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APPROXIMATION PROPERTIES OF CERTAIN LINEAR POSITIVE OPERATORS IN POLYNOMIAL WEIGHTED SPACES OF FUNCTIONS OF ONE AND TWO VARIABLES

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We consider certain linear positive operators in polynomial weighted spaces of functions of one and two variables and study approximation properties of these operators, including theorems on the degree of approximation.

1. APPROXIMATION OF FUNCTION OF ONE VARIABLE

1.1. Introduction. Approximation properties of the Szasz-Mirakyan operators

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\Big(\frac{k}{n}\Big) \quad \big(x\in\mathbb{R}_0=[0,+\infty),\ n\in\mathbb{N}\big), \tag{1}$$

in polynomial weighted spaces C_p were examined in [**1**]. The space C_p, $p\in\mathbb{N}_0 := \{0,1,2,\ldots\}$, considered in [**1**] is associated with the weight function

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1}, \ \text{ if } p\ge 1, \tag{2}$$

and consists of all real-valued functions f, continuous on $\mathbb{R}_0$ and such that w_pf is uniformly continuous and bounded on $\mathbb{R}_0$. The norm on C_p is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x\in\mathbb{R}_0} w_p(x)\,|f(x)|. \tag{3}$$

These operators are very interesting approximation processes and have many nice properties.

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In this note we introduce in the space C_p, $p \in \mathbb{N}_0$ a new modification of the Szasz-Mirakyan operators.

Let C_p be the space given above and let for fixed $m \in \mathbb{N}$

$$C_p^m := \left\{ f \in C_p : f^{(k)} \in C_p,\ k = 1, 2, \ldots, m \right\}.$$

For $f \in C_p$ we define the modulus of continuity $\omega_1(f; \cdot)$ as usual ([**2**]) by

$$\omega_1(f; C_p; t) := \sup_{0 \le h \le t} \|\Delta_h f(\cdot)\|_p \qquad (t \in \mathbb{R}_0), \tag{4}$$

where $\Delta_h f(x) := f(x+h) - f(x)$ for $x, h \in \mathbb{R}_0$. From the above it follows that

$$\lim_{t \to 0+} \omega_1(f; C_p; t) = 0, \tag{5}$$

for every $f \in C_p$.

We introduce the following class of operators in C_p, $p \in \mathbb{N}$.

Definition 1. *Fix $r \in \mathbb{N}$ and $p \in \mathbb{N}_0$. We define the class of operators A_n by the formula*

$$A_n(f; r; x) := \frac{1}{g\,(nx; r)} \sum_{k=0}^{\infty} \frac{(nx)^k}{(k+r)!} f\Big(\frac{k+r}{n}\Big) \qquad (x \in \mathbb{R}_0,\ n \in \mathbb{N}), \tag{6}$$

where

$$g(t; r) := \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!} \qquad (t \in \mathbb{R}_0). \tag{7}$$

Observe that

$$g(0; r) = \frac{1}{r!}, \qquad g(t, r) = \frac{1}{t^r}\left(e^t - \sum_{j=0}^{r-1} \frac{t^j}{j!}\right) \quad \textit{if} \quad t > 0.$$

The operator A_n is linear and positive. In Section 2 we shall prove that A_n is an operator from the space C_p into C_p for every fixed $p \in \mathbb{N}_0$.

1.2. Auxiliary results. In this section we shall give some properties of above operators, which we shall apply to the proofs of the main theorems.

From (6)–(7) we derive the following:

Lemma 1. *For each $n, r \in \mathbb{N}$ and $x \in \mathbb{R}_0$ we have*

$$
\begin{aligned}
&A_n(1;r;x) = 1, \quad A_n(t;r;x) = x + \frac{1}{n(r-1)!\,g(nx;r)}\,,\\
&A_n(t^2;r;x) = x^2 + \frac{x}{n}\left(1 + \frac{1}{(r-1)!\,g(nx;r)}\right) + \frac{r}{n^2(r-1)!\,g(nx;r)}\,,\\
&A_n(t^3;r;x) = x^3 + \frac{x^2}{n}\left(3 + \frac{1}{(r-1)!\,g(nx;r)}\right) + \frac{x}{n^2}\left(1 + \frac{r+2}{(r-1)!\,g(nx;r)}\right)\\
&\qquad + \frac{r^2}{n^3(r-1)!\,g(nx;r)},\\
&A_n(t^4;r;x) = x^4 + \frac{x^3}{n}\left(6 + \frac{1}{(r-1)!\,g(nx;r)}\right) + \frac{x^2}{n^2}\left(7 + \frac{r+5}{(r-1)!\,g(nx;r)}\right)\\
&\qquad + \frac{x}{n^3}\left(1 + \frac{r^2+3r+3}{(r-1)!\,g(nx;r)}\right) + \frac{r^3}{n^4(r-1)!\,g(nx;r)}\,.
\end{aligned} \tag{8}
$$

Applying Lemma 1 it is easy to prove the following two lemmas.

