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SOME REFINEMENTS OF HERMITE-HADAMARD INEQUALITIES FOR CONVEX FUNCTIONS

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In this paper, we define two mappings associated with Hermite-Hadamard inequalities for convex functions, give their properties, and obtain refinements of Hermite-Hadamard inequalities by these properties.

1. INTRODUCTION

Let f be a continuous convex function on a closed interval $[a,b] (a<b)$. Then

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}\,. \tag{1}$$

The inequalities (1) are called Hermite-Hadamard inequalities (see[**1-3**]). The inequalities (1) are equivalent to

$$2(b-a)f\Big(\frac{a+b}{2}\Big) \le 2\int_a^b f(x)\,\mathrm{d}x \le (b-a)\big(f(a)+f(b)\big). \tag{2}$$

If f is a continuous function on $[a,b]$, for any $x,y\in[a,b]$, $x\le y$, we define two mappings:

$$h(x,y) = (y-x)\big(f(x)+f(y)\big) - 2\int_x^y f(t)\,\mathrm{d}t,$$

and

$$H(x,y) = \int_x^y f(t)\,\mathrm{d}t - (y-x)f\Big(\frac{x+y}{2}\Big),$$

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where $h(x,y)$ and $H(x,y)$ are generated by difference of right side and left side for (2), respectively.

Let f be continuous convex on $[a,b]$, any $x \in (a,b)$, for above h and H, S. S. Dragomir and P. Agarwal ([**4**]) showed the following properties:

(a) $h(a,y)$ is convex increasing with y on $[a,b]$,

(b) $H(a,y)$ is increasing with y on $[a,b]$,

and the following inequalities:

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \le \frac{x-a}{b-a}\cdot\frac{f(a)+f(x)}{2} + \frac{1}{b-a}\int_x^b f(t)\,\mathrm{d}t \le \frac{f(a)+f(b)}{2}, \tag{3}$$

$$\begin{aligned} f\Big(\frac{a+b}{2}\Big) &\le \frac{1}{b-a}\int_a^x f(t)\,\mathrm{d}t - \frac{x-a}{b-a} f\Big(\frac{a+x}{2}\Big) + f\Big(\frac{a+b}{2}\Big) \\ &\le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t. \end{aligned} \tag{4}$$

In this paper, we give special properties of $h(x,b)$ and $H(x,b)$, and obtain some refinements of (1) by their properties and (3)–(4). For other refinements of (1), see [**5-6**].

In Section 2, we will use the following lemma.

Lemma. *Let g be a continuous function on $[a,b]$, if g has right derivative g'_+ and g'_+ is increasing on (a,b), then g is convex function on $[a,b]$* (see [**7; 8**, p.44]).

2. MAIN RESULTS

Theorem 1. *Let f be a continuous convex function on $[a,b]$. Then*

(c) $h(x,b)$ is convex decreasing with x on $[a,b]$,

(d) $H(x,b)$ is decreasing with x on $[a,b]$.

Proof. (c) By the continuity of f, we get that $h(x,b)$ is a continuous function with x on $[a,b]$. Let $h'_+(x,b)$ is right derivative of $h(x,b)$ for x. for any $x \in (a,b)$, using properties of derivative, we have

$$h'_+(x,b) = f(x) - f(b) + (b-x)f'_+(x). \tag{5}$$

For any $x_1, x_2 \in (a,b)$, $x_1 < x_2$, using the convexity of f and (5) we have

$$\begin{aligned} &h'_+(x_2,b) - h'_+(x_1,b) \\ &\quad = f(x_2) - f(x_1) + (x_1-x_2)f'_+(x_1) + (b-x_2)\big(f'_+(x_2) - f'_+(x_1)\big) \\ &\quad \ge (x_2-x_1)f'_+(x_1) + (x_1-x_2)f'_+(x_1) + (b-x_2)\big(f'_+(x_2) - f'_+(x_1)\big) \\ &\quad \ge 0, \end{aligned}$$

which shows that $h'_+(x,b)$ is increasing with x on (a,b). By Lemma, it is shown that $h(x,b)$ is convex with x on $[a,b]$. (By same method, we can prove that $h(a,y)$ is convex with y on $[a,b]$. This method is simpler than [**4**]).

