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MAXIMUM MODULE VALUES OF POLYNOMIALS ON $|z| = R$ $(R > 1)$

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Let $f(z)$ and $g(z)$ be two polynomials of degrees $m \geq 1$ and $n \geq 1$ respectively on $|z| \leq R$ $(R > 1)$, and $\mathcal{M}_f = \max\limits_{|z|=R} |f(z)|$, $\mathcal{M}_g = \max\limits_{|z|=R} |g(z)|$ and $\mathcal{M}_{fg} = \max\limits_{|z|=R} |f(z)\, g(z)|$. If $z = 0$ is not a root of given polynomials, it is shown that $\mathcal{M}_{fg} \geq \delta_1 \mathcal{M}_f \mathcal{M}_g$, where $\delta_1 = \dfrac{1}{2^m}\,\dfrac{1}{2^n}$. On the other hand, if $z = 0$ is k-multiple root of $f(z)$ and a r-multiple root of $g(z)$, then it is proved that $\mathcal{M}_{fg} \geq \varepsilon \mathcal{M}_f \mathcal{M}_g$ with $\varepsilon = \dfrac{1}{2^{m-k}}\,\dfrac{1}{2^{n-r}}$. Moreover, some generalizations have been obtained for n similar polynomials.

1. INTRODUCTION

Let $f,\, g:\ \mathbb{C} \to \mathbb{C}$ be complex-valued polynomial functions of degrees $m \geq 1$, $n \geq 1$, respectively, of a complex variable z, and $M_f = \max\limits_{|z|=1} |f(z)|$, $M_g = \max\limits_{|z|=1} |g(z)|$ and $M_{fg} = \max\limits_{|z|=1} |f(z)g(z)|$. It is shown (see [**1**]) that

$$M_{fg} \geq \nu M_f M_g \quad \text{with} \quad \nu = \sin^m \frac{\pi}{8m} \sin^n \frac{\pi}{8n}\,.$$

Let $f_1,\, f_2, \ldots, f_n : \mathbb{C} \to \mathbb{C}$ be complex-valued polynomial functions of degrees $d_1,\, d_2, \ldots, d_n$, respectively, of a complex variable z. In [**2**] the following inequality is obtained:

$$M_{f_1} M_{f_2} \cdots M_{f_n} \geq M_{f_1 f_2 \cdots f_n} \geq k\, M_{f_1} M_{f_2} \cdots M_{f_n},$$

with $k = \left(\sin\left(\dfrac{2}{n}\,\dfrac{\pi}{8d_1}\right)\right)^{d_1} \cdot \left(\sin\left(\dfrac{2}{n}\,\dfrac{\pi}{8d_2}\right)\right)^{d_2} \cdots \left(\sin\left(\dfrac{2}{n}\,\dfrac{\pi}{8d_n}\right)\right)^{d_n}.$

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It shown in [**3**] that $M_{fg} > \nu M_f M_g$ with $\nu = \dfrac{1}{2^m} \cdot \dfrac{1}{2^n}$.

If $f(z)$ and $g(z)$ accept $z = 0$ as k-multiple and r-multiple roots, respectively, then in [**4**] the following inequality is obtained:

$$M_{fg} \geq \delta M_f M_g \quad \text{with} \quad \delta = \frac{1}{2^{m-k}} \frac{1}{2^{n-r}} .$$

In [**5**], some generalizations of the results of [**3**] and [**4**], have been obtained for n similar polynomials.

2. MAXIMUM MODULE VALUES OF POLYNOMIALS NOT ADMITTING $z = 0$ AS A ROOT ON $|z| \leq R$ ($R > 1$)

Firstly, we give two lemmas in order to facilitate the development of our work.

Lemma 2.1. *For $|z| = R$ and $|\gamma| \neq 1/R$, we have* $\left| \dfrac{R^2\gamma - z}{1 - \overline{\gamma} z} \right| = R.$

For the proof, it is enough to take the module of the both sides of $\dfrac{R^2\gamma - z}{1 - \overline{\gamma} z} = \dfrac{R^2\gamma - z}{\dfrac{z}{R^2}\left(\overline{z} - R^2\,\overline{\gamma}\right)}$.

