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AN OSTROWSKI TYPE INEQUALITY FOR CONVEX FUNCTIONS

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An Ostrowski type integral inequality for convex functions and applications for quadrature rules and integral means are given. A refinement and a counterpart result for Hermite–Hadamard inequalities are obtained and some inequalities for pdf's and (HH) −divergence measure are also mentioned.

1. INTRODUCTION

The following result is known in the literature as Ostrowski's inequality [**1**].

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \le M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right| \le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)\,M \tag{1.1}$$

for all $x \in [a,b]$.

The constant $1/4$ is the best possible in the sense that it cannot be replaced by a smaller constant.

A simple proof of this fact can be done by using the identity:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \frac{1}{b-a}\int_a^b p(x,t) f'(t)\,\mathrm{d}t, \quad x \in [a,b], \tag{1.2}$$

where

$$p(x,t) := \begin{cases} t-a & \text{if} \quad a \le t \le x \\ t-b & \text{if} \quad x < t \le b \end{cases}$$

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which holds for absolutely continuous functions $f : [a, b] \to \mathbb{R}$.

The following OSTROWSKI type result holds (see [**2**], [**3**] and [**4**]).

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous on $[a, b]$. Then, for all $x \in [a, b]$, we have:*

$$
\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, \mathrm{d}t \right| \tag{1.3}
$$

$$
\leq \begin{cases} \left(\dfrac{1}{4} + \left(\dfrac{x - \frac{a+b}{2}}{b-a} \right)^2 \right) (b-a) \left\| f' \right\|_\infty & \textit{if } f' \in L_\infty [a, b]; \\[2ex] \dfrac{1}{(p+1)^{1/p}} \left(\left(\dfrac{x-a}{b-a} \right)^{p+1} + \left(\dfrac{b-x}{b-a} \right)^{p+1} \right)^{1/p} (b-a)^{1/p} \left\| f' \right\|_q & \\ \qquad\qquad \textit{if } f' \in L_q [a, b],\ \dfrac{1}{p} + \dfrac{1}{q} = 1,\ p > 1; & \\[2ex] \left(\dfrac{1}{2} + \left| \dfrac{x - \frac{a+b}{2}}{b-a} \right| \right) \left\| f' \right\|_1; & \end{cases}
$$

where $\|\cdot\|_r$ $(r \in [1, \infty])$ are the usual Lebesgue norms on $L_r [a, b]$, i.e.,

$$
\|g\|_\infty := ess \sup_{t \in [a,b]} |g(t)|
$$

and

$$
\|g\|_r := \left(\int_a^b |g(t)|^r \, dt \right)^{1/r}, \; r \in [1, \infty).
$$

The constants $1/4$, $1/(p+1)^{1/p}$ and $1/2$ respectively are sharp in the sense presented in Theorem 1.

The above inequalities can also be obtained from FINK's result in [**5**] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is HÖLDER continuous, then one may state the result (see [**6**]):

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be of $r - H-$Hölder type, i.e.,*

$$
|f(x) - f(y)| \leq H |x - y|^r, \; \textit{for all } \; x, y \in [a, b], \tag{1.4}
$$

where $r \in (0, 1]$ and $H > 0$ are fixed. Then for all $x \in [a, b]$ we have the inequality:

$$
\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, \mathrm{d}t \right| \tag{1.5}
$$

$$
\leq \frac{H}{r+1} \left(\left(\frac{b-x}{b-a} \right)^{r+1} + \left(\frac{x-a}{b-a} \right)^{r+1} \right) (b-a)^r.
$$

The constant $1/(r+1)$ *is also sharp in the above sense.*

Note that if $r=1$, i.e., f is LIPSCHITZ continuous, then we get the following version of OSTROWSKI's inequality for Lipschitzian functions (with L instead of H) (see [**7**])

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leq\left(\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right)(b-a)\,L. \tag{1.6}$$

Here the constant $1/4$ is also best.

Moreover, if one drops the continuity condition of the function, and assumes that it is of bounded variation, then the following result may be stated (see [**8**]).

Theorem 4. *Assume that* $f:[a,b]\to\mathbb{R}$ *is of bounded variation and denote by* $\bigvee_a^b(f)$ *its total variation. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right|\leq\left(\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right)\bigvee_a^b(f) \tag{1.7}$$

for all $x\in[a,b]$.