Fox any $x_1, x_2 \in [a,b]$, $x_1 < x_2$, when $x_2 < b$, since f and $h(x,b)$ is convex, from (1) it follows that

$$\begin{aligned}\frac{h(x_2,b)-h(x_1,b)}{x_2-x_1} &\le \frac{h(x_2,b)-h(b,b)}{x_2-b}\\ &= \frac{h(x_2,b)}{x_2-b} = -\big(f(x_2)+f(b)\big) + \frac{2}{b-x_2}\int_{x_2}^{b} f(t)\,\mathrm{d}t\\ &\le 0,\end{aligned}$$

when $b = x_2$, from (1) it follows that

$$h(x_1,b) \ge 0 = h(b,b) = h(x_2,b),$$

hence $h(x_1,b) \ge h(x_2,b)$. This shows that $h(x,b)$ is decreasing with x on $[a,b]$.

(d) For any $x_1, x_2 \in [a,b]$, $x_1 < x_2$, when $x_2 < b$, since f is convex, from (1) it follows that

$$\begin{aligned}&H(x_1,b)-H(x_2,b)\\ &= \int_{x_1}^{x_2} f(t)\,\mathrm{d}t + (b-x_2)\Big(f\Big(\frac{x_2+b}{2}\Big) - f\Big(\frac{x_1+b}{2}\Big)\Big) - (x_2-x_1)\,f\Big(\frac{x_1+b}{2}\Big)\\ &\ge (x_2-x_1)\Big(f\Big(\frac{x_2+x_1}{2}\Big) - f\Big(\frac{x_1+b}{2}\Big)\Big)\\ &\qquad + (b-x_2)\Big(f\Big(\frac{x_2+b}{2}\Big) - f\Big(\frac{x_1+b}{2}\Big)\Big)\\ &\ge \frac{(x_2-x_1)(x_2-b)}{2}\,f'_+\Big(\frac{x_1+b}{2}\Big) + \frac{(x_2-x_1)(b-x_2)}{2}\,f'_+\Big(\frac{x_1+b}{2}\Big) = 0,\end{aligned}$$

when $x_2 = b$, from (1) it follows that

$$H(x_1,b) \ge 0 = H(b,b) = H(x_2,b),$$

hence $H(x_1,b) \ge H(x_2,b)$. This shows that $h(x,b)$ is decreasing with x on $[a,b]$. The proof is completed.

Theorem 2. *Let f be defined as in Theorem* 1, *then we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x &\le \frac{1}{2}\Big(f(a) + f\Big(\frac{a+b}{2}\Big)\Big)\\ &\qquad - \frac{2}{b-a}\int_a^{(a+b)/2} f(x)\,\mathrm{d}x + \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\\ &\le \frac{1}{2}\,f(a) + \frac{1}{(b-a)^2}\int_a^b x f(x)\,\mathrm{d}x + \frac{b-2a}{(b-a)^2}\int_a^b f(x)\,\mathrm{d}x\\ &\qquad - \frac{2}{(b-a)^2}\int_a^b \Big(\int_a^x f(t)\,\mathrm{d}t\Big)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}\,,\end{aligned} \tag{6}$$

and

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{1}{2}\Big(f\Big(\frac{a+b}{2}\Big)+f(b)\Big)-\frac{2}{b-a}\int_{(a+b)/2}^b f(x)\,\mathrm{d}x \\
&\quad +\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{1}{2}\,f(b)-\frac{1}{(b-a)^2}\int_a^b xf(x)\,\mathrm{d}x \\
&\quad +\frac{2b-a}{(b-a)^2}\int_a^b f(x)\,\mathrm{d}x-\frac{2}{(b-a)^2}\int_a^b\Big(\int_x^b f(t)\,\mathrm{d}t\Big)\mathrm{d}x \le \frac{f(a)+f(b)}{2}\,.
\end{aligned}
\tag{7}
$$