Lemma 2.2. *We have*

(i) $(z - R^2\gamma) = \left(z - \dfrac{1}{\overline{\gamma}}\right) \overline{\gamma}\, \dfrac{R^2\gamma - z}{1 - \overline{\gamma} z}$ *for* $|\gamma| \neq \dfrac{1}{R}$.

(ii) *All roots of the polynomials $f(z) = (z - R^2\alpha_1)(z - R^2\alpha_2) \cdots (z - R^2\alpha_k)$ of degree $m \geq 1$ satisfy $|z| \leq R$, where $|\alpha_k| \leq 1/R$, $k = 1, 2, \ldots, m$.*

Proof. (i) is obvious and (ii) follows from the hypothesis and $|R^2\alpha_k| = R^2|\alpha_k|$, $k = 1, 2, \ldots, m$.

Theorem 2.1. *Let $\mathcal{M}_f = \max\limits_{|z|=R} |f(z)|$, $\mathcal{M}_g = \max\limits_{|z|=R} |g(z)|$, $\mathcal{M}_{fg} = \max\limits_{|z|=R} |f(z)g(z)|$ be the maximum module values of the polynomials*

$$f(z) = \prod_{i=1}^{m} (z - R^2\alpha_i) \qquad (\alpha_i \neq 0,\ |\alpha_i| \leq 1/R)$$

and

$$g(z) = \prod_{j=1}^{n} (z - R^2\beta_j) \qquad (\beta_j \neq 0,\ |\beta_j| \leq 1/R)$$

on $|z| = R$. Then

(1) $$\mathcal{M}_{fg} \leq \delta_1 \mathcal{M}_f \mathcal{M} g, \quad \text{where } \delta_1 = \frac{1}{2^m} \frac{1}{2^m} .$$

Proof. Consider the polynomial

$$h(z) = \prod_{k=1}^{\ell} (z - z_k) \qquad (z_k \neq 0,\ |z_k| \leq R). \tag{2}$$

Then we have $\mathcal{M}_h = R^\ell \max\limits_{|z|=R} \left\{ \left| \dfrac{h(z)}{z^\ell} \right| \right\} = R^\ell \max\limits_{|z|=R} \left| \prod\limits_{k=1}^{\ell} \Big(1 - \dfrac{z_k}{z}\Big) \right|$. If we put $t = R/z$, than taking $s(t) = \prod\limits_{k=1}^{\ell} \Big(1 - \dfrac{z_k t}{R}\Big)$ it comes $\mathcal{M}_h = R^\ell \max\limits_{|t|\leq 1} |s(t)|$, where $s(0) = 1$, and we obtain from the *Maximum module principle* $\mathcal{M}_h \geq R^\ell$. Furthermore, by the definition of $\mathcal{M}_h$ it is clear that $\mathcal{M}_h \leq 2^\ell R^\ell$.

Since $f(z)$ and $g(z)$ are polynomials of the type (2) similar argument yields $\mathcal{M}_f \leq 2^m R^m$, $\mathcal{M}_g \leq 2^n R^n$.

If $z_1 = z_2 = \cdots = z_\ell = Re^{i\theta_0}$ $(\theta_0 \in \mathbb{R})$, then $\mathcal{M}_h = 2^\ell R^\ell$. On the other hand, let us consider the following sequences:

$$\begin{aligned} &R^2\alpha_1, \ldots, R^2\alpha_{p-1}/\alpha_p, \ldots, \alpha_m; \quad |\alpha_p| > 1/R, \ldots, |\alpha_m| > 1/R, \\ &R^2\beta_1, \ldots, R^2\beta_{q-1}/\beta_q, \ldots, \beta_n; \quad |\beta_q| > 1/R, \ldots, |\beta_n| > 1/R. \end{aligned}$$