The constant $1/2$ *is the best possible.*

If we assume more about f, i.e., f is monotonically increasing, then the inequality (1.7) may be improved in the following manner [**9**] (see also [**10**]).

Theorem 5. *Let* $f:[a,b]\to\mathbb{R}$ *be monotonic nondecreasing. Then for all* $x\in[a,b]$, *we have the inequality*:

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| &\leq\frac{1}{b-a}\left(\big(2x-(a+b)\big)f(x)+\int_a^b \operatorname{sgn}(t-x)f(t)\,\mathrm{d}t\right)\\ &\leq\frac{1}{b-a}\Big((x-a)\big(f(x)-f(a)\big)+(b-x)\big(f(b)-f(x)\big)\Big)\\ &\leq\left(\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right)\big(f(b)-f(a)\big).\end{aligned} \tag{1.8}$$

All the inequalities in (1.8) *are sharp and the constant* $1/2$ *is the best possible.*

For recent generalisations of OSTROWSKI inequality in various directions and for n-time differentiable functions, see [**14**] and the research monograph [**13**] where further references are provided.

In this paper we establish an OSTROWSKI type inequality for convex functions. Applications for quadrature rules, for integral means, for probability distribution functions, and for $HH-$divergences in Information Theory are also considered.

2. THE RESULTS

The following theorem providing a lower bound for the OSTROWSKI difference $\int_a^b f(t)\,\mathrm{d}t-(b-a)f(x)$ holds.

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in(a,b)$ we have the inequality*:

$$\frac{1}{2}\left((b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right)\le\int_a^b f(t)\,\mathrm{d}t-(b-a)f(x). \tag{2.1}$$

The constant $1/2$ in the left hand side of (2.1) is sharp in the sense that it cannot be replaced by a larger constant.

Proof. It is easy to see that for any locally absolutely continuous function $f:(a,b)\to\mathbb{R}$, we have the identity

$$\int_a^x (t-a)f'(t)\,\mathrm{d}t+\int_x^b (t-b)f'(t)\,\mathrm{d}t=f(x)-\int_a^b f(t)\,\mathrm{d}t, \tag{2.2}$$

for any $x\in(a,b)$ where f' is the derivative of f which exists a.e. on (a,b).

Since f is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any $x\in(a,b)$, we have the inequalities

$$f'(t)\le f'_-(x)\quad\text{for a.e. } t\in[a,x] \tag{2.3}$$

and

$$f'(t)\ge f'_+(x)\quad\text{for a.e. } t\in[x,b]. \tag{2.4}$$

If we multiply (2.3) by $t-a\ge 0$, $t\in[a,x]$, and integrate on $[a,x]$, we get

$$\int_a^x (t-a)f'(t)\,\mathrm{d}t\le\frac{1}{2}(x-a)^2 f'_-(x) \tag{2.5}$$

and if we multiply (2.4) by $b-t\ge 0$, $t\in[x,b]$, and integrate on $[x,b]$, we also have

$$\int_x^b (b-t)f'(t)\,\mathrm{d}t\ge\frac{1}{2}(b-x)^2 f'_+(x). \tag{2.6}$$

Finally, if we subtract (2.6) from (2.5) and use the representation (2.2) we deduce the desired inequality (2.1).

Now, assume that (2.1) holds with a constant $C>0$ instead of $1/2$, i.e.,

$$C\left((b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right)\le\int_a^b f(t)\,\mathrm{d}t-(b-a)f(x). \tag{2.7}$$

Consider the convex function $f_0(t):=k\left|t-\frac{a+b}{2}\right|$, $k>0$, $t\in[a,b]$. Then

$$f'_{0+}\left(\frac{a+b}{2}\right)=k,\quad f'_{0-}\left(\frac{a+b}{2}\right)=-k,\quad f_0\left(\frac{a+b}{2}\right)=0$$

and

$$\int_a^b f_0(t)\,\mathrm{d}t = \frac{1}{4}\,k(b-a)^2.$$

If in (2.7) we choose f_0 as above and $x=(a+b)/2$, then we get

$$C\left(\frac{1}{4}\,(b-a)^2k+\frac{1}{4}\,(b-a)^2k\right)\le\frac{1}{4}\,k(b-a)^2,$$

which gives $C\le 1/2$, and the sharpness of the constant is proved. $\square$

Now, recall that the following inequality, which is well known in the literature as the *Hermite–Hadamard inequality* for convex functions, holds:

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\le\frac{f(a)+f(b)}{2}. \tag{HH}$$

The following corollary which improves the first Hermite–Hadamard inequality (HH) holds.

