ D_{HH}^f(p, q) := \int_{\chi} \frac{p^2(x)}{q(x) - p(x)} \left( \int_1^{q(x)/p(x)} f(t) \, dt \right) \, d\mu(x), \, p, q \in \Omega, \]

and, by the use of the Hermite-Hadamard inequality for convex functions, proved the following basic inequality

(6.4) \[ D_f \left( p, \frac{1}{2} p + \frac{1}{2} q \right) \leq D_{HH}^f(p, q) \leq \frac{1}{2} D_f(p, q), \]

provided that \( f \) is convex and normalised, i.e., \( f(1) = 0 \).

The following result in estimating the difference

\[ D_{HH}^f(p, q) - D_f \left( p, \frac{1}{2} p + \frac{1}{2} q \right) \]

holds.
Theorem 11. Let $f : [0, \infty) \to \mathbb{R}$ be a convex function and $p, q \in \Omega$. Then we have the inequality:

\begin{equation}
0 \leq \frac{1}{8} \left( D_{f^+}^\prime \cdot \|\frac{1}{a+b}\| (p, q) - D_{f^-}^\prime \cdot \|\frac{1}{a+b}\| (p, q) \right)
\end{equation}

\begin{align*}
& \leq D_{f^H}^H (p, q) - D_f \left( p, \frac{1}{2} p + \frac{1}{2} q \right) \\
& \leq \frac{1}{8} D_{f^\prime (-1)} (p, q).
\end{align*}

Proof. Using the double inequality

\begin{align*}
0 & \leq \frac{1}{8} \left( f_+^\prime \left( \frac{a+b}{2} \right) - f_-^\prime \left( \frac{a+b}{2} \right) \right) |b-a| \\
& \leq \frac{1}{b-a} \int_a^b f(t) \, dt - f \left( \frac{a+b}{2} \right) \\
& \leq \frac{1}{8} (f_-(b) - f_+(a))(b-a)
\end{align*}

for the choices $a = 1, b = \frac{q(x)}{p(x)}, x \in \chi$, multiplying with $p(x) \geq 0$ and integrating over $x$ on $\chi$ we get

\begin{align*}
0 & \leq \frac{1}{8} \int_{\chi} \left( f^\prime_+ \left( \frac{p(x) + q(x)}{2p(x)} \right) - f^\prime_- \left( \frac{p(x) + q(x)}{2p(x)} \right) \right) |q(x) - p(x)| \, d\mu(x) \\
& \leq D_{f^H}^f (p, q) - D_f \left( p, \frac{1}{2} p + \frac{1}{2} q \right) \\
& \leq \frac{1}{8} \int_{\chi} \left( f_-(q(x) \frac{2p(x)}{p(x)} - f^\prime_+(1)) (q(x) - p(x)) \, d\mu(x),
\end{align*}

which is clearly equivalent to (6.5).

Corollary 7. With the above assumptions and if $f$ is differentiable on $(0, \infty)$, then

\begin{equation}
0 \leq D_{f^H}^f (p, q) - D_f \left( p, \frac{1}{2} p + \frac{1}{2} q \right) \leq \frac{1}{8} D_{f^\prime (-1)} (p, q).
\end{equation}

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