Lemma 2. *Let $r \in \mathbb{N}$ be a fixed number. Then for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we have*

$$
\begin{aligned}
A_n(t-x;r;x) &= \frac{1}{n(r-1)!\,g(nx;r)}\,,\\
A_n\big((t-x)^2;r;x\big) &= \frac{x}{n}\left(1 - \frac{1}{(r-1)!\,g(nx;r)}\right) + \frac{r}{n^2(r-1)!\,g(nx;r)}\,,\\
A_n\big((t-x)^3;r;x\big) &= \left(\frac{r}{n} - x\right)^2 \frac{1}{n(r-1)!\,g(nx;r)} + \frac{x}{n^2}\left(1 + \frac{2}{(r-1)!\,g(nx;r)}\right),\\
A_n\big((t-x)^4;r;x\big) &= \left(\frac{r}{n} - x\right)^3 \frac{1}{n(r-1)!\,g(nx;r)} + \frac{x^2}{n^2}\left(3 - \frac{3}{(r-1)!\,g(nx;r)}\right)\\
&\qquad + \frac{x}{n^3}\left(1 + \frac{3r+3}{(r-1)!\,g(nx;r)}\right).
\end{aligned}
$$

Lemma 3. *Fix $s \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Then there exist coefficients $\alpha_{s,j}$, depending only on j, s and $\beta_{s,j}(r)$, depending only on r, j and s, $0 \le j \le s$ such that*

$$
A_n(t^s;r;x) = \sum_{j=0}^{s} \frac{x^j}{n^{s-j}}\left(\alpha_{s,j} + \frac{\beta_{s,j}(r)}{g(nx;r)}\right), \tag{9}
$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$. Moreover $\alpha_{0,0} = 1$, $\beta_{0,0}(r) = 0$ and $\alpha_{s,0} = \beta_{s,s}(r) = 0$, $\alpha_{s,s} = 1$, $\beta_{s,0}(r) = r^{s-1}/(r-1)!$ for $s \in \mathbb{N}$.

Proof. We shall use mathematical induction for s.The formula (9) for $s = 0, 1, 2, 3$ is given above. Let (9) hold for $f(x) = x^j$, $0 \le j \le s$, with fixed $s \in \mathbb{N}$. We shall prove (9) for $f(x) = x^{s+1}$. From (6) and (7) it follows that

$$\begin{aligned}A_n(t^{s+1};r;x) &= \frac{r^s}{n^{s+1}(r-1)!\,g(nx;r)} + \frac{1}{g(nx;r)}\sum_{k=1}^{\infty}\frac{(nx)^k}{(k+r-1)!}\,\frac{(k+r)^s}{n^{s+1}}\\ &= \frac{r^s}{n^{s+1}(r-1)!\,g(nx;r)} + \frac{x}{g(nx;r)}\sum_{k=0}^{\infty}\frac{(nx)^k}{(k+r)!}\,\frac{(k+r+1)^s}{n^s}\\ &= \frac{r^s}{n^{s+1}(r-1)!\,g(nx;r)} + \frac{x}{g(nx;r)}\sum_{k=0}^{\infty}\frac{(nx)^k}{(k+r)!}\,\frac{1}{n^s}\sum_{\mu=0}^{s}\binom{s}{\mu}(k+r)^{\mu}\\ &= \frac{r^s}{n^{s+1}(r-1)!\,g(nx;r)} + x\sum_{\mu=0}^{s}\binom{s}{\mu}\frac{1}{n^{s-\mu}}A_n(t^{\mu};r;x).\end{aligned}$$

By our assumption we get

$$\begin{aligned}A_n(t^{s+1};r;x) &= \frac{r^s}{n^{s+1}(r-1)!\,g(nx;r)} + x\sum_{\mu=0}^{s}\binom{s}{\mu}\sum_{j=0}^{\mu}\frac{x^j}{n^{s-j}}\left(\alpha_{\mu,j}+\frac{\beta_{\mu,j}(r)}{g(nx;r)}\right)\\ &= \frac{r^s}{n^{s+1}(r-1)!\,g(nx;r)} + x\sum_{j=0}^{s}\frac{x^j}{n^{s-j}}\sum_{\mu=j}^{s}\binom{s}{\mu}\left(\alpha_{\mu,j}+\frac{\beta_{\mu,j}(r)}{g(nx;r)}\right)\\ &= \frac{r^s}{n^{s+1}(r-1)!\,g(nx;r)} + x\sum_{j=1}^{s+1}\frac{x^{j-1}}{n^{s+1-j}}\left(\sum_{\mu=j-1}^{s}\binom{s}{\mu}\alpha_{\mu,j-1}\right.\\ &\qquad\left.+\frac{1}{g(nx;r)}\sum_{\mu=j-1}^{s}\binom{s}{\mu}\beta_{\mu,j-1}(r)\right)\\ &= \sum_{j=0}^{s+1}\frac{x^j}{n^{s+1-j}}\left(\alpha_{s+1,j}+\frac{\beta_{s+1,j}(r)}{g(nx;r)}\right)\end{aligned}$$

and $\alpha_{s+1,0}=\beta_{s+1,s+1}(r)=0$, $\alpha_{s+1,s+1}=1$, $\beta_{s+1,0}(r)=r^s/(r-1)!$, which proves (10) for $f(x)=x^{s+1}$. Hence the proof of (10) is completed.

Next we shall prove

Lemma 4. *Let $p\in\mathbb{N}_0$ and $r\in\mathbb{N}$ be fixed numbers. Then there exists a positive constant $M_2\equiv M_2(p,r)$, depending only on the parameters p and r such that*

$$\|A_n(1/w_p(t);r;\cdot)\|_p \le M_2 \qquad (n\in\mathbb{N}). \tag{10}$$

Moreover for every $f\in C_p$ we have

$$\|A_n(f;r;\cdot)\|_p \le M_2\|f\|_p \qquad (n\in\mathbb{N}). \tag{11}$$

Formula (6) and inequality (11) show that A_n, $n\in\mathbb{N}$, is a positive linear operator from the space C_p into C_p, for every $p\in\mathbb{N}_0$.

