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SOME REMARKS REGARDING PITMAN CLOSENESS

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We give an equivalent definition of Pitman's closeness criterion, in terms of medians of the difference of loss functions. Based on that definition, we present some relations between Pitman's closeness and usual decision theoretical framework, then a result which enables comparison of estimators with a presence of a positive or negative association, and a Rao-Blackwell type result related to improving estimators by a conditioning.

1. INTRODUCTION

Let T_1 and T_2 be two estimators of a same parameter θ in a parameter space Θ. E. J. G. Pitman [**17**] introduced the concept of closeness as follows: We say that T_1 is closer to θ then T_2 if

$$P_\theta\big(|T_1-\theta| < |T_2-\theta|\big) > \frac{1}{2}\,, \tag{1}$$

for all $\theta \in \Theta$. In the paper [**16**] he actually found the best estimators in the sense of (1) for several cases of interest. Besides Pitman's original definition (1), there are several variations on the theme. According to Rao's definition [**18**], T_1 is closer than T_2 if

$$P_\theta(|T_1-\theta| \le |T_2-\theta|) \ge \frac{1}{2}\,, \tag{2}$$

for all $\theta \in \Theta$, with a strict inequality for at least one θ. Nayak [**12**] requires that

$$P_\theta(|T_1-\theta| \le |T_2-\theta|) \ge P_\theta(|T_2-\theta| \le |T_1-\theta|) \tag{3}$$

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for each $\theta \in \Theta$. Closely related to these definitions is PITMAN's measure of closeness (nearness):

$$PN(T_1, T_2, \theta) = P_\theta\big(|T_1 - \theta| \le |T_2 - \theta|\big). \tag{4}$$

In all these definitions one can replace the absolute difference with any loss function in two variables. This is called generalized PITMAN's criterion [**15, 12**]. For instance, a generalized PITMAN's measure of nearness would be

$$GPN(T_1, T_2, \theta) = P_\theta\big(L(T_1, \theta) \le L(T_2, \theta)\big). \tag{5}$$

The approach with generalized PITMAN's criterion gives an obvious way of dealing with multidimensional parameters.

When T_1 is closer to T_2 in the sense of some of the above definitions, then we say that T_1 is P-better that T_2 or $T_1 \prec T_2$. It is usually hard to show the "for at least one θ" part, and so, the vast majority of work is concentrated on showing that one of (1)–(3) holds for every $\theta \in \Theta$.

Finally, it is clear that we need not restrict to parameters and their estimators in order to measure closeness. For instance, if Z is a random variable and X, Y are other two random variables, then we may define

$$GPN(X, Y, Z) = P\big(d(X, Z) \le d(Y, Z)\big),$$

where d is some appropriate measure of distance.

PITMAN's criterion is closely related to medians, and we will give some details in subsequent sections. For a convenience, recall that median of a random variable X is any real number m such that

$$P(X \le m) \ge \frac{1}{2} \quad \text{and} \quad P(X \ge m) \ge \frac{1}{2}\,.$$

Therefore, it can be seen that definition (2), or its generalized version via the loss function, can be expressed as

$$\mathrm{Med}\big(L(T_0, \theta) - L(T, \theta)\big) \le 0, \tag{6}$$

which gives a decision-theoretic flavor to PITMAN's criterion. Of course, here a mean is replaced by a median and the usual loss function is replaced by a difference of two loss functions (comparative loss function in the sense of [**19**]. This simple observation will be explored later in the paper.

We would occasionally need some conditions on the loss function. These are as follows.

1.1. Conditions on loss functions–I. The loss function $L(a, \theta)$ is for any fixed θ a non-increasing (non-decreasing) function of a in the domain $a \le \theta$ ($a \ge \theta$). In some details we will need another assumption:

1.2. Conditions on loss functions–II. The loss function $L(a, \theta)$ is for any fixed a a non-increasing (non-decreasing) function of θ in the domain $\theta \le a$ ($\theta \ge a$).

Note that for a symmetric loss ($L(a,\theta) = L(\theta,a)$) conditions I and II are equivalent. In what follows we will sometimes need stronger versions of the above conditions in the sense that non-increasing should be replaced by increasing and non-decreasing should be increasing. We will refer to those conditions as *strict conditions.*

2. SOME PROPERTIES OF PITMAN'S CRITERION

Besides the comparison which one of two estimators is P-better, the Pitman criterion provides another, more useful information which is contained in the value of PN or GPN, which are defined by (4) and (5). Indeed, if PN is close to 0.5, then for most practical purposes it is really irrelevant which estimator will be used, from the viewpoint of this criterion. In rare cases, the evaluation of PN is analytically tractable, but mostly one has to use numerical methods or simulation. There are some quantitative examples in the literature, cf. [**4**], [**6**], [**7**], [**11**]. In the present paper we contribute some more numerical examples.

Pitman's criterion naturally leads to medians in much the same way as the mean square error leads to means. We will illustrate this feature by a simple example. Suppose we want to determine a single number b which gives the best description of a given random variable X. In the sense of MSE criterion, we need to find b so that

$$\mathrm{E}(X-b)^2 \le \mathrm{E}(X-a)^2 \quad \text{for each } a \in \mathbb{R},$$

and then we get $b = \mathrm{E}X$, if the expectation exists. In the sense of Pitman's criterion, it is not difficult to show that

$$P(|X-m| \le |X-a|) \ge \frac{1}{2} \quad \text{for each } a \in \mathbb{R},$$

where $m = \mathrm{Med}(X)$ is a median of X. In fact, a more general result holds, which we give in the following Lemma.

2.1. Lemma. *Let X be a random variable with a median m and let L be any loss which satisfies Conditions* 1.2. *Then*

$$\mathrm{Med}\big(L(X,m) - L(X,c)\big) \le 0 \quad \textit{for any } c.$$

If strict conditions in 1.2. *are satisfied, then*

$$\mathrm{Med}(L(X,m) < L(X,c)) < 0 \quad \textit{for any } c \ne m.$$

Proof. Suppose that $m < c$ and let $X \le m$. Then by 1.2, $L(X,m) \le L(X,c)$. Therefore, in this case, the event $\{X \le m\}$ implies the event $\{L(X,m) \le L(X,c)$, and so

$$\frac{1}{2} \le P(X \le m) \le P\big(L(X,m) \le L(X,c)\big).$$

If $m > c$ then we can see that the event $\{X \geq m\}$ implies the event $\{L(X,m) \leq L(X,c)\}$, hence

$$\frac{1}{2} \leq P(X \geq m) \leq P\big(L(X,m) \leq L(X,c)\big). \qquad \square$$

We remark that Lemma 2.1. was essentially known to PITMAN [**17**] for $L(a,\theta) = |a-\theta|$ (and probably was known much before in this form). NAYAK [**12**] proved it for $L(a,\theta) = h(a-\theta)$, where h is decreasing for negative values of arguments and increasing for positive values.

Therefore (with some conditions to avoid trivial cases), a median is P-closer to X than any other real number.

In passing, let us note that the median is obtained also as a result of L_1 criterion-minimizing the mean absolute deviation. However, the PITMAN'S criterion and L_1 are generally different.

3. FRAMEWORK OF DECISION THEORY

A drawback of PITMAN's criterion is that it does not fit into the usual framework of decision theory, that is, it does not follow from the evaluation of risk. We will show that, under some additional assumptions, some kind of decision theoretical approach is possible. This is a complement to the work of PEDDADA [**15, 16**], LEE [**10**], and [RUKHIN] [**19**].

Suppose that we have a class $\mathcal{T}$ of estimators of θ and let T_0 be a P-best estmator in $\mathcal{T}$, in the sense that (6) holds for any $T \in \mathcal{T}$. Then let T_0 be an estimator in $\mathcal{T}$ with the least median of loss, that is

$$\text{Med}\big(L(T_0,\theta)\big) - \text{Med}\big(L(T,\theta)\big) \leq 0, \tag{7}$$

for each $T \in \mathcal{T}$. If we were dealing with expectations, (6) and (7) would yield the same estimator, i.e.,

$$\text{E}\big(L(T_0,\theta)\big) - \text{E}\big(L(T,\theta)\big) \leq 0. \tag{8}$$

However, for medians we have to impose some additional conditions to ensure any one way implication between these three. For median unbiased estimators, relation (7) is an analogue of variance comparision in the usual MSE setup, since $\text{Med}\, L(T,\theta)$ is a measure of a dispersion of T arround θ. Let us firstly consider an example.

3.1. Example. Suppose that we have an iid sample of size n from a normal $\mathcal{N}(0,\sigma^2)$ distribution. Let $\mathcal{T} = \{T_\alpha\}$ be the class of estimators for σ^2 which are of the form

$$T_\alpha = \frac{1}{\alpha}\sum_{i=1}^{n} {X_i}^2,$$

and suppose, for simplicity, that we want to estimate σ^2 with the absolute difference loss, $L(T,\theta) = |T-\theta|$. Then it is well known (in fact, it was stated in PITMAN's paper [**17**], but later several times rediscovered) that the P-best estimator here is T_{α_0} with $\alpha_0 = \mathrm{Med}\,(\chi_n^2)$. Now if we propose to find an estimator with the smallest median of deviations, that is

$$\mathrm{Med}\left|\frac{1}{\alpha}\sum_{i=1}^{n} {X_i}^2 - \sigma^2\right| \;\to\; \min,$$

then it obviously leads to the problem of finding $\alpha = \alpha_0' > 0$ that minimizes

$$\varphi(\alpha) = \mathrm{Med}\left|\frac{Y}{\alpha} - 1\right|, \tag{9}$$

where $Y \sim \chi_n^2$.These two problems have different solutions, in general. For example, if $n = 5$, then $\alpha_0 = \mathrm{Med}({\chi_5}^2) = 4.351$ and $\alpha_0' = 5.321$ (the latter number is obtained numerically). It is interesting to note the discrepancy between the minimum value in (9) which is 0.416 (at $\alpha = \alpha_0'$) and the value at $\alpha = \alpha_0$, which is 0.594. The best estimator in the sense of (8) is again the one with $\alpha = \alpha_0$. □

Still, there is a connection between (6), (7) and (8), under some additional conditions. One approach would be suitable if the distribution of $L(T_0, \theta) - L(T, \theta)$ is unimodal. Then we may recall the well known mean-median-mode inequality which states that, under certain conditions, the mean, median and mode occur in this or in reversed order, see [**2**] or [**3**] for more details. Let μ, m, M be the mean, median and mode for the distribution of $L(T_0, \theta) - L(T, \theta)$. If

$$\mu \le m \le M \tag{10}$$

then (6) implies (8) and is implied by $M \le 0$. If inequalities in (10) are reversed, then (8) implies (6). Therefore, we have the following result.

3.2. Theorem. *If the distribution of $L(T_0, \theta) - L(T, \theta)$ is unimodal with a negative mode, if*

$$\mathrm{E}\big(L(T_0, \theta)\big) \le \mathrm{E}\big(L(T, \theta)\big)$$

and if the mean-median-mode inequality holds, then (6) *holds.*

The reason why we state this theorem is that the mean-median-mode inequality is observed in all cases of some practical interest. For the sake of completeness, we give a sufficient condition for this result to hold (cf. [**2**] or [**3**]).

Theorem. [**3**] *Let X be a unimodal random variable. If $(X - m)^+$ is stochastically larger than $(X - m)^-$, then X has a mode M satisfying $M \le m \le \mu$..*

A second approach, to find a connection between (6) and (7), is based on the following simple lemma.

3.3. Lemma. *Let X and Y be jointly distributed random variables such that*

$$m_X \le m_Y, \qquad \textit{where } m_X = \mathrm{Med}(X),\; m_Y = \mathrm{Med}(Y), \tag{11}$$

and

$$\text{Med}\,(|X - m_X| + |Y - m_Y|) \le m_Y - m_X. \tag{12}$$

Then $\text{Med}(Y - X) \le 0$.

Proof. Let Q be the square with the center in (m_X, m_Y) and one side on the line $y = x$. Then (12) simply states that more than $1/2$ of the mass of joint distribution for (X, Y) is concentrated in Q. By the assumption (11), Q is in the upper half plane and the assertion follows. □

Note that the expression at the left hand side of (12) is a quantitative measure of dispersion: in one dimension it is known as median absolute deviation.

Based on the Lemma 3.3, we can conclude that for (6) to hold, it suffices that (7) holds and that the joint distribution of $L(T_0, \theta)$ and $L(T, \theta)$ is sufficiently concentrated (in the sense of (12)) in the neighborhood of the intersection of their respective "median lines". This requires either strong concentration of both marginal distributions around their medians, or a relatively strong dependence between T_0 and T.

3.4. Example. If the distribution of X is symmetric, then both the sample mean and sample median (in the case of an odd sample size) are unbiased equivariant estimators (under appropriate invariant loss) of $\theta = \text{E}X$. Since PITMAN's criterion usually favors median unbiased estimators, it is interesting to compare two median unbiased estimators. We will consider a normal and an exponential case.

Normal distribution. Let $X \sim \mathcal{N}(\mu, 1)$ and let $X_1, \dots, X_{2n+1}$ be a sample drawn from the distribution of X. Let $\widehat{\mu}$ and $\widehat{m} = X_{(n+1)}$ be the sample mean and median, respectively. In the existing literature, comparision of these two estimators under PC has not been made. FOUNTAIN [**4**] compared X_1 vs. $\widehat{m}$, although his method is applicable for comparison of $\widehat{\mu}$ vs. $\widehat{m}$. A Monte Carlo simulation reveals that under the squared error loss, $\widehat{\mu}$ is P-better than $\widehat{m}$. In the Table we present results based on a Monte Carlo simulation of size 10000. $PN(\widehat{m}, \widehat{\mu}, \mu)$ is the probability that sample median is closer to $\mu = 0$ than the sample mean.

Sample size	$PN(\widehat{m}, \widehat{\mu}, \mu)$
5	0.39
15	0.40
25	0.39
35	0.39
45	0.39
55	0.39
65	0.39
75	0.38
85	0.39
95	0.39
105	0.38

Laplace distribution. In the case of LAPLACE distribution, with density

$$f_\theta(x) = \frac{1}{2}\, e^{-|x-\theta|}, \quad -\infty < x < +\infty,$$

FOUNTAIN [**4**] obtained exact results which show that $\widehat{m}$ is P-better than $\widehat{\mu}$.

T-distribution. As shown in the table below, the results for the T-distribution depend on sample size and on degrees of freedom. The sample median generally dominates the sample mean for large sample sizes and small number of degrees of freedom.

n/df	1	2	3	4	5
5	0.72	0.57	0.53	0.49	0.47
15	0.83	0.64	0.55	0.51	0.48
25	0.86	0.66	0.56	0.51	0.48
35	0.88	0.67	0.57	0.51	0.49
45	0.90	0.68	0.57	0.52	0.49
55	0.90	0.69	0.57	0.51	0.49
65	0.91	0.69	0.58	0.52	0.48
75	0.91	0.70	0.57	0.52	0.50
85	0.92	0.70	0.59	0.52	0.49
95	0.92	0.70	0.58	0.52	0.48
105	0.93	0.70	0.59	0.52	0.49

$PN(\widehat{m}, \widehat{\mu}, \mu)$ *(probability that the sample median is closer to* $\mu = 0$ *than the sample mean) for different sample sizes* n *and degrees of freedom df of* T *distribution.*

Discussion. Normal and LAPLACE case can be explained theoretically: normal case by [**12**, Theorem 3.3] (or, since there is an ancillary statistics involved, by KUBOKAWA's general result [**9**]). However, in the light of results obtained in this section, we may offer a heuristic explanation, which may deserve a further research. Note that in both cases prefered estimators are MLE ones. Since MLE estimators have the minimal variance, a similar property should hold for medians of loss, and via Lemma 3.3, for PITMAN's comparision. The behavior of estimators in the case of STUDENT's distribution is also interesting, because here neither the sample median nor the sample mean are MLE.

4. ASSOCIATION AND PITMAN CLOSENESS

4.1. Definition. *We say that random variables* X *and* Y *are positively associated if*

$$P(X \le y, Y \le y) \ge P(X \le x)P(Y \le y), \quad \textit{for all } x, y \in \mathbb{R}, \tag{13}$$

and they are negatively associated if

$$P(X \le y, Y \le y) \le P(X \le x)P(Y \le y), \quad \textit{for all } x, y \in \mathbb{R}. \tag{14}$$

It is easy to check that these conditions are respectively equivalent to

$$P(X > y, Y > y) \ge P(X > x)P(Y > y), \quad \text{for all } x, y \in \mathbb{R}, \tag{15}$$

$$P(X > y, Y > y) \le P(X > x)P(Y > y), \quad \text{for all } x, y \in \mathbb{R}. \tag{16}$$

Due to existence of left limits of probability, we may interchange $\leq$ and $<$ and also $\geq$ and $>$ in all events which probability is evaluated. Further, it is not difficult to see that X and Y are negatively associated if and only if X and $-Y$ are positively associated.

Association is obviously a more complex condition than correlation, but has a similar interpretation (see [**1, 3**] for more details). In particular, any two independent random variables are both positively and negatively associated. It was proved by Sibiuya [**20**] that minimum and maximum in a sample are positively associated. More than that, it can be proved that any two order statistics are positively associated, but the proof would not be along the lines of the present paper.

For a convenience, we give here a definition of a median unbiased estimator.

4.2. Definition. *An estimator T of a parameter θ is called median unbiased if* $\mathrm{Med}_\theta(T) = \theta$, *or, equivalently, if*

$$P_\theta(T \leq \theta) \geq \frac{1}{2} \quad \textit{and} \quad P_\theta(T \geq \theta) \geq \frac{1}{2}\,.$$

4.3. Theorem. *Let T be a median unbiased estimator of a parameter $\theta \in \Theta$ and let either* (i) $U = T + Z$, *or* (ii) $U = T(1 + Z)$, *where T and Z are positively associated. Then*

$$\mathrm{Med}_\theta\big(L(T,\theta) - L(U,\theta)\big) \leq 0 \tag{17}$$

for any $\theta \in \theta$ and any loss which satisfies Condition 1.1.

Proof. We give the proof for case (i) only, since (ii) is similar. Let events A and B be defined as

$$A = \{T \leq \theta,\ Z < 0\}, \quad B = \{T \geq \theta,\ Z \geq 0\}.$$

It is easy to see that, due to the assumptions we adopted for L, both events imply that $L(T,\theta) \leq L(U,\theta)$ and so,

$$\begin{aligned} P\big(L(T,\theta) \leq L(U,\theta)\big) &\geq P(A \cup B) = P(A) + P(B) \\ &\geq P(T \leq \theta)P(Z < 0) + P(T \geq \theta)P(Z \geq 0) \\ &\geq \frac{1}{2}\,P(Z < 0) + \frac{1}{2}\,P(Z > 0) = \frac{1}{2}\,. \end{aligned}$$

Remarks. For a case of T and Z being independent, a similar result was proved by Ghosh and Sen [**5**], for the absolute value loss.

We can also estimate the difference between the probability in Theorem 4.3 and 1/2. Indeed, the event $L(T,\theta) \leq L(U,\theta)$ is also implied by the following two events:

$$C = \{T \leq \theta,\ 2T + Z \geq 0\} \quad \text{and} \quad D = \{T \geq \theta,\ 2T + Z < 2\theta\},$$

and so,

$$P_\theta\big(L(T,\theta) \le L(U,\theta)\big) - \frac{1}{2} \ge P(C) + P(D).$$

4.4. Example. Let $T = \widehat{m}$ be the sample median in the case of an odd sample size. Let $Z = \alpha X_{(i)}$, where $X_{(i)}$ is any order statistics and $\alpha > 0$. Then T and Z are positively associated, T is median unbiased estimate of a median, hence we conclude, using Theorem 4.3, that $\operatorname{Med}\big(L(\widehat{m}, m) - L(\widehat{m} + \alpha X_{(i)}, m)\big) \le 0$, if the loss satisfies the stated conditions. □

If we interchange events A, B and C, D (see notations in Theorem 4.3 and remarks below it), then we can get another related result, as follows.

4.5. Theorem. *In the setup of theorem* 4.3, *if* $-T$ *and* $U + T$ *are positively associated, then* (17) *holds.*

Proof. Here we consider events C and D and we note that

$$\begin{aligned} P\big(L(T,\theta) \le L(U,\theta)\big) &\ge P(C \cup D) = P(C) + P(D) \\ &= P(-T \ge -\theta, U + T \ge \theta) + P(-T \le \theta, U + T < 2\theta) \\ &\ge \frac{1}{2}\, P(Z < 0) + \frac{1}{2}\, P(Z > 0) = \frac{1}{2}\,, \end{aligned}$$

by the same argument as in Theorem 4.3.

5. ESTIMATORS BASED ON CONDITIONING

In this section we give the following Rao-Blackwell type result.

5.1. Theorem. *Let* S *be any estimator of* θ *and let* T *be any other statistics. Let* S^* *be a conditional median of* S *given* T, *that is*

$$P(S^* \ge S \mid T) \ge \frac{1}{2}\,, \quad P(S^* \le S \mid T) \ge \frac{1}{2}\,.$$

Then for any loss such that 1.1. *is satisfied, we have that*

$$\operatorname{Med}\big(L(S^*, \theta) - L(S, \theta)\big) \le 0.$$

Proof. It is easy to see that events

$$E = \{\theta \le S^*(T) \le S\} \quad \text{and} \quad F = \{S \le S^*(T) < \theta\}$$

imply the event $L(S^*, \theta) \le L(S, \theta)$. Hence, for any given $T = t$ we have that

$$P_\theta\big(L(S^*, \theta) \le L(S, \theta) \mid T = t\big) \ge P_\theta(E \cup F \mid T = t).$$

With $T = t$, S^* is a number which is either $\ge \theta$ or $< \theta$. In the former case, the event $E \cup F$ reduces to the event $\{S^* \le S\}$, in the latter case, it reduces to $\{S^* \ge S\}$. The conditional probability in both cases is $\ge 1/2$, so we have that

$$P_\theta(E \cup F \mid T = t) \ge \frac{1}{2}$$

and by taking the expectation here we get the desired result.

REMARK. Theorem 5.1 is a generalization (with a different proof) of a result of NAYAK [**13**], who proved it under the assumption that T is a sufficient statistics and that the conditional distribution of S given T is continuous. Of course, theorem 5.1 is useful only if the conditional median S^* does not depend on actual value of θ, but T need not be a sufficient statistics for that purpose. Let us consider an example.

5.2. Example. Let f_θ be a family of densities with the same median m and a mean θ. Suppose that we need to estimate θ based on a sample X_1, X_2 from f_θ. A usual estimator would be $S = (X_1 + X_2)/2$. Conditional median of S given X_1 is $(X_1 + m)/2$; it does not depend on θ, although X_1 is not a sufficient statistics. So, on the basis of Theorem 5.1, we conclude that $(X_1 + m)/2$ is an improvement over $(X_1 + X_2)/2$ in PITMAN's sense.

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