Proof. From (1), we obtain inequality of left side for (6). Applying convexity of $h(a,y)$ (see (a)) to (1), we obtain

$$
\begin{aligned}
&\frac{b-a}{2}\Big(f(a)+f\Big(\frac{a+b}{2}\Big)\Big)-2\int_a^{(a+b)/2} f(x)\,\mathrm{d}x \le \frac{1}{2}(b-a)f(a) \\
&\quad +\frac{1}{b-a}\int_a^b xf(x)\,\mathrm{d}x-\frac{a}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{2}{b-a}\int_a^b\Big(\int_a^x f(t)\,\mathrm{d}t\Big)\mathrm{d}x \\
&\quad \le \frac{1}{2}(b-a)\big(f(a)+f(b)\big)-\int_a^b f(x)\,\mathrm{d}x.
\end{aligned}
\tag{8}
$$

Manipulating (8) we get two inequalities of the right side for (6).

Using $h(x,b)$ and the same method as proof of (6), we get (7).

Corollary. *Let f be defined as in Theorem* 1, *for any positive n, then we have*

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x &\le \frac{1}{2}\Big(f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big) \\
&\le \frac{1}{2}\Big(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x+\frac{f(a)+f(b)}{2}\Big) \\
&\le \cdots \le \frac{1}{2^n}\,f\Big(\frac{a+b}{2}\Big)+\frac{2^n-1}{2^n}\cdot\frac{f(a)+f(b)}{2} \\
&\le \frac{1}{2^n}\cdot\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x+\frac{2^n-1}{2^n}\cdot\frac{f(a)+f(b)}{2} \\
&\le \frac{1}{2^{n+1}}f\Big(\frac{a+b}{2}\Big)+\frac{2^{n+1}-1}{2^{n+1}}\cdot\frac{f(a)+f(b)}{2} \\
&\le \cdots \le \frac{f(a)+f(b)}{2}\,.
\end{aligned}
\tag{9}
$$

Proof. Expression (6) plus (7) with a simple manipulation yields

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x &\le \frac{1}{2}\Big(f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big) \\
&\le \frac{1}{2}\Big(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x+\frac{f(a)+f(b)}{2}\Big) \le \frac{f(a)+f(b)}{2}\,.
\end{aligned}
\tag{10}
$$

From (1) and (10), for any positive integer n, we have

$$\begin{aligned}
&\frac{1}{2^n} f\Big(\frac{a+b}{2}\Big) + \frac{2^n-1}{2^n} \cdot \frac{f(a)+f(b)}{2} \\
&\le \frac{1}{2^n} \cdot \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x + \frac{2^n-1}{2^n} \cdot \frac{f(a)+f(b)}{2} \\
&\le \frac{1}{2^n} \cdot \frac{1}{2} \Big(f\Big(\frac{a+b}{2}\Big) + \frac{f(a)+f(b)}{2} \Big) + \frac{2^n-1}{2^n} \cdot \frac{f(a)+f(b)}{2} \\
&= \frac{1}{2^{n+1}} f\Big(\frac{a+b}{2}\Big) + \frac{2^{n+1}-1}{2^{n+1}} \cdot \frac{f(a)+f(b)}{2} \\
&\le \frac{1}{2^{n+1}} \cdot \frac{f(a)+f(b)}{2} + \frac{2^{n+1}-1}{2^{n+1}} \cdot \frac{f(a)+f(b)}{2} = \frac{f(a)+f(b)}{2}.
\end{aligned} \tag{11}$$

Combination of (10) and (11) yields (9).