Proof. The inequality (10) is obvious for $p=0$ by (2), (3) and (8). Let $p\in\mathbb{N}$. From (7) we get

$$\frac{1}{g(t;r)} \le r! \quad \text{for} \ \ t\in\mathbb{R}_0. \tag{12}$$

From (12) and by (2) and (6)–(9) we have

$$\begin{aligned} w_p(x)A_n\big(1/w_p(t);r;x\big) &= w_p(x)\big(1+A_n(t^p;r;x)\big) \\ &= \frac{1}{1+x^p} + \sum_{j=0}^{p} \frac{x^j}{n^{p-j}(1+x^p)} \left(\alpha_{p,j} + \frac{\beta_{p,j}(r)}{g(nx;r)}\right) \\ &\le 1 + \sum_{j=0}^{p} \frac{x^j}{1+x^p}\big(\alpha_{p,j} + r!\,\beta_{p,j}(r)\big) \le M_2(p,r), \end{aligned}$$

for $x \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $r \in \mathbb{N}$, where $M_2(p,r)$ is a positive constant depending only p and r. From this follows (10).

The formulas (6)–(7) and (2) imply

$$\|A_n\big(f(t);r;\cdot\big)\|_p \le \|f\|_p \|A_n(1/w_p(t);r;\cdot)\|_p \qquad (n \in \mathbb{N},\ \ r \in \mathbb{N}),$$

for every $f \in C_p$. Applying (10), we obtain (11).

Lemma 5. *Let $p \in \mathbb{N}_0$ and $r \in \mathbb{N}$ be fixed numbers. Then for all $x \in \mathbb{R}_0$ there exists a positive constant $M_3 \equiv M_3(p,r)$ such that*

$$w_p(x)A_n\left(\frac{(t-x)^2}{w_p(t)};r;x\right) \le M_3\,\frac{x+1}{n} \quad \textit{for all } n \in \mathbb{N}. \tag{13}$$

Proof. The formulas given in Lemma 2 and (2) imply (13) for $p=0$. By (2) and (8) we have

$$A_n\big((t-x)^2/w_p(t);r;x\big) = A_n\big((t-x)^2;r;x\big) + A_n\big(t^p(t-x)^2;r;x\big),$$

for $p,n,r \in \mathbb{N}$. If $p=1$ then by the equality we get

$$A_n\big((t-x)^2/w_1(t);r;x\big) = A_n\big((t-x)^3;r;x\big) + (1+x)A_n\big((t-x)^2;r;x\big),$$

which by (2) and Lemma 2 yields (13) for $p=1$. Let $p \ge 2$. Applying Lemma 3 and (2), we get

$$\begin{aligned} &w_p(x)\,A_n\big(t^p(t-x)^2;r;x\big) \\ &\quad= w_p(x)\Big(A_n(t^{p+2};r;x) - 2xA_n(t^{p+1};r;x) + x^2A_n(t^p;r;x)\Big) \\ &\quad= w_p(x)\Bigg(\sum_{j=0}^{p}\frac{x^j}{n^{p+2-j}}\left(\alpha_{p+2,j} + \frac{\beta_{p+2,j}(r)}{g(nx;r)}\right) \\ &\qquad -2\sum_{j=0}^{p-1}\frac{x^{j+1}}{n^{p+1-j}}\left(\alpha_{p+1,j} + \frac{\beta_{p+1,j}(r)}{g(nx;r)}\right) + \sum_{j=0}^{p-2}\frac{x^{j+2}}{n^{p-j}}\left(\alpha_{p,j} + \frac{\beta_{p,j}(r)}{g(nx;r)}\right) \\ &\qquad + \frac{x^{p+1}}{n}\left(\left(\alpha_{p+2,p+1} + \frac{\beta_{p+2,p+1}(r)}{g(nx;r)}\right) - 2\left(\alpha_{p+1,p} + \frac{\beta_{p+1,p}(r)}{g(nx;r)}\right)\right. \end{aligned}$$

$$+\left(\alpha_{p,p-1}+\frac{\beta_{p,p-1}(r)}{g(nx;r)}\right)\Bigg)\Bigg)$$

$$\le \frac{1}{n^2}\left(\sum_{j=0}^{p}\frac{x^j}{1+x^p}\left(\alpha_{p+2,j}+\frac{\beta_{p+2,j}(r)}{g(nx;r)}\right)-2\sum_{j=0}^{p-1}\frac{x^{j+1}}{1+x^p}\left(\alpha_{p+1,j}+\frac{\beta_{p+1,j}(r)}{g(nx;r)}\right)\right.$$

$$+\left.\sum_{j=0}^{p-2}\frac{x^{j+2}}{1+x^p}\left(\alpha_{p,j}+\frac{\beta_{p,j}(r)}{g(nx;r)}\right)\right)+\frac{x^p}{1+x^p}\,\frac{x}{n}\left(\left(\alpha_{p+2,p+1}+\frac{\beta_{p+2,p+1}(r)}{g(nx;r)}\right)\right.$$

$$\left.-2\left(\alpha_{p+1,p}+\frac{\beta_{p+1,p}(r)}{g(nx;r)}\right)+\left(\alpha_{p,p-1}+\frac{\beta_{p,p-1}(r)}{g(nx;r)}\right)\right),$$

which by (12) implies

$$w_p(x)A_n\big(t^p(t-x)^2;r;x\big)\le M_4(p,r)\,\frac{x+1}{n}\,,$$

for $x\in\mathbb{R}_0$, $n,r\in\mathbb{N}$. Thus the proof is completed.