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned}0&\le\frac{1}{8}\left(f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right)(b-a)\\&\le\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-f\left(\frac{a+b}{2}\right).\end{aligned}\tag{2.8}$$

The constant $1/8$ is sharp.

The proof is obvious by the above theorem. The sharpness of the constant is obtained for $f_0(t):=k\left|t-\dfrac{a+b}{2}\right|$, $t\in[a,b]$, $k>0$.

When x is a point of differentiability, we may state the following corollary as well.

Corollary 2. *Let f be as in Theorem 6. If $x\in(a,b)$ is a point of differentiability for f, then*

$$\left(\frac{a+b}{2}-x\right)f'(x)\le\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t-f(x). \tag{2.9}$$

Remark 1. *If $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is convex on I and if we choose $x\in\mathring{I}$ ($\mathring{I}$ is the interior of I), $b=x+\dfrac{h}{2}$, $a=x-\dfrac{h}{2}$, $h>0$ is such that $a,b\in I$, then from (2.1) we may write*

$$0\le\frac{1}{8}\,h^2\big(f'_+(x)-f'_-(x)\big)\le\int_{x-\frac{h}{2}}^{x+\frac{h}{2}}f(t)\,\mathrm{d}t-hf(x), \tag{2.10}$$

and the constant $1/8$ is sharp in (2.10).

The following result providing an upper bound for the Ostrowski difference $\int_a^b f(t)\,\mathrm{d}t - (b-a)f(x)$ also holds.

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $x \in [a,b]$, we have the inequality:*

$$\int_a^b f(t)\,\mathrm{d}t - (b-a)f(x) \le \frac{1}{2}\left((b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right). \tag{2.11}$$

The constant $1/2$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. If either $f'_+(a) = -\infty$ or $f'_-(b) = +\infty$, then the inequality (2.11) evidently holds true.

Assume that $f'_+(a)$ and $f'_-(b)$ are finite.

Since f is convex on $[a,b]$, we have

$$f'(t) \ge f'_+(a) \text{ for a.e. } t \in [a,x] \tag{2.12}$$

and

$$f'(t) \le f'_-(b) \text{ for a.e. } t \in [x,b]. \tag{2.13}$$

If we multiply (2.12) by $t-a \ge 0$, $t \in [a,x]$, and integrate on $[a,x]$, then we deduce

$$\int_a^x (t-a)f'(t)\,\mathrm{d}t \ge \frac{1}{2}(x-a)^2 f'_+(a) \tag{2.14}$$

and if we multiply (2.13) by $b-t \ge 0$, $t \in [x,b]$, and integrate on $[x,b]$, then we also have

$$\int_x^b (b-t)f'(t)\,\mathrm{d}t \le \frac{1}{2}(b-x)^2 f'_-(b). \tag{2.15}$$

Finally, if we subtract (2.14) from (2.15) and use the representation (2.2), we deduce the desired inequality (2.11).

Now, assume that (2.11) holds with a constant $D > 0$ instead of $1/2$, i.e.,

$$\int_a^b f(t)\,\mathrm{d}t - (b-a)f(x) \le D\left((b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right). \tag{2.16}$$

If we consider the convex function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = k\left|t - \frac{a+b}{2}\right|$, then we have $f'_-(b) = k$, $f'_+(a) = -k$ and by (2.16) we deduce for $x = \frac{a+b}{2}$ that

$$\frac{1}{4}k(b-a)^2 \le D\left(\frac{1}{4}k(b-a)^2 + \frac{1}{4}k(b-a)^2\right),$$

giving $D \ge 1/2$, and the sharpness of the constant is proved. □

The following corollary related to the HERMITE–HADAMARD inequality is interesting as well.

Corollary 3. *Let $f : [a,b] \to \mathbb{R}$ be convex on $[a,b]$. Then*

$$0 \leq \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}\left(f'_-(b) - f'_+(a)\right)(b-a) \tag{2.17}$$

and the constant $1/8$ is sharp.

