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A NOTE ON COMPACT OPERATORS

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Let H be a separable complex Hilbert space and let $\mathcal{B}(H)$ be the algebra of bounded linear operators on H. Recall that an operator $T \in \mathcal{B}(H)$ is said to be compact if for every bounded sequence $\{x_n\}$ of vectors in H, the sequence $\{Tx_n\}$ contains a converging subsequence. An operator T is said to be of finite rank if the range of T, $R(T)$ is finite dimensional. It is easily seen that every finite rank operator is compact, however, the converse is false. The operator T is said to be of almost finite rank of T is the limit, in the norm topology of $\mathcal{B}(H)$, of a sequence of finite rank operators. Finally, the operator T is said to be completely continuous (or C.C.) if for every weakly convergent sequence $\{x_n\}$, the sequence $\{Tx_n\}$ converges. A sequence $\{x_n\}$ in H is said to converge weakly if the sequence $\{\langle x_n, y\rangle\}$ of numbers converges for all y. For these concepts see [**1**], [**2**], [**3**], [**6**].

The following result is well known (see [**4**], [**7**]):

Theorem. *Let $T \in \mathcal{B}(H)$. The following statements are equivalent*:
1. *T is compact.* 2. *T is of almost finite rank.* 3. *T is completely continuous.*

In this note we introduce the concepts of a quasi-compact operator and semi-compact operator and we show the equivalence of these concepts with compactness.

1. QUASI-COMPACT OPERATORS AND SEMI-COMPACT OPERATORS

We start this by definitions:

Definition 1.1. *An operator $T \in \mathcal{B}(H)$ is said to be quasi-compact if for every sequence $\{x_n\}$ in H that converges weakly to the zero vektor* 0, *the sequence $\{\langle Tx_n, x_n\rangle\}$ converges to* 0.

Note that this definition is equivalent to saying that when $x_n \to x$ weakly, then $\langle Tx_n, x_n\rangle \to \langle Tx, x\rangle$.

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Definition 1.2. *An operator $T \in \mathcal{B}(H)$ is said to be semi-compact if for every orthonormal sequence $\{e_n\}$ in H, the sequence $\{\langle Te_n, e_n\rangle\}$ converges to* 0.

Since every orthonormal sequence converges weakly to 0, then it is clear that every quasi-compact operator is semi-compact. Our main result in this note is the following.

Theorem 1.3. *Let $T \in \mathcal{B}(H)$. The following statements are equivalent*:

1. *T is compact.* 2. *T is quasi-compact.* 3. *T is semi-compact.*

For the proof of the theorem we need the following lemmas:

Lemma 1.4. *If A is self adjoint operator ($A = A^*$) in $\mathcal{B}(H)$ which is quasi-compact, then A is C.C., and hence is compact.*

Proof. Let each of $\{x_n\}$ and $\{y_n\}$ be a weakly convergent sequence in H that converges to 0. Thus $\langle Ax_n, x_n\rangle \to 0$ and $\langle Ay_n, y_n\rangle \to 0$. It is clear that the sequences $\{x_n + y_n\}$ and $\{x_n - y_n\}$ are weakly converging to 0. Thus

$$\langle A(x_n + y_n), x_n + y_n\rangle \to 0 \text{ and } \langle A(x_n - y_n), x_n - y_n\rangle \to 0.$$

Thus $\langle Ax_n, x_n\rangle + \langle Ax_n, y_n\rangle + \langle Ay_n, x_n\rangle + \langle Ay_n, y_n\rangle \to 0$. But $\langle Ax_n, x_n\rangle \to 0$ and $\langle Ay_n, y_n\rangle \to 0$, hence $\langle Ax_n, y_n\rangle + \langle Ay_n, x_n\rangle \to 0$. Because $A = A^*$, then $\langle Ay_n, x_n\rangle = \langle y_n, Ax_n\rangle = \overline{\langle Ax_n, y_n\rangle}$. Thus $\langle Ax_n, y_n\rangle + \overline{\langle Ax_n, y_n\rangle} = 2\,\mathrm{Re}\,\langle Ax_n, y_n\rangle \to 0$.

Similarly, $\langle Ax_n, y_n\rangle - \overline{\langle Ax_n, y_n\rangle} = 2\,\mathrm{Im}\,\langle Ax_n, y_n\rangle \to 0$.