1.3. Theorems. In this part we shall some estimates of the rate of convergence of A_n. We shall use the classical modulus of continuity defined by (4).

We shall apply the method used in [**1**].

Theorem 1. *Let $p\in\mathbb{N}_0$ and $r\in\mathbb{N}$ be fixed numbers. Then there exists a positive constant $M_5\equiv M_5(p,r)$ such that for every $f\in C_p^1$ we have*

$$w_p(x)|A_n(f;r;x)-f(x)|\le M_5\|f'\|_p\sqrt{\frac{x+1}{n}}\,, \tag{14}$$

for all $x\in\mathbb{R}_0$ and $n\in\mathbb{N}$.

Proof. Fix $x\in\mathbb{R}_0$. Then for $f\in C_p^1$ we have

$$f(t)-f(x)=\int_x^t f'(u)\,\mathrm{d}u \qquad (t\in R_0).$$

From this and by (6), (7) and (8) we get

$$A_n\big(f(t);r;x\big)-f(x)=A_n\left(\int_x^t f'(u)\,\mathrm{d}u;r;x\right) \qquad (n\in\mathbb{N}).$$

But by (2) and (3) we have

$$\left|\int_x^t f'(u)\,\mathrm{d}u\right|\le\|f'\|_p\left(\frac{1}{w_p(t)}+\frac{1}{w_p(x)}\right)|t-x| \qquad (t\in\mathbb{R}_0),$$

which implies

$$w_p(x)|A_n(f;r;x)-f(x)|\le\|f'\|_p\left(A_n\left(|t-x|;r;x\right)+w_p(x)A_n\left(\frac{|t-x|}{w_p(t)};r;x\right)\right) \tag{15}$$

for $n \in \mathbb{N}$. By the HÖLDER inequality and by (8) and Lemmas 2, 4, 5 it follows that

$$A_n\left(|t-x|;r;x\right) \le \Big(A_n\left((t-x)^2;r;x\right)A_n\left(1;r;x\right)\Big)^{1/2} \le M_3\sqrt{\frac{x+1}{n}}$$

$$w_p(x)A_n\left(\frac{|t-x|}{w_p(t)};r;x\right) \le \left(w_p(x)A_n\left(\frac{(t-x)^2}{w_p(t)};r;x\right)\right)^{1/2}\times$$
$$\times\left(w_p(x)A_n\left(\frac{1}{w_p(t)};r;x\right)\right)^{1/2} \le M_6(p,r)\sqrt{\frac{x+1}{n}}$$

for $n \in \mathbb{N}$. From this and by (15) we immediately obtain (14).

Theorem 2. *Fix $p \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Then there exists $M_7 \equiv M_7(p,r)$ such that for every $f \in C_p$ and $n \in \mathbb{N}$ we have*

$$w_p(x)|A_n(f;r;x)-f(x)| \le M_7\omega_1\left(f;C_p;\sqrt{\frac{x+1}{n}}\right) \tag{16}$$

for all $x \in \mathbb{R}_0$.

Proof. We use STEKLOV function f_h of $f \in C_p$

$$f_h(x) := \frac{1}{h}\int_0^h f(x+t)\,\mathrm{d}t \qquad (x \in \mathbb{R}_0,\ h>0). \tag{17}$$

From (17) we get

$$f_h(x)-f(x) = \frac{1}{h}\int_0^h \Delta_t f(x)\,\mathrm{d}t, \quad f_h'(x)\,\frac{1}{h}\,\Delta_h f(x) \quad (x\in\mathbb{R}_0,\ \ h>0),$$

which imply

$$\|f_h - f\|_p \le \omega_1\left(f;C_p;h\right), \tag{18}$$

$$\|f_h'\|_p \le h^{-1}\omega\left(f;C_p;h\right), \tag{19}$$

for $h>0$. From this we deduce that $f_h \in C_p^1$ if $f \in C_p$ and $h>0$. Hence we can write

$$w_p(x)|A_n(f;x)-f(x)| \le w_p(x)\big(|A_n(f-f_h;x)|$$
$$+|A_n(f_h;x)-f_h(x)|+|f_h(x)-f(x)|\big) : K_1(x)+K_2(x)+K_3(x),$$

for $n\in\mathbb{N}$, $h>0$ and $x\in\mathbb{R}_0$. From (11) and (18) we get

$$K_1(x) \le M_2\|f-f_h\|_p \le M_2\omega_1(f;C_p;h), \quad K_3(x) \le \omega_1(f;C_p;h).$$

By Theorem 1 and (19) it follows that

$$K_2(x) \le M_5\|f_h'\|_p\sqrt{\frac{x+1}{n}} \le M_5h^{-1}\sqrt{\frac{x+1}{n}}\,\omega_1(f;C_p;h).$$

Consequently

$$w_p(x)|A_n(f;r;x)-f(x)|\le\left(1+M_2+\frac{M_5}{h}\sqrt{\frac{x+1}{n}}\right)\omega_1(f;C_p;h).$$

Setting $h=\sqrt{(x+1)/n}$ we obtain the assertion of Theorem 2.

From Theorem 1 and Theorem 2 and by (5) we obtain

Corollary 1. *For every fixed* $r\in\mathbb{N}$ *and* $f\in C_p$, $p\in\mathbb{N}_0$, *we have*

$$\lim_{n\to\infty}\big(A_n(f;r;x)-f(x)\big)=0\qquad(x\in\mathbb{R}_0). \tag{20}$$

Moreover (20) *holds uniformly on every interval* $[x_1,x_2]$, $x_2>x_1\ge 0$.