REMARK 2. *Denote $B := f'_-(b)$, $A := f'_+(a)$ and assume that $B \neq A$, i.e., f is not constant on (a,b). Then*

$$(b-x)^2 B - (x-a)^2 A = (B-A)\left(x - \left(\frac{bB-aA}{B-A}\right)\right)^2 - \frac{AB}{B-A}(b-a)^2$$

and by (2.11) we get

$$\int_a^b f(t)\,\mathrm{d}t - (b-a)f(x) \leq \frac{1}{2}(B-A)\left(\left(x - \left(\frac{bB-aA}{B-A}\right)\right)^2 - \frac{AB}{(B-A)^2}(b-a)^2\right) \tag{2.18}$$

for any $x \in [a,b]$.

If $A \geq 0$ then $x_0 = \dfrac{bB-aA}{B-A} \in [a,b]$ and by (2.18) we get, choosing $x = \dfrac{bB-aA}{B-A}$, that

$$0 \leq \frac{1}{2}\frac{AB}{B-A}(b-a) \leq f\left(\frac{bB-aA}{B-A}\right) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t, \tag{2.19}$$

which is an interesting inequality in itself.

REMARK 3. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on I and if we choose $x \in \mathring{I}$, $b = x + \frac{h}{2}$, $a = x - \frac{h}{2}$, $h > 0$ is such that $a, b \in I$, then from (2.11) we deduce:*

$$0 \leq \int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f(t)\,\mathrm{d}t - hf(x) \leq \frac{1}{8}h^2\left(f'_-\left(x+\frac{h}{2}\right) - f'_+\left(x-\frac{h}{2}\right)\right), \tag{2.20}$$

and the constant $1/8$ is sharp.

3. THE COMPOSITE CASE

Consider the division $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and denote $h_i := x_{i+1} - x_i$, $i = \overline{0, n-1}$. If $\xi_i \in [x_i, x_{i+1}]$ $(i = \overline{0, n-1})$ are intermediate points, then we will denote by

$$R_n(f; I_n, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} h_i f(\xi_i) \tag{3.1}$$

the RIEMANN sum associated to f, I_n and $\boldsymbol{\xi}$.

The following theorem providing upper and lower bounds for the remainder in approximating the integral $\int_a^b f(t)\,\mathrm{d}t$ of a convex function f in terms of a general Riemann sum holds.

Theorem 8. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and I_n and $\boldsymbol{\xi}$ be as above. Then we have*:

$$\int_a^b f(t)\,dt = R_n(f;I_n,\boldsymbol{\xi}) + W_n(f;I_n,\boldsymbol{\xi}), \tag{3.2}$$

where $R_n(f;I_n,\boldsymbol{\xi})$ is the Riemann sum defined by (3.1) *and the remainder $W_n(f;I_n,\xi)$ satisfies the estimate*:

$$\begin{aligned}\frac{1}{2}\left(\sum_{i=0}^{n-1}(x_{i+1}-\xi_i)^2 f'_+(\xi_i) - \sum_{i=0}^{n-1}(\xi_i-x_i)^2 f'_-(\xi_i)\right)&\\ \le W_n(f;I_n,\boldsymbol{\xi})&\\ \le \frac{1}{2}\Bigg((b-\xi_{n-1})^2 f'_-(b) + \sum_{i=1}^{n-1}\Big((x_i-\xi_{i-1})^2 f'_-(x_i)&\\ -(\xi_i-x_i)^2 f'_+(x_i)\Big) - (\xi_0-a)^2 f'_+(a)\Bigg).&\end{aligned} \tag{3.3}$$

Proof. If we write the inequalities (2.1) and (2.11) on the interval $[x_i,x_{i+1}]$ and for the intermediate points $\xi_i\in[x_i,x_{i+1}]$, then we have

$$\begin{aligned}&\frac{1}{2}\Big((x_{i+1}-\xi_i)^2 f'_+(\xi_i) - (\xi_i-x_i)^2 f'_-(\xi_i)\Big)\\ &\qquad\le \int_{x_i}^{x_{i+1}} f(t)\,\mathrm{d}t - h_i f(\xi_i)\\ &\qquad\le \frac{1}{2}\Big((x_{i+1}-\xi_i)^2 f'_-(x_{i+1}) - (\xi_i-x_i)^2 f'_+(x_i)\Big).\end{aligned}$$

