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ON THE RELATION BETWEEN THE DIGITAL SUM AND PRODUCT OF A NATURAL NUMBER

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For an integer $n \geq 1$ one denotes by $s_b(n)$ the sum of its digits and by $p_b(n)$ the product of its digits with respect to the basis b. In the present paper, one compares $s_b(n)$ and $p_b(n)$ in the general case, as well as in several special cases, e.g., when n has a given number of digits or when n is the square of a natural number.

1. INTRODUCTION

For an integer $n \geq 1$ one denotes by $s_b(n)$ the sum of its digits and by $p_b(n)$ the product of its digits with respect to the basis b. When $b = 10$, we use the notation $s(n)$ and $p(n)$, respectively.

Several problems have been raised in connection with the digits of a number. Some of them have already been solved, but there are still open problems in this field. In this connection, we recall the conjecture

$$s(2^n) < 2n \text{ for all } n > 3, \tag{1}$$

which is cited in [**1**].

A nice problem due to A. Cohn is reproduced by Pólya and Szegö in [**2**]. Cohn proved the following:

If a prime p is expressed in the decimal system as

$$p = \sum_{k=0}^{n} a_k 10^k \qquad (0 \leq a_k \leq 9), \tag{2}$$

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then the polynomial $\sum_{k=0}^{n} a_k X^k$ *is irreducible in* $\mathbb{Z}[X]$.

This example shows once again that the interest in this field is not without importance.

The problems we are going to approach in the present paper concern the simultaneous taking into account of the numbers $s(a)$ and $p(a)$, and the pointing out of a relationship between them in an appropriate framework.

2. ON $r_b(n) = p_b(n)/s_b(n)$ AND $d_b(n) = p_b(n) - s_b(n)$

For $b \geq 2$, we denote $r_b(n) = p_b(n)/s_b(n)$ and $d_b(n) = p_b(n) - s_b(n)$. We first study the sequence $\big(r_b(n)\big)_{n\geq 1}$.

For $b = 2$, the sequence $\big(r_2(n)\big)_{n\geq 1}$ contains a subsequence consisting of 0 and the subsequence $(1/k)_{k\geq 1}$. Thus $\lim\limits_{n\to\infty} r_2(n) = 0$.

For $b \geq 3$ the sequence $\big(r_b(n)\big)_{n\geq 1}$ has no limit. More precisely we have:

Property 1. *For* $b \geq 3$, *the set of limit points of the sequence* $\big(r_b(n)\big)_{n\geq 1}$ *is* $[0, \infty]$.

Proof. Consider the number with k digits $x_k = 11\ldots 1$. We have $r_b(x_k) = 1/k$ hence $\lim\limits_{k\to\infty} r_b(x_k) = 0$.

For the number with k digits $y_k = 22\ldots 2$ we have $r_b(y_k) = 2^k/(2k)$, whence $\lim\limits_{k\to\infty} r_b(y_k) = \infty$.

Now let $0 < \alpha < \infty$ and k such that $2^k > 2\alpha k$. Consider $z_k = 11\ldots 122\ldots 2$, where the digit 2 occurs k times, while the digit 1 occurs h times, where $h = [2^k/\alpha - 2k]$. It follows that $r_b(z_k) = 2^k/(h + 2k)$. Since $2^k/\alpha - 1 < h + 2k < 2^k/\alpha$, we have $\alpha < r_b(z) < \alpha/(1 - \alpha/2^k)$, whence $\lim\limits_{k\to\infty} r_b(z_k) = \alpha$. $\square$

For $n \geq 1$ and $b \geq 2$ we denote $d_b(n) = p_b(n) - s_b(n)$. If $b = 2$ then $d_b(n) \leq 0$. For $b \geq 3$, it is easy to see that we have $d_b(n) > 0$ for infinitely many values of n, and $d_b(n) < 0$ also for infinitely many values of n. In connection with this remark, we will determine the extrema of $d_b(n)$ as a function of the number of digits of n.

Property 2. *If* $b \geq 3$ *and the number* $n \geq 1$ *has* m *digits, then*

$$0 \geq d_2(n) \geq 1 - m \quad \textit{for} \quad m \geq 1 \tag{3}$$

and

$$(b-1)^m - m(b-1) \geq d_b(n) \geq (1-m)(b-1) \quad \textit{for} \quad m \geq 3. \tag{4}$$

Proof. If $b = 2$ and $m = 1$, then $n = 1$ and $d_2(n) = 1 - 1 = 0$, hence (3) holds in this case. If $b = 2$ and $m \geq 2$ then there can arise one of the following cases:

(i) $n = 11\dots1$, whence $d_2(n) = 1 - m$.

(ii) n has k digits 0 $(1 \le k \le m-1)$. In this case $d_2(n) = -s_2(n) = k - m$.

In both cases, the relation (3) holds.