Now, we shall give the Voronovskaya type theorem for A_n.

Theorem 3. *Suppose that* $p\in\mathbb{N}_0$, $r\in\mathbb{N}$ *are fixed numbers and* $f\in C_p^2$. *Then*

$$\lim_{n\to\infty}n\big(A_n(f;r;x)-f(x)\big)=\frac{x}{2}f''(x) \tag{21}$$

for every $x>0$.

Proof. Let $x>0$ be a fixed point. Then by the Taylor formula we have

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+\varepsilon(t;x)(t-x)^2$$

for $t\in\mathbb{R}_0$, where $\varepsilon(t)\equiv\varepsilon(t;x)$ is a function belonging to C_p and $\varepsilon(x)=0$. Hence by (6) and (8) we get

$$\begin{aligned}A_n(f;r;x)&=f(x)+f'(x)A_n(t-x;r;x)\\&+\frac{1}{2}f''(x)A_n\big((t-x)^2;r;x\big)+A_n\big(\varepsilon(t)(t-x)^2;r;x\big)\qquad(n\in\mathbb{N}),\end{aligned} \tag{22}$$

which by Lemma 2 yields

$$\lim_{n\to\infty}n\big(A_n(f;r;x)-f(x)\big)=\frac{x}{2}f''(x)+\lim_{n\to\infty}nA_n\big(\varepsilon(t)(t-x)^2;r;x\big). \tag{23}$$

By the Hölder inequality we have

$$\left|A_n\big(\varepsilon(t)(t-x)^2;r;x\big)\right|\le\Big(A_n\big((\varepsilon^2(t);x\big)\Big)^{1/2}\Big(A_n\big((t-x)^4;x\big)\Big)^{1/2}.$$

The properties of ε and Corollary 1 imply that

$$\lim_{n\to\infty}A_n\big(\varepsilon^2(t);r;x\big)=\varepsilon^2(x)=0.$$

From this and by Lemma 2 we deduce that

$$\lim_{n\to\infty}nA_n\big(\varepsilon(t)(t-x)^2;r;x\big)=0$$

and from (23) follows (21).

2. APPROXIMATION OF FUNCTIONS OF TWO VARIABLES

2.1. Preliminaries. Let $p, q \in \mathbb{N}_0$ and let

$$w_{p,q}(x,y) := w_p(x) w_q(y) \qquad \big((x,y) \in \mathbb{R}_0^2 := \mathbb{R}_0 \times \mathbb{R}_0\big), \tag{24}$$

where $w_p(\cdot)$ is defined by (2). Denote by $C_{p,q}$ the weighted space of all real-valued functions f continuous on $\mathbb{R}_0^2$ for which $w_{p,q}f$ is uniformly continuous and bounded on $\mathbb{R}_0^2$. The norm on $C_{p,q}$ is defined by

$$\|f\|_{p,q} \equiv \|f(\cdot,\cdot)\|_{p,q} := \sup_{(x,y)\in\mathbb{R}_0^2} w_{p,q}(x,y)\,|f(x,y)|. \tag{25}$$

The modulus of continuity of $f \in C_{p,q}$ we define as usual by the formula

$$\omega(f, C_{p,q}; t, s) := \sup_{0\le h\le t,\, 0\le \delta\le s} \|\Delta_{h,\delta} f(\cdot,\cdot)\|_{p,q} \qquad (t, s \ge 0), \tag{26}$$

where $\Delta_{h,\delta} f(x,y) := f(x+h, y+\delta) - f(x,y)$ and $(x+h, y+\delta) \in \mathbb{R}_0^2$.

From (26) it follows that

$$\lim_{t,s\to 0+} \omega(f, C_{p,q}; t, s) = 0 \tag{27}$$

for every $f \in C_{p,q}$, $p, q \in \mathbb{N}_0$. Moreover let $C_{p,q}^m$ denotes the set of all functions $f \in C_{p,q}$ which the partial derivatives $f^{(k)}_{x^j, y^{k-j}}$, $k = 1, \dots, m$, belong also to $C_{p,q}$.

We introduce the following

Definition 2. *Fix $r, s \in \mathbb{N}$. For functions $f \in C_{p,q}$, $p, q \in \mathbb{N}_0$, we define operators*

$$A_{m,n}(f; r, s; x, y) := \frac{1}{g(mx; r)\, g(ny; s)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(mx)^j}{(j+r)!} \frac{(ny)^k}{(k+s)!} f\left(\frac{j+r}{m}, \frac{k+s}{n}\right) \tag{28}$$

for $(x,y) \in \mathbb{R}_0^2$, $m, n \in \mathbb{N}$, where $g(\cdot\,; r)$ is defined by (7).

From (28) and (6)–(7) we deduce that $A_{m,n}(f; r, s)$ are well defined in every space $C_{p,q}$, $p, q \in \mathbb{N}_0$. Moreover for fixed $r, s \in \mathbb{N}$ we have

$$A_{m,n}(1; r, s; x, y) = 1 \quad \text{for} \quad (x,y) \in \mathbb{R}_0^2 \;\; (m, n \in \mathbb{N}), \tag{29}$$

and if $f \in C_{p,q}$ and $f(x,y) = f_1(x) f_2(y)$ for all $(x,y) \in \mathbb{R}_0^2$, then

$$A_{m,n}(f; r, s; x, y) = A_m(f_1; r; x) A_n(f_2; s; y) \tag{30}$$

for all $(x,y) \in \mathbb{R}_0^2$ and $m, n \in \mathbb{N}$.