Summing the above inequalities over i from 0 to $n-1$, we deduce

$$\begin{aligned}&\frac{1}{2}\sum_{i=0}^{n-1}\Big((x_{i+1}-\xi_i)^2 f'_+(\xi_i) - (\xi_i-x_i)^2 f'_-(\xi_i)\Big)\\ &\quad\le \int_a^b f(t)\,\mathrm{d}t - R_n(f;I_n,\boldsymbol{\xi})\\ &\quad\le \frac{1}{2}\left(\sum_{i=0}^{n-1}(x_{i+1}-\xi_i)^2 f'_-(x_{i+1}) - \sum_{i=0}^{n-1}(\xi_i-x_i)^2 f'_+(x_i)\right).\end{aligned} \tag{3.4}$$

However,

$$\begin{aligned}\sum_{i=0}^{n-1}(x_{i+1}-\xi_i)^2 f'_-(x_{i+1}) &= (b-\xi_{n-1})^2 f'_-(b) + \sum_{i=0}^{n-2}\Big((x_{i+1}-\xi_i)^2 f'_-(x_{i+1})\Big)\\ &= (b-\xi_{n-1})^2 f'_-(b) + \sum_{i=1}^{n-1}\Big((x_i-\xi_{i-1})^2 f'_-(x_i)\Big)\end{aligned}$$

and

$$\sum_{i=0}^{n-1}(\xi_i - x_i)^2 f'_+(x_i) = \sum_{i=1}^{n-1}(\xi_i - x_i)^2 f'_+(x_i) + (\xi_0 - a)^2 f'_+(a)$$

and then, by (3.4), we deduce the desired estimate (3.3). □

The following corollary may be useful in practical applications.

Corollary 4. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable convex function on (a,b). Then we have the representation* (3.1) *and the remainder $W_n(f; I_n, \boldsymbol{\xi})$ satisfies the estimate*:

$$\begin{aligned}(3.5)\quad \sum_{i=0}^{n-1}\left(\frac{x_i + x_{i+1}}{2} - \xi_i\right) h_i f'(\xi_i) &\le W_n(f; I_n, \xi_i) \\ &\le \frac{1}{2}\Big((b - \xi_{n-1})^2 f'_-(b) - (\xi_0 - a)^2 f'_+(a)\Big) \\ &\quad + \sum_{i=1}^{n-1}\left(x_i - \frac{\xi_i + \xi_{i-1}}{2}\right)(\xi_i - \xi_{i-1}) f'(x_i).\end{aligned}$$

We may also consider the mid-point quadrature rule:

$$M_n(f, I_n) := \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right). \tag{3.6}$$

Using Corollaries 1 and 2, we may state the following result as well.

Corollary 5. *Assume that $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$ and I_n is a division as above. Then we have the representation*:

$$\int_a^b f(x)\,\mathrm{d}x = M_n(f, I_n) + S_n(f, I_n), \tag{3.7}$$

where $M_n(f, I_n)$ is the mid-point quadrature rule given in (3.6) *and the remainder $S_n(f, I_n)$ satisfies the estimates*:

$$\begin{aligned}(3.8)\quad 0 &\le \frac{1}{8}\sum_{i=0}^{n-1}\left(f'_+\left(\frac{x_i + x_{i+1}}{2}\right) - f'_-\left(\frac{x_i + x_{i+1}}{2}\right)\right) h_i^2 \\ &\le S_n(f, I_n) \le \frac{1}{8}\sum_{i=0}^{n-1}\Big(f'_-(x_{i+1}) - f'_+(x_i)\Big) h_i^2.\end{aligned}$$

The constant $1/8$ is sharp in both inequalities.

4. INEQUALITIES FOR INTEGRAL MEANS

We may prove the following result in comparing two integral means.