Hence the sequence $\{\langle Ax_n, y_n\rangle\} \to 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ that converge to 0 weakly. In particular, if $\{y_n\} = \{Ax_n\}$, then $\{\langle Ax_n, Ax_n\rangle\} = \{\|Ax_n\|^2\}$ converges to 0. And the operator A is a C.C. operator.

Lemma 1.5. *If A is a self adjoint operator which is semi-compact, then A is of almost finite rank.*

We postpone the proof of the lemma to Section 3.

Theorem 1.3 now follows from the following.

Theorem 1.6. *Let $T \in \mathcal{B}(H)$, then the following statements are equivalent*:

1. *T is a compact operator.* 2. *T is a C.C. operator.*

3. *T is a quasi-compact operator.* 4. *T is a semi-compact operator.*

5. *T is of almost finite rank.*

Proof. The equivalence of 1 and 2 is well known.

(2) $\Rightarrow$ (3) : Let $\{x_n\}$ be a weakly convergent sequence to 0. Since T is C.C., $Tx_n \to 0$. By the continuity of the inner product, $\langle Tx_n, x_n\rangle \to 0$, thus T is quasi-compact.

(3) $\Rightarrow$ (2) : Let $T = A + iB$, where $A = \frac{1}{2}(T + T^*)$ and $B = \frac{1}{2i}(T - T^*)$, it is clear that A and B are self adjoint. Let $\{x_n\}$ be a weakly convergent sequence to 0. Since T is quasi-compact, then $\langle Tx_n, x_n\rangle \to 0$, which implies $\langle T^*x_n, x_n\rangle \to 0$. Hence

$$\langle Ax_n, x_n\rangle = \frac{1}{2}\langle Tx_n, x_n\rangle + \frac{1}{2}\langle T^*x_n, x_n\rangle \to 0.$$

Thus A is a quasi-compact operator. But A is self adjoint, hence by Lemma 1.4, A is C.C. By the same argument B is C.C., and hence T is C.C.

(3) $\Rightarrow$ (4) : Follows from the fact that every orthonormal sequence converges weakly to 0.

(4) $\Rightarrow$ (5) : Let $T = A+iB$, where $A = \frac{1}{2}(T+T^*)$ and $B = \frac{1}{2i}(T-T^*)$. Let $\{e_n\}$ be an orthonormal sequence in H. Since T is semi-compact, $\langle Te_n, e_n\rangle \to 0$, which implies $\langle T^* e_n, e_n\rangle \to 0$. Consequently,

$$\langle Ae_n, e_n\rangle = \frac{1}{2}\langle Te_n, e_n\rangle + \frac{1}{2}\langle T^* e_n, e_n\rangle \to 0.$$

Similarly $\langle Be_n, e_n\rangle \to 0$. Thus by Lemma 1.5, each of A and B is of almost finite rank. Hence $T = A + iB$ is almost finite rank.

(5) $\Rightarrow$ (1) : Follows from the fact that each finite rank operator is compact and the set of compact operators is closed, [**6**].

Now, for the proof of the Lemma 1.5, we need the functional calculus.

2. FUNCTIONAL CALCULUS

For a general reference on functional calculus see [**3**], [**4**], [**6**], [**7**].

Let A be a self-adjoint operator in $\mathcal{B}(H)$. It is easily seen that $\langle Ax, x\rangle$ is real for all $x \in H$. It is known that $\|A\| = \sup_{\|x\|=1}\{|\langle Ax, x\rangle|\}$, [**6**, Prop. 68.5]. Let $m = m(A) = \inf_{\|x\|=1}\{\langle Ax, x\rangle\}$, $M = M(A) = \sup_{\|x\|=1}\{\langle Ax, x\rangle\}$. These numbers are called the lower and the upper bounds of A, respectively. It is known that $\|A\| = \text{Max}\{|m|, |M|\}$. Let P be the linear space of all polynomial functions with real coefficients defined on the interval $[n, M]$. Let $p(t) = a_0 + a_1 t + \cdots + A_n t^n \in P$, $p(A) = a_0 I + a_1 A + \cdots + a_n A^n$.

Recall that a self adjoint operator A is called positive, $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. Consequently, if A, B are self adjoint operators on H, we say that $A \leq B$ if $B - A$ is positive, i.e. $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$.

Next we turn to continuous functions on $[m, M]$. Since every continuous function can be approximated uniformly by the sequence of polynomial functions (Weierstrass approximation theorem), the map $p \to p(A)$ can be extended to the Banach space of all continuous functions on $[m, M]$.