Next let $b \ge 3$ and $m \ge 3$. If $p_b(n) = 0$, then $d_b(n) = -s_b(n) < 0 \le (b-1)^m - m(b-1)$.

If $p_b(n) > 0$, then $s_b(n) \ge m$ and $d_b(n) \le p_b(n) - m$.

If not all of the digits of n are $b-1$, then $d_b(n) \le (b-1)^{m-1}(b-2) - m$. In this case we have $(b-1)^{m-1}(b-2) - m \le (b-1)^m - m(b-1)$, which is equivalent to $m(b-2) \le (b-1)^{m-1}$. The latter relation holds because for $m \ge 4$ we have $b-1 > b-2$ and $(b-1)^{m-2} \ge 2^{m-2} \ge m$, while for $m = 3$ we have $(b-1)^2 \ge 3(b-2)$, which reduces to $b^2 - 5b + 7 > 0$.

If each digit of n equals $b-1$, then $d_b(n) = (b-1)^m - m(b-1)$, and thus the inequality $d_b(n) \le (b-1)^m - m(b-1)$ is completely proved.

If $p_b(n) = 0$, then $d_b(n) = -s_b(n) \ge -(m-1)(b-1)$.

Now let $p_b(n) > 0$ and x, y digits of b. It follows that $s_b(n) \le (m-2)(b-1) + x + y$, where $x, y \in \overline{1, b-1}$. We have $d_b(n) \ge xy - (m-2)(b-1) - x - y$ $= (x-1)(y-1) - (m-2)(b-1) - 1 \ge -(m-2)(b-1) - 1 > (1-m)(b-1)$. □

3. ON THE EQUALITY $s_b(n) = p_b(n)$

We are going to approach the case $r_b(n) = 1$, that is, $d_b(n) = 0$. If $b = 2$, $n \ge 2$ and n has m digits, then for $n = 11\dots1$ we have $s_b(n) = m > 1 = p_b(n)$. On the other hand, if n has k digits equal to 0 $(1 \le k \le m-1)$, then $s_b(n) = m - k$ $> 0 = p_b(n)$, hence the equality $s_b(n) = p_b(n)$ holds only in the case $n = 1$.

In the case $b \ge 3$ we have:

Property 3. *If $b \ge 3$, then for infinitely many values of m there exist natural numbers n with m digits such that $s_b(n) = p_b(n)$, and also for infinitely many values of m we have $s_b(n) \ne p_b(n)$ for each natural number n with m digits.*

Proof. For $m = 2^k - k$, we consider the number

$$n = \underbrace{22\dots2}_{k}\underbrace{11\dots1}_{2^k-2k}$$

and then $s_b(n) = 2k + 2^k - 2k = 2^k = p_b(n)$.

For the other part of the proof we will suppose that that for every m there exists n such that $s_b(n) = p_b(n)$. It is obvious that such a number n has no digit equal to 0. Denote by x_k the number of digits of n which are equal to k $(1 \le k \le b-1)$. We then have

$$x_1 + x_2 + \dots + x_{b-1} = m. \tag{5}$$

The equality $p_b(n)=s_b(n)$ can be written under the form

$$p_b(n)=\prod_{i=2}^{b-1} i^{x_i}=m+\sum_{i=2}^{b-1}(i-1)x_i. \tag{6}$$

We choose $m=\big((b-1)!\big)^k$ and then (6) implies that $\prod\limits_{i=2}^{b-1} i^{x_i}>m=\prod\limits_{i=2}^{b-1} i^k$. Hence there exists $j\in\overline{2,b-1}$ such that $x_j>k$. It follows that j^k divides $p_b(n)$, hence $\sum\limits_{i=2}^{b-1}(i-1)x_i\,\vdots\, j^k$, which implies that

$$\sum_{i=2}^{b-1}(i-1)x_i\geq j^k\geq 2^k.$$

Consequently $\sum\limits_{i=2}^{b-1}(i-1)x_i\leq (b-2)\sum\limits_{i=2}^{b-1}x_i$, and thus

$$(b-2)\sum_{i=2}^{b-1}x_i\geq 2^k. \tag{7}$$

Furthermore we have the inequalities $s_b(n)\leq m(b-1)$ and

$$p_b(n)\geq\prod_{i=2}^{b-1} i^{x_i}\geq 2^{\sum\limits_{i=2}^{b-1}x_i}$$

and since $s_b(n)=p_b(n)$, we get $2^{\sum\limits_{i=2}^{b-1}x_i}\leq m(b-1)$.

Taking into account (7), it follows that $2^{2^k/(b-2)}\leq(b-1)\big((b-1)!\big)^k$, that is,

$$\frac{2^k}{b-2}\log 2\leq\log(b-1)+k\log(b-1)!. \tag{8}$$

Since $\lim\limits_{k\to\infty}2^k/k=\infty$, it follows that the inequality (8) is false for k big enough. Consequently, for $m=\big((b-1)!\big)^k$ and k big enough we have $s_b(n)\neq p_b(n)$. $\square$

4. SPECIAL CASES

We will consider two special sequences $(x_n)_{n\geq1}$ of natural numbers, for which we will study properties of $s_b(x_n)$ and $p_b(x_n)$.