Let

$$F(z) = \prod_{i=1}^{p-1} (z - R^2\alpha_i) \prod_{j=p}^{m} \Big(z - \frac{1}{\overline{\alpha}_j}\Big), \quad G(z) = \prod_{i=1}^{q-1} (z - R^2\beta_i) \prod_{j=q}^{n} \Big(z - \frac{1}{\overline{\beta}_j}\Big)$$

be polynomials on $|z| \leq R$ $(R > 1)$ with $m, n \geq 1$. Then, if we write $\mathcal{A} = \overline{\alpha}_p \cdots \overline{\alpha}_m$, $\mathcal{B} = \overline{\beta}_q \cdots \overline{\beta}_n$, we have by means of Lemma 2.1 and Lemma 2.2

$$f(z) = \mathcal{A}\, F(z) \prod_{\mu=p}^{m} \left(\frac{R^2\alpha_\mu - z}{1 - \overline{\alpha}_\mu z} \right), \quad g(z) = \mathcal{B}\, G(z) \prod_{\eta=q}^{n} \left(\frac{R^2\beta_\eta - z}{1 - \overline{\beta}_\eta z} \right).$$

It is easily deduced from the last equalities that we have

$$\mathcal{M}_f = |\mathcal{A}|\, \mathcal{M}_F\, R^{m-p}, \quad \mathcal{M}_g = |\mathcal{B}|\, \mathcal{M}_G\, R^{n-q}, \quad \mathcal{M}_{fg} = |\mathcal{A}|\, |\mathcal{B}|\, \mathcal{M}_{FG}\, R^{m-p+n-q}$$

and hence

$$\frac{\mathcal{M}_{fg}}{\mathcal{M}_f \mathcal{M}_g} = \frac{\mathcal{M}_{FG}}{\mathcal{M}_F \mathcal{M}_G}\,. \tag{3}$$

Since $F(z)$ and $G(z)$ are polynomials of type (2), we obtain $\mathcal{M}_F \leq 2^m R^m$, $\mathcal{M}_G \leq 2^n R^n$ and $\mathcal{M}_{FG} \geq R^{m+n}$, and thus (1) is found from (3).

Corollary 2.1. *Let $f_1(z), f_2(z), \ldots, f_n(z)$ be polynomials of degrees $m_1, m_2, \ldots, m_n$, respectively, on $|z| \leq R$ $(R > 1)$. Suppose that $z = 0$ is not a root of these polynomials. Then*

$$\mathcal{M}_{f_1 f_2 \cdots f_n} \geq \varepsilon\, \mathcal{M}_{f_1} \mathcal{M}_{f_2} \cdots \mathcal{M}_{f_n}, \quad \textit{where } \varepsilon = \frac{1}{2^{m_1}} \frac{1}{2^{m_2}} \cdots \frac{1}{2^{m_n}}\,.$$

3. MAXIMUM MODULE VALUES OF POLYNOMIALS HAVING $z=0$ AS A ROOT ON $|z| \le R$ ($R>1$)

In this section, we will give some relations concerning maximum module values of polynomials which admit $z=0$ as a simple or multiple root on $|z|=R$ $(R>1)$.

Theorem 3.1. *Let*

$$f(z) = z\prod_{i=1}^{m-1}(z-R^2\alpha_i) \qquad (\alpha_i \neq 0,\ |\alpha_i| \le 1/R)$$

and

$$g(z) = z\prod_{j=1}^{n}(z-R^2\beta_j) \qquad (\beta_j \neq 0,\ |\beta_j| \le 1/R)$$

be polynomials on $|z| \le R$ $(R>1)$ *with* $m-1,\, n-1 \ge 1$. *Then*

$$\mathcal{M}_{fg} \ge \delta_2\,\mathcal{M}_f\mathcal{M}_g, \quad \text{where } \delta_2 = \frac{1}{2^{m-1}}\,\frac{1}{2^{n-1}}\,. \tag{5}$$

Proof. Consider

$$h(z) = z\prod_{k=1}^{\ell-1}(z-z_k) \qquad (z_k \neq 0,\ |z_k| \le R). \tag{6}$$

If we apply the technique used in Theorem 2.1, then we have $\mathcal{M}_h \ge R^\ell$ and $\mathcal{M}_h \le 2^{\ell-1}R^\ell$. Similarly, we can find $\mathcal{M}_f \le 2^{m-1}R^m$, $\mathcal{M}_g \le 2^{n-1}R^n$.