Theorem 3. *Let f be defined as in Theorem 1, for $x \in (a,b)$, then we have*

$$\frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \le \frac{b-x}{b-a} \cdot \frac{f(x)+f(b)}{2} + \frac{1}{b-a} \int_a^x f(t)\,\mathrm{d}t \le \frac{f(a)+f(b)}{2}, \tag{12}$$

$$\begin{aligned}
\frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t &\le \frac{1}{4} \Big(\frac{x-a}{b-a} f(a) + f(x) + \frac{b-x}{b-a} f(b) \Big) \\
&\quad + \frac{1}{2\,(b-a)} \int_a^b f(t)\,\mathrm{d}t \le \frac{f(a)+f(b)}{2}.
\end{aligned} \tag{13}$$

Proof. For $x \in (a,b)$, Using monotonicity of $h(x,b)$, we get

$$\begin{aligned}
0 = h(b,b) \le h(x,b) &= (b-x)\big(f(x)+f(b)\big) - 2 \int_x^b f(t)\,\mathrm{d}t \\
&\le h(a,b) = (b-a)\,(f(a)+f(b)) - 2 \int_a^b f(x)\,\mathrm{d}x.
\end{aligned} \tag{14}$$

Manipulating (14) we get (12). Expression (3) plus (12) with simple manipulation yields (13).

Theorem 4. *Let f be defined as in Theorem 1, for $x \in (a,b)$. Then we have*

$$\begin{aligned}
f\Big(\frac{a+b}{2}\Big) &\le \frac{1}{b-a} \int_x^b f(t)\,\mathrm{d}t - \frac{b-x}{b-a} f\Big(\frac{x+b}{2}\Big) + f\Big(\frac{a+b}{2}\Big) \\
&\le \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&f\Big(\frac{a+b}{2}\Big) \le \frac{1}{2\,(b-a)} \int_a^b f(t)\,\mathrm{d}t + f\Big(\frac{a+b}{2}\Big) \\
&- \frac{x-a}{2(b-a)} f\Big(\frac{a+x}{2}\Big) - \frac{b-x}{2\,(b-a)} f\Big(\frac{x+b}{2}\Big) \le \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t.
\end{aligned} \tag{16}$$

Proof. Using monotonicity of $H(x, b)$, (4) and the same method as the proof of Theorem 3, we can prove Theorem 4.

REMARK. (3), (6), (7), (9), (12) and (13) are refinements of right side for (1); (4), (15) and (16) are refinements of left side for (1).

REFERENCES

1. J. HADAMARD: *Etude sur les proprieties des fonctions entieres et en particulier d'une fonction consideree par Riemann.* J. Math. Pures Appl., **58** (1893), 171–215.
2. D. S. MITRINOVIĆ, I. B. LACKOVIĆ: *Hermite and convexity.* Aequationes Math., **28** (1985), 229–232.
3. G. S. YANG, K. L. TSENG: *On certain integral inequalities related to Hermite-Hadamard inequalities.* J. Math. Anal. Appl., **239** (1999), 180–187.
4. S. S. DRAGOMIR, P. AGARWAL: *Two new mappings associated with Hadamard's inequalities for convex functions.* Appl. Math. Lett., **11**, No.3 (1998), 33–38.
5. S. S. DRAGOMIR: *Two mappings in connection to Hadamard's inequalities.* J. Math. Anal. Appl., **167** (1992), 49–56.
6. L. C. WANG: *On extensions and refinements of Hermite-Hadamard inequalities for convex functions.* Math. Ineq. Appl., **6** (2003), in the press.
7. L. C. WANG, S. G. WANG: *Some decision theories of convex function in linear spaces.* J. Chongqing Teachers' College, **10**, No.3 (1993), 38–34. (Chinese)
8. L. C. WANG: *Convex functions and their inequalities.* Sichuan Univ. Press, Sichuan, China. 2001. (Chinese)

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