2.2. Main results. Applying Lemma 4 we shall prove the main lemma on $A_{m,n}$ defined by (28).

Lemma 6. *For fixed $p,q\in\mathbb{N}_0$ and $r,s\in\mathbb{N}$ there exists a positive constant $M_8\equiv M_8(p,q,r,s)$ such that*

$$\left\|A_{m,n}\big(1/w_{p,q}(t,z);r,s;\cdot,\cdot\big)\right\|_{p,q}\le M_8 \quad \textit{for} \quad m,n\in\mathbb{N}. \tag{31}$$

Moreover for every $f\in C_{p,q}$ we have

$$\|A_{m,n}(f;r,s;\cdot,\cdot)\|_{p,q}\le M_8\,\|f\|_{p,q} \quad \textit{for} \quad m,n\in\mathbb{N},\ r,s\in\mathbb{N}. \tag{32}$$

Formula (28) and inequality (32) show that $A_{m,n}$, $m,n\in N$, are linear positive operators from the space $C_{p,q}$ into $C_{p,q}$.

Proof. The inequality (31) follows immediately from (24), (30) and (10).

From (28) and (25) we get for $f\in C_{p,q}$ and $r,s\in\mathbb{N}$

$$\|A_{m,n}(f;r,s)\|_{p,q}\le\|f\|_{p,q}\,\|A_{m,n}(1/w_{p,q};r,s)\|_{p,q} \quad (m,n\in\mathbb{N}),$$

which by (31) implies (32).

Now we shall give two theorems on the degree of approximation of functions by $A_{m,n}$ defined by (28).

Theorem 4. *Suppose that $f\in C^1_{p,q}$ with fixed $p,q\in\mathbb{N}_0$. Then for fixed $r,s\in\mathbb{N}$ there exists a positive constant $M_9=M_9(p,q,r,s)$ such that for all $m,n\in\mathbb{N}$ and $(x,y)\in{\mathbb{R}_0}^2$*

$$\begin{aligned} &w_{p,q}(x,y)\,|A_{m,n}(f;r,s;x,y)-f(x,y)|\\ &\qquad\le M_9\Bigg(\|f'_x\|_{p,q}\sqrt{\frac{x+1}{m}}+\|f'_y\|_{p,q}\sqrt{\frac{y+1}{n}}\Bigg). \end{aligned}\tag{33}$$

Proof. Fix $(x,y)\in{\mathbb{R}_0}^2$. Then for $f\in C^1_{p,q}$

$$f(t,z)-f(x,y)=\int_x^t f'_u(u,z)\,\mathrm{d}u+\int_y^z f'_v(x,v)\,\mathrm{d}v \qquad \big((t,z)\in{\mathbb{R}_0}^2\big).$$

Thus by (29)

$$\begin{aligned} A_{m,n}\big(f(t,z);r,s;x,y\big)-f(x,y)&=A_{m,n}\left(\int_x^t f'_u(u,z)\,\mathrm{d}u;r,s;x,y\right)\\ &\quad+A_{m,n}\left(\int_y^z f'_v(x,v)\,\mathrm{d}v;r,s;x,y\right). \end{aligned}\tag{34}$$

By (2), (24)–(25) we have

$$\left|\int_x^t f'_u(u,z)\,\mathrm{d}u\right|\le\|f'_x\|_{p,q}\left|\int_x^t\frac{\mathrm{d}u}{w_{p,q}(u,z)}\right|\le\|f'_x\|_{p,q}\left(\frac{1}{w_{p,q}(t,z)}+\frac{1}{w_{p,q}(x,z)}\right)|t-x|,$$

which by (2), (24), (28)–(30) and (6), (8) implies

$$\begin{aligned} w_{p,q}(x,y)\left|A_{m,n}\left(\textstyle\int_x^t f'_u(u,z)\,\mathrm{d}u;r,s;x,y\right)\right| \\ \le w_{p,q}(x,y)A_{m,n}\left(\left|\textstyle\int_x^t f'_u(u,z)\,\mathrm{d}u\right|;r,s;x,y\right) \\ \le \|f'_x\|_{p,q}\,w_{p,q}(x,y)\Bigg(A_{m,n}\left(\frac{|t-x|}{w_{p,q}(t,z)};r,s;x,y\right) \\ + A_{m,n}\left(\frac{|t-x|}{w_{p,q}(x,z)};r,s;x,y\right)\Bigg) \\ \le \|f'_x\|_{p,q}\,w_q(y)A_n\left(\frac{1}{w_q(z)};s;y\right)\Bigg(w_p(x)A_m\left(\frac{|t-x|}{w_p(t)};r;x\right) \\ + A_m\left(|t-x|;r;x\right)\Bigg). \end{aligned}$$