Let $A \in \mathcal{B}(H)$ be a self adjoint operator, and let f be a continuous function on $[m, M]$. Then there exists a unique operator $f(A)$ which is bounded and self adjoint.

Next we extend the notion $f(A)$ to functions which are not necessarily continuous. But first we need a definition.

Definition 2.1. [**6**] *A sequence* $\{T_n\}$ *of self adjoint operators is said to be monotonic increasing* (*decreasing*) *if* $T_1 \leq T_2 \leq \cdots \leq L$ $(T_1 \geq T_2 \geq T_3 \geq \cdots)$. *It is* *said to be bounded from above (below) if there exists a self adjoint operator B with $T_n \leq B$ ($T_n \geq B$) for all n. A sequence is said to be bounded if it is bounded from below and above.*

The proof of the following proposition is not difficult (see [**6**]).

Proposition 2.2. *Every monotonic and bounded sequence of self adjoint operators on H converges strongly (i.e. pointwise) to a self adjoint operator, i.e. exists a self adjoint operator T on H such that $\|T_n x - Tx\| \to 0$ for each $x \in H$.*

Let K_1 be the class of functions $f : [m, M] \to \mathbb{R}$ for which the following holds: There exists a sequence of continuous functions $\{f_n\}$ on $[m, M]$ with $f_n(t) \geq f_{n+1}(t) \geq 0$ and $f_n(t) \to f(t)$ for every $t \in [m, M]$.

Remark 2.3. By Weierstrass approximation theorem, we can replace the functions $\{f_n\}$ by polynomials $\{p_n\}$.

Proposition 2.4. *Let $A \in \mathcal{B}(H)$ be a self adjoint operator. Let $f \in K_1$ and $h_n(t)$ be a monotone decreasing sequence of polynomials that converges pointwise to f. Then the sequence of operators $\{h_n(A)\}$ converges strongly to an operator denoted by $f(A)$, moreover, $f(A)$ is self adjoint and does not depend on the choice of $\{h_n\}$.*

Proof. See [**6**] and [**7**].

Proposition 2.5. *Let $A \in \mathcal{B}(H)$ be a self adjoint operator. Then the mapping $f \to f(A)$, where $f \in K_1$ has the following properties:*

(1) *If $f, g \in K_1$ and $f(t) \leq g(t)$ for all $t \in [m, M]$, then $f(A) \leq g(A)$.*

(2) $(\alpha f)(A) = \alpha\big(f(A)\big)$ *for* $\alpha \geq 0$. (3) $(f + g)(A) = f(A) + g(A)$.

(4) $fg(A) = f(A)g(A)$.

Proof. Proof is simple.

3. PROOF OF LEMMA 1.5

In this section we apply the tools constructed in Section 2 to prove Lemma 1.5. We start by the following:

Proposition 3.1. *Let $\mu \in \mathbb{R}$, define the characteristic function e_μ as follows:*

$$e_\mu(\lambda) = \begin{cases} 0 & (\textit{for } \lambda \leq \mu) \\ 1 & (\textit{for } \lambda > \mu) \end{cases}$$

Then the function e_μ belongs to the class K_1 and $e_\mu(A)$ is a projection operator.

Proof. Define a sequence $\{f_n\}$ of continuous functions on $\mathbb{R}$ as follows: $f_n(\lambda) = 1$ for $\lambda \leq \mu$, $f_n(\lambda) = 0$ for $\lambda \geq \mu + \dfrac{1}{n}$ and on $\left(\mu,\, \mu + \dfrac{1}{n}\right)$ the function f_n is linear.

Now, if $A \in \mathcal{B}(H)$ is self adjoint with m, M defined as above, then by Proposition 2.4, the operator $e_\mu(A)$ is defined in $\mathcal{B}(H)$ and is self adjoint. Moreover, since $e_\mu^2(\lambda) = e_\mu(\lambda)$, then by Proposition 2.5 $e_\mu^2(A) = e_\mu(A)$. Thus the operator

$e_\mu(A)$ is a projection operator. Notice that $e_\mu(A) = 0$, the zero operator if $\mu < m$ and $e_\mu(A) = I$, the identity operator if $\mu > M$.