Theorem 9. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $c, d \in [a,b]$ with $c < d$. Then we have the inequalities*

$$\begin{aligned}
&\frac{a+b}{2} \cdot \frac{f(d)-f(c)}{d-c} - \frac{df(d)-cf(c)}{d-c} + \frac{1}{d-c}\int_c^d f(x)\,\mathrm{d}x \\
&\quad \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(x)\,\mathrm{d}x \\
&\quad \le \frac{f'_-(b)\big((b-d)^2+(b-d)(b-c)+(b-c)^2\big)}{6\,(b-a)} \\
&\qquad\quad - \frac{f'_+(a)\big((d-a)^2+(d-a)(c-a)+(c-a)^2\big)}{6\,(b-a)}.
\end{aligned} \tag{4.1}$$

Proof. Since f is convex, then for *a.e.* $x \in [a,b]$, we have (by (2.9)) that

$$\left(\frac{a+b}{2} - x\right) f'(x) \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f(x). \tag{4.2}$$

Integrating (4.2) on $[c,d]$ we deduce

$$\frac{1}{d-c}\int_c^d \left(\frac{a+b}{2} - x\right) f'(x)\,\mathrm{d}x \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(x)\,\mathrm{d}x. \tag{4.3}$$

Since

$$\begin{aligned}
&\frac{1}{d-c}\int_c^d \left(\frac{a+b}{2} - x\right) f'(x)\,\mathrm{d}x \\
&\quad = \frac{1}{d-c}\left(\left(\frac{a+b}{2} - d\right) f(d) - \left(\frac{a+b}{2} - c\right) f(c) + \int_c^d f(x)\,\mathrm{d}x\right)
\end{aligned}$$

then by (4.3) we deduce the first part of (4.1).

Using (2.11), we may write for any $x \in [a,b]$ that

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f(x) \le \frac{1}{2\,(b-a)}\big((b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\big). \tag{4.4}$$

Integrating (4.4) on $[c,d]$, we deduce

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{d-c}\int_c^d f(x)\,\mathrm{d}x \tag{4.5}$$
$$\leq \frac{1}{2(b-a)}\left(f'_-(b)\,\frac{1}{d-c}\int_c^d (b-x)^2\,\mathrm{d}x - f'_+(a)\,\frac{1}{d-c}\int_c^d (x-a)^2\,\mathrm{d}x\right).$$

Since

$$\frac{1}{d-c}\int_c^d (b-x)^2\,\mathrm{d}x = \frac{(b-d)^2+(b-d)(b-c)+(b-c)^2}{3}$$

and

$$\frac{1}{d-c}\int_c^d (x-a)^2\,\mathrm{d}x = \frac{(d-a)^2+(d-a)(c-a)+(c-a)^2}{3},$$

then by (4.5) we deduce the second part of (4.1). □

Remark 4. *If we choose* $f(x)=x^p$, $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ *or* $f(x)=1/x$ *or even* $f(x)=-\ln x$, $x\in[a,b]\subset(0,\infty)$, *in the above inequalities, then a great number of interesting results for* $p-$*logarithmic, logarithmic and identric means may be obtained. We leave this as an exercise to the interested reader.*

5. APPLICATIONS FOR P.D.F.S

Let X be a random variable with the *probability density function* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}_+$ and with *cumulative distribution function* $F(x)=\Pr(X\leq x)$.

The following theorem holds.

Theorem 10. *If* $f:[a,b]\subset\mathbb{R}\to\mathbb{R}_+$ *is monotonically increasing on* $[a,b]$, *then we have the inequality*:

$$\frac{1}{2}\left((b-x)^2 f_+(x) - (x-a)^2 f_-(x)\right) \tag{5.1}$$
$$\leq b - E(X) - (b-a)F(x)$$
$$\leq \frac{1}{2}\left((b-x)^2 f_-(b) - (x-a)^2 f_+(a)\right)$$

for any $x\in(a,b)$, *where* $f_-(\alpha)$ *means the left limit in* α *while* $f_+(\alpha)$ *means the right limit in* α *and* $E(X)$ *is the expectation of* X.

The constant $1/2$ *is sharp in both inequalities.*

The second inequality also holds for $x=a$ *or* $x=b$.

Proof. Follows by Theorem 6 and 7 applied for the convex cdf function $F(x)=\int_a^x f(t)\,\mathrm{d}t$, $x\in[a,b]$ and taking into account that

$$\int_a^b F(x)\,\mathrm{d}x = b - E(X).$$ □

Finally, we may state the following corollary in estimating the probability $\Pr\left(X\leq\frac{a+b}{2}\right)$.