Applying the HÖLDER inequality, (8), (10), (13) and Lemma 2, we get

$$A_m\big(|t-x|;r;x\big) \le \Big(A_m\big((t-x)^2;r;x\big)A_m(1;r;x)\Big)^{1/2} \le M_{10}(p,r)\sqrt{\frac{x+1}{m}}\,,$$

$$\begin{aligned} w_p(x)A_m\left(\frac{|t-x|}{w_p(t)};r;x\right) \le \left(w_p(x)A_m\left(\frac{(t-x)^2}{w_p(t)};r;x\right)\right)^{1/2} \times \\ \times\left(w_p(x)A_m\left(\frac{1}{w_p(t)};r;x\right)\right)^{1/2} \le M_{11}(p,r)\sqrt{\frac{x+1}{m}} \end{aligned}$$

for $x \in \mathbb{R}_0$ and $m \in \mathbb{N}$. Consequently

$$\begin{aligned} w_{p,q}(x,y)\left|A_{m,n}\left(\textstyle\int_x^t f'_u(u,z)\,\mathrm{d}u;r,s;x,y\right)\right| \\ \le M_{12}(p,q,r,s)\|f'_x\|_{p,q}\sqrt{\frac{x+1}{m}} \quad (m \in \mathbb{N}). \end{aligned}$$

Analogously we obtain

$$\begin{aligned} w_{p,q}(x,y)\left|A_{m,n}\left(\textstyle\int_y^z f'_v(x,v)\,\mathrm{d}v;r,s;x,y\right)\right| \\ \le M_{13}(p,q,r,s)\|f'_y\|_{p,q}\sqrt{\frac{y+1}{n}} \quad (n \in \mathbb{N}). \end{aligned}$$

Combining these, we derive from (34)

$$\begin{aligned} w_{p,q}(x,y)\,|A_{m,n}(f;r,s;x,y)-f(x,y)| \\ \le M_9\left(\|f'_x\|_{p,q}\sqrt{\frac{x+1}{m}} + \|f'_y\|_{p,q}\sqrt{\frac{y+1}{n}}\right), \end{aligned}$$

for all $m, n \in \mathbb{N}$, where $M_9 = M_9(p,q,r,s) = \text{const} > 0$. Thus the proof of (33) is completed.

Theorem 5. *Suppose that $f \in C_{p,q}$, $p, q \in \mathbb{N}_0$. Then there exists a positive constant $M_{14} \equiv M_{14}(p,q,r,s)$ such that for all $(x,y) \in \mathbb{R}_0^2$*

$$
\begin{aligned}
& w_{p,q}(x,y)\left|A_{m,n}(f;r,s;x,y) - f(x,y)\right| \\
& \qquad \le M_{14}\,\omega\left(f, C_{p,q}; \sqrt{\frac{x+1}{m}}, \sqrt{\frac{y+1}{n}}\right) \qquad (m,n \in \mathbb{N},\ r,s \in \mathbb{N}).
\end{aligned} \tag{35}
$$

Proof. We apply the Steklov function $f_{h,\delta}$ for $f \in C_{p,q}$

$$
f_{h,\delta}(x,y) := \frac{1}{h\delta}\int_0^h \mathrm{d}u \int_0^\delta f(x+u, y+v)\,\mathrm{d}v \quad \big((x,y) \in \mathbb{R}_0^2,\ h,\delta > 0\big). \tag{36}
$$

From (36) it follows that

$$
\begin{aligned}
f_{h,\delta}(x,y) - f(x,y) &= \frac{1}{h\delta}\int_0^h \mathrm{d}u \int_0^\delta \Delta_{u,v} f(x,y)\,\mathrm{d}v, \\
(f_{h,\delta})'_x(x,y) &= \frac{1}{h\delta}\int_0^\delta \big(\Delta_{h,v} f(x,y) - \Delta_{0,v} f(x,y)\big)\,\mathrm{d}v, \\
(f_{h,\delta})'_y(x,y) &= \frac{1}{h\delta}\int_0^h \big(\Delta_{u,\delta} f(x,y) - \Delta_{u,0} f(x,y)\big)\,\mathrm{d}u.
\end{aligned}
$$

Thus

$$
\|f_{h,\delta} - f\|_{p,q} \le \omega\left(f, C_{p,q}; h, \delta\right), \tag{37}
$$

$$
\left\|(f_{h,\delta})'_x\right\|_{p,q} \le 2h^{-1}\omega\left(f, C_{p,q}; h, \delta\right), \tag{38}
$$

$$
\left\|(f_{h,\delta})'_y\right\|_{p,q} \le 2\delta^{-1}\omega\left(f, C_{p,q}; h, \delta\right), \tag{39}
$$

for all $h, \delta > 0$, which show that $f_{h,\delta} \in C^1_{p,q}$ if $f \in C_{p,q}$ and $, h, \delta > 0$.

Now, for $A_{m,n}$ defined by (28), we can write

$$
\begin{aligned}
& w_{p,q}(x,y)\left|A_{m,n}(f;r,s;x,y) - f(x,y)\right| \\
& \quad \le w_{p,q}(x,y)\Big(\left|A_{m,n}\big(f(t,z) - f_{h,\delta}(t,z); r,s;x,y\big)\right| \\
& \qquad\quad + \left|A_{m,n}\big(f_{h,\delta}(t,z); r,s;x,y\big) - f_{h,\delta}(x,y)\right| + \left|f_{h,\delta}(x,y) - f(x,y)\right|\Big) \\
& \quad := T_1(x) + T_2(x) + T_3(x).
\end{aligned}
$$

By (25), (32) and (37) obtain

$$
\begin{aligned}
& T_1(x) \le \left\|A_{m,n}\left(f - f_{h,\delta}; r,s;\cdot,\cdot\right)\right\|_{p,q} \le M_8\left\|f - f_{h,\delta}\right\|_{p,q} \le M_8\,\omega\left(f, C_{p,q}; h, \delta\right), \\
& T_3(x) \le \omega\left(f, C_{p,q}; h, \delta\right).
\end{aligned}
$$