Lemma 3.2. *Let A be a self adjoint operator and α is non negative real number. Let*

$$p_\alpha(\lambda) = \begin{cases} 0 & (\text{for } |\lambda| < \alpha) \\ 1 & (\text{for } |\lambda| \geq \alpha) \end{cases}.$$

Then $p_\alpha(A)$ is a self adjoint operator.

Proof. Define the characteristic functions $q_\alpha(\lambda)$ and $r_\alpha(\lambda)$ as follows:

$$q_\alpha(\lambda) = \begin{cases} 0 & (\text{for } \lambda \leq -\alpha) \\ 1 & (\text{for } \lambda > -\alpha) \end{cases}, \qquad r_\alpha(\lambda) = \begin{cases} 0 & (\text{for } \lambda \geq \alpha) \\ 1 & (\text{for } \lambda < \alpha) \end{cases}.$$

It is clear that $p_\alpha(\lambda) = q_\alpha(\lambda) + r_\alpha(\lambda)$. By Proposition 3.1, each of the operators $r_\alpha(A)$ and $q_\alpha(A)$ is defined and self adjoint. By Proposition 2.5, the operator $p_\alpha(A) = q_\alpha(A) + r_\alpha(A)$ is self adjoint.

Lemma 3.3. *If A is a self adjoint operator which is semi compact and $p_\alpha(A)$ is the function defined in Lemma* 3.2, *then $p_\alpha(A)$ is an operator of finite rank.*

Proof. It is enough to show that each of the operators $r_\alpha(A)$ and $q_\alpha(A)$ is of finite rank. Assume that $r_\alpha(A)$ is not of finite rank, i.e. its range $r_\alpha(A)H$ is infinite dimensional. Let $\{e_n\}$ be an infinite orthonormal sequence in $r_\alpha(A)H$. Since $r_\alpha(A)$ is a projection by Proposition 3.1, then $r_\alpha(A)e_n = e_n$ for each n. Thus $\langle Ae_n, e_n\rangle = \langle Ar_\alpha(A)e_n, r_\alpha(A)e_n\rangle$.

Now, define the function

$$t_\alpha(\lambda) = \begin{cases} \lambda & (\text{for } \lambda \geq \alpha) \\ \alpha & (\text{for } \lambda < \alpha) \end{cases}.$$

Notice that $\lambda r_\alpha(\lambda) = t_\alpha(\lambda) r_\alpha(\lambda)$. Using Proposition 2.5, we get $Ar_\alpha(A) = t_\alpha(A) r_\alpha(A)$. Note also, because $t_\alpha(\lambda)$ is non negative for all $\lambda \in \mathbb{R}$, then $t_\alpha(A)$ is a positive operator. Moreover, since $t_\alpha(A)\lambda \geq \alpha$ for all real λ, then by Proposition 2.5, $t_\alpha(A) \geq \alpha I$. Consequently $\langle t_\alpha(A)u, u\rangle \geq \alpha\langle u, u\rangle$ for all $u \in H$. In particular, if $u = r_\alpha(A)e_n$, we get

$$\langle Ae_n, e_n\rangle = \langle t_\alpha(A) r_\alpha(A) e_n, r_\alpha(A)\rangle \geq \alpha\langle r_\alpha(A)e_n, r_\alpha(A)e_n\rangle = \alpha\langle e_n, e_n\rangle = \alpha.$$

But this relation is true for each $\alpha > 0$, thus $\langle Ae_n, e_n\rangle \to 0$, and hence A is not semi compact which is a contradiction. Thus $r_\alpha(A)$ is an operator of finite rank. Similarly $q_\alpha(A)$ has finite rank.

We are now in a position to prove Lemma 1.5.

Proof of Lemma 1.5. Let $p_\alpha(A)$ be the function defined as in Lemma 3.2. Consider the function $\lambda\big(1 - p_\alpha(\lambda)\big) = \lambda - \lambda p_\alpha(\lambda)$. Since $\big|\lambda - \lambda p_\alpha(\lambda)\big| \leq \alpha$ for all real

λ, then by Proposition 2.5, $\|A - Ap_\alpha(A)\| \leq \alpha$. Since the operator $p_\alpha(A)$ has finite rank, so is $Ap_\alpha(A)$. Hence for each $n \in \mathbb{N}$, take $\alpha = 1/n$. Then $\|A - Ap_{1/n}(A)\| \leq 1/n$. Thus $Ap_{1/n}(A) \to A$, and hence A is almost finite rank.

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