Corollary 6. *With the above assumptions, we have*

$$b - E(X) - \frac{1}{8}(b-a)^2\big(f_-(b) - f_+(a)\big) \tag{5.2}$$
$$\leq \Pr\left(X \leq \frac{a+b}{2}\right)$$
$$\leq b - E(X) - \frac{1}{8}(b-a)^2\left(f_+\left(\frac{a+b}{2}\right) - f_-\left(\frac{a+b}{2}\right)\right).$$

6. APPLICATIONS FOR HH−DIVERGENCE

Assume that a set χ and the $\sigma-$finite measure μ are given. Consider the set of all probability densities on μ to be

$$\Omega := \left\{p | p : \Omega \to \mathbb{R},\ p(x) \geq 0,\ \int_\chi p(x)\,\mathrm{d}\mu(x) = 1\right\}. \tag{6.1}$$

Csiszár's $f-$divergence is defined as follows [**11**]

$$D_f(p,q) := \int_\chi p(x) f\left(\frac{q(x)}{p(x)}\right)\mathrm{d}\mu(x),\ p,q \in \Omega, \tag{6.2}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived.

In [**12**], Shioya and Da-te introduced the generalised Lin-Wong $f-$divergence $D_f\Big(p, \frac{1}{2}p + \frac{1}{2}q\Big)$ and the Hermite-Hadamard (HH) divergence

$$D_{HH}^f(p,q) := \int_\chi \frac{p^2(x)}{q(x)-p(x)}\left(\int_1^{q(x)/p(x)} f(t)\,\mathrm{d}t\right)\mathrm{d}\mu(x),\ p,q \in \Omega, \tag{6.3}$$

and, by the use of the Hermite-Hadamard inequality for convex functions, proved the following basic inequality

$$D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right) \leq D_{HH}^f(p,q) \leq \frac{1}{2}D_f(p,q), \tag{6.4}$$

provided that f is convex and normalised, i.e., $f(1) = 0$.

The following result in estimating the difference

$$D_{HH}^f(p,q) - D_f\left(p, \frac{1}{2}p + \frac{1}{2}q\right)$$

holds.

Theorem 11. *Let $f : [0, \infty) \to \mathbb{R}$ be a convex function and $p, q \in \Omega$. Then we have the inequality*:

$$
\begin{aligned}
0 &\le \frac{1}{8}\left(D_{f'_+ \cdot \left|\frac{\cdot+1}{2}\right|}(p,q) - D_{f'_- \cdot \left|\frac{\cdot+1}{2}\right|}(p,q)\right) \\
&\le D^f_{HH}(p,q) - D_f\left(p, \frac{1}{2}\,p + \frac{1}{2}\,q\right) \\
&\le \frac{1}{8} D_{f'_- \cdot (\cdot - 1)}(p,q).
\end{aligned} \tag{6.5}
$$

Proof. Using the double inequality

$$
\begin{aligned}
0 &\le \frac{1}{8}\left(f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right)|b-a| \\
&\le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f\left(\frac{a+b}{2}\right) \\
&\le \frac{1}{8}\big(f_-(b) - f'_+(a)\big)(b-a)
\end{aligned}
$$

for the choices $a = 1$, $b = \dfrac{q(x)}{p(x)}$, $x \in \chi$, multiplying with $p(x) \ge 0$ and integrating over x on χ we get

$$
\begin{aligned}
0 &\le \frac{1}{8}\int_\chi \left(f'_+\left(\frac{p(x)+q(x)}{2p(x)}\right) - f'_-\left(\frac{p(x)+q(x)}{2p(x)}\right)\right)|q(x)-p(x)|\,\mathrm{d}\mu(x) \\
&\le D^f_{HH}(p,q) - D_f\left(p, \frac{1}{2}\,p + \frac{1}{2}\,q\right) \\
&\le \frac{1}{8}\int_\chi \left(f'_-\left(\frac{q(x)}{p(x)}\right) - f'_+(1)\right)\big(q(x)-p(x)\big)\,\mathrm{d}\mu(x),
\end{aligned}
$$

which is clearly equivalent to (6.5). □

Corollary 7. *With the above assumptions and if f is differentiable on* $(0, \infty)$, *then*

$$
0 \le D^f_{HH}(p,q) - D_f\left(p, \frac{1}{2}\,p + \frac{1}{2}\,q\right) \le \frac{1}{8}\,D_{f' \cdot (\cdot - 1)}(p,q). \tag{6.6}
$$

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