Applying Theorem 4 and (38) and (39), we get

$$\begin{aligned} T_2(x) &\le M_9\Big(\big\|(f_{h,\delta})'_x\big\|_{p,q} \sqrt{\frac{x+1}{m}} + \big\| (f_{h,\delta})'_y \big\|_{p,q} \sqrt{\frac{y+1}{n}} \Big) \\ &\le 2M_9\omega\left(f, C_{p,q}; h, \delta\right) \left(h^{-1}\sqrt{\frac{x+1}{m}} + \delta^{-1}\sqrt{\frac{y+1}{n}}\right). \end{aligned}$$

Consequently there exists $M_{15} \equiv M_{15}(p,q,r,s)$ such that

$$\begin{aligned} &w_{p,q}(x,y)\left|A_{m,n}(f;r,s;x,y) - f(x,y)\right| \\ &\qquad \le M_{15}\omega\left(f, C_{p,q}; h, \delta\right) \left(1 + h^{-1}\sqrt{\frac{x+1}{m}} + \delta^{-1}\sqrt{\frac{y+1}{n}}\right). \end{aligned} \tag{40}$$

Setting $h = \sqrt{(x+1)/m}$ and $\delta = \sqrt{(y+1)/n}$ to (40), we obtain (35).

From Theorem 5 and the property (27) follows

Corollary 3. *Let $f \in C_{p,q}$, $p,q \in \mathbb{N}_0$. Then for $r,s \in \mathbb{N}$*

$$\lim_{m,n\to\infty} A_{m,n}(f;r,s;x,y) = f(x,y) \qquad \big((x,y) \in \mathbb{R}_0^2\big). \tag{41}$$

Moreover (41) *holds uniformly on every rectangle* $0 \le x \le x_0$, $0 \le y \le y_0$.

In this part we give the Voronovskaya type theorem for operators $A_{n,n}$, $n \in \mathbb{N}$.

Theorem 6. *Suppose that $f \in C^2_{p,q}$, $p,q \in \mathbb{N}_0$. Then for fixed $r,s \in \mathbb{N}$ and for every $(x,y) \in \mathbb{R}^2_+ : \{(x,y) : x > 0, y > 0\}$* we have

$$\lim_{n\to\infty} n\left(A_{n,n}(f;r,s;x,y) - f(x,y)\right) \frac{x}{2}\, f''_{xx}(x,y) + \frac{y}{2}\, f''_{yy}(x,y). \tag{42}$$

Proof. Choosing $(x,y) \in \mathbb{R}^2_+$, we have by the Taylor formula for $f \in C^2_{p,q}$

$$\begin{aligned} f(t,z) &= f(x,y) + f'_x(x,y)(t-x) + f'_y(x,y)(z-y) \\ &+ \frac{1}{2}\big(f''_{xx}(x,y)(t-x)^2 + 2f''_{xy}(x,y)(t-x)(z-y) + f''_{yy}(x,y)(z-y)^2\big) \\ &+ \psi(t,z;x,y)\sqrt{(t-x)^4 + (z-y)^4} \quad \big((t,z) \in \mathbb{R}_0^2, \end{aligned}$$

where $\psi(t,z) = \psi(t,z;x,y)$ is a function from $C_{p,q}$ and $\psi(x,y) = 0$. From this and by (6), (8), (28)–(30) we get

$$\begin{aligned} &A_{n,n}\big(f(t,z);r,s;x,y\big) = f(x,y) + f'_x(x,y)A_n(t-x;r;x) \\ &+f'_y(x,y)A_n(z-y;s;y) + \frac{1}{2}\Big(f''_{xx}(x,y)A_n\big((t-x)^2;r;x\big) \\ &+f''_{xy}(x,y)A_n(t-x;r;x)A_n(z-y;s;y) + f''_{yy}(x,y)A_n\big((z-y)^2;s;y\big)\Big) \\ &+A_{n,n}\Big(\psi(t,z)\sqrt{(t-x)^4+(z-y)^4}\,;r,s;x,y\Big) \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

Next,using Lemma 2, we can write

$$\begin{aligned}\lim_{n\to\infty} n\big(A_{n,n}(f;r,s;x,y)-f(x,y)\big) &= \frac{x}{2}\,f''_{xx}(x,y)+\frac{y}{2}\,f''_{yy}(x,y)\\ &+\lim_{n\to\infty} nA_{n,n}\Big(\psi(t,z)\sqrt{(t-x)^4+(z-y)^4}\,;r,s;x,y\Big).\end{aligned} \tag{43}$$

By the HÖLDER inequality, (6), (8), (28)–(30) and Lemma 2 we have

$$\begin{aligned}&\Big|A_{n,n}\Big(\psi(t,z)\sqrt{(t-x)^4+(z-y)^4};r,s;x,y\Big)\Big|\\ &\le\Big(A_{n,n}\big(\psi^2(t,z);r,s;x,y\big)\Big)^{1/2}\Big(A_n\big((t-x)^4;r;x\big)+A_n\big((z-y)^4;s;y\big)\Big)^{1/2}.\end{aligned} \tag{44}$$

The properties of ψ and Corollary 2 imply that

$$\lim_{n\to\infty} A_{n,n}\left(\psi^2(t,z);r,s;x,y\right)\psi^2(x,y)=0. \tag{45}$$

Using (45) and Lemma 2, we obtain from (44)

$$\lim_{n\to\infty} n\,A_{n,n}\Big(\psi(t,z)\sqrt{(t-x)^4+(z-y)^4};r,s;x,y\Big)=0. \tag{46}$$

From (46) and (43) follows (42).

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