If $z_1 = z_2 = \cdots = z_{\ell-1} = Re^{i\theta_0}$ $(\theta_0 \in \mathbb{R})$, then $\mathcal{M}_h = 2^{\ell-1}R^\ell$. Now, let us write the following sequences:

$$0, R^2\alpha_1, \ldots, R^2\alpha_{p-1}/\alpha_p, \ldots, \alpha_{m-1}; \quad |\alpha_p| > 1/R, \ldots, |\alpha_{m-1}| > 1/R,$$
$$0, R^2\beta_1, \ldots, R^2\beta_{q-1}/\beta_q, \ldots, \beta_{n-1}; \quad |\beta_q| > 1/R, \ldots, |\beta_{n-1}| > 1/R.$$

As in Theorem 2.1, consider

$$F_1(z) = z\prod_{i=1}^{p-1}(z-R^2\alpha_i)\prod_{j=p}^{m-1}\left(z-\frac{1}{\overline{\alpha}_j}\right), \quad G_1(z) = z\prod_{i=1}^{q-1}(z-R^2\beta_i)\prod_{j=q}^{n-1}\left(z-\frac{1}{\overline{\beta}_j}\right).$$

Putting $\mathcal{A}_1 = \overline{\alpha}_p\cdots\overline{\alpha}_{m-1}$, $\mathcal{B}_1 = \overline{\beta}_q\cdots\overline{\beta}_{n-1}$, we can write

$$f(z) = \mathcal{A}_1\,F_1(z)\prod_{\mu=p}^{m-1}\left(\frac{R^2\alpha_\mu - z}{1-\overline{\alpha}_\mu z}\right), \quad g(z) = \mathcal{B}_1\,G_1(z)\prod_{\eta=q}^{n-1}\left(\frac{R^2\beta_\eta - z}{1-\overline{\beta}_\eta z}\right),$$

and hence

$$\mathcal{M}_f = |\mathcal{A}_1|\,\mathcal{M}_{F_1}\,R^{m-1-p}, \quad \mathcal{M}_g = |\mathcal{B}_1|\,\mathcal{M}_{G_1}\,R^{n-1-q},$$
$$\mathcal{M}_{fg} = |\mathcal{A}_1|\,|\mathcal{B}_1|\,\mathcal{M}_{F_1G_1}\,R^{m+n-p-q-2}.$$

It is clear that the following equation results from the last equalities:

$$\frac{\mathcal{M}_{fg}}{\mathcal{M}_f \mathcal{M}_g} = \frac{\mathcal{M}_{F_1 G_1}}{\mathcal{M}_{F_1} \mathcal{M}_{G_1}}\,. \tag{7}$$

Since $F_1(z)$ and $G_1(z)$ are polynomials of the type (6), we have $\mathcal{M}_{F_1} \le 2^{m-1} R^m$, $\mathcal{M}_{G_1} \le 2^{n-1} R^n$ and $\mathcal{M}_{F_1 G_1} \ge R^{m+n}$. Thus (5) is obtained from (7).

Corollary 3.1. *Let $f_1(z)$, $f_2(z), \dots, f_n(z)$ be polynomials of degrees m_1, m_2, $\dots, m_n$, respectively, on $|z| \le R$ $(R > 1)$. Suppose that $z = 0$ is not simple zero of these polynomials. Then*

$$\mathcal{M}_{f_1 f_2 \cdots f_n} \ge \varepsilon_1 \, \mathcal{M}_{f_1} \mathcal{M}_{f_2} \cdots \mathcal{M}_{f_n}, \quad \textit{where } \varepsilon_1 = \