Property 4. *Let $b\geq2$. If $(y_n)_{n\geq1}$ is an arithmetic progression of positive numbers and $x_n=[y_n]$, then there exist infinitely many indices $n_1,n_2,\dots,n_k,\dots$ for which the sequences $\big(s_b(x_{n_k})\big)_{k\geq1}$ and $\big(p_b(x_{n_k})\big)_{k\geq1}$ are constant.*

Proof. Let $y_m = \alpha m + \beta$. For $\alpha = 0$ we have $x_m = [\beta]$, and the conclusion is obvious.

For $\alpha > 0$, we choose $m_h = [(b^h + \alpha - \beta)/\alpha]$. Then

$$\alpha\left(\frac{b^h + \alpha - \beta}{\alpha} - 1\right) + \beta < y_{m_k} \le \alpha\left(\frac{b^h + \alpha - \beta}{\alpha}\right) + \beta$$

that is, $b^h < y_{m_k} \le b^h + \alpha$, which implies $b^h \le x_{m_h} \le b^h + [\alpha]$. Consequently for $x_{m_h} = b^h + i$, where $0 \le i \le [\alpha]$ and $b^{h-1} > i$, we deduce that x_{m_h} has at least one digit equal to 0, whence $p_b(x_{m_h}) = 0$ for $h \ge M$. In these conditions, we also have $s(x_{m_h}) = s(b^h + i) = 1 + s(i)$. Since i takes only finitely many values, there exists a subsequence

$$n_1, n_2 \ldots, n_k, \ldots$$

of $(m_h)_{h\ge M}$ such that $s_b(x_{n_k})$ has the same value for all $k \ge 1$; moreover, as we have already seen $p_b(x_{n_k}) = 0$. □

We now consider the case when $x_n = n^2$, but we take the basis to be 10.

Proposition 5. *For every $m \ge 2$ there exist natural numbers n_1 and n_2 such that both n_1 and n_2 have m digits and*

$$p(n_1^2) < s(n_1^2) \tag{9}$$

and

$$p(n_2^2) > s(n_2^2). \tag{10}$$

Proof. If $m = 2k + 1$, $k \ge 2$, we choose $n_1 = 10^k$ and then $n_1^2 = 10^{2k}$ has $2k + 1$ digits and $p(n_1^2) = 0$, while $s(n_1^2) = 1$.

If $m = 2k$, $k \ge 2$, we choose $n_1 = 5 \cdot 10^{k-1} + 1$, and then $n_1^2 = 25 \cdot 10^{2k-2}$ $+10^k + 1$ has $2k$ digits and $p(n_1^2) = 0$, while $s(n_1^2) = 9$.

For $m = 2$, we can choose $n_1 = 4$ and then $s(n_1^2) = 7 > 6 = p\,(n_1^2)$. If $m = 3$, we can take $n_1 = 10$, and then $s(n_1^2) = 1 > 0 = p\,(n_1^2)$.

To prove the inequality (10) let us consider $m = 2k$ $k \ge 3$, and $n_2 = (10^k$ $+2)/3$. We have

$$n_2^2 = \underbrace{11\ldots1}_{k}\underbrace{55\ldots5}_{k-1}6$$

and $p(n_2^2) = 6 \cdot 5^{k-1}$ and $s(n_2^2) = k + 5(k-1) + 6 = 6k + 1$. It follows that $p(n_2^2) > s(n_2^2)$.

For $m = 2k + 1$, $k \ge 3$, we choose $n_2 = (4 \cdot 10^k + 8)/3$ and then

$$n_2^2 = 1\underbrace{77\ldots7}_{k-2}84\underbrace{88\ldots8}_{k-2}96$$

and we have $p(n_2^2) = 7^{k-2} \cdot 8^{k-1} \cdot 4 \cdot 6 \cdot 9$ and

$$s(n_2^2) = 1 + 7(k-2) + 8 + 4 + 8(k-2) + 9 + 6 = 15k - 2,$$

hence $p(n_2^2) > s(n_2^2)$. Finally for $m = 2$ we can take $n_2 = 5$, for $m = 3$ we can take $n_2 = 12$, for $m = 4$ we can take $n_2 = 34$ and for $m = 5$ we can take $n_2 = 111$.

Open problem. For every $b \geq 3$ and every $m \geq 2$ there exist natural numbers n_1, n_2, n_3 such that each of the numbers n_1^2, n_2^2, n_3^2 has m digits and

$$p_b(n_1^2) < s_b(n_1^2), \qquad p_b(n_2^2) > s_b(n_2^2), \qquad p_b(n_3^2) = s_b(n_3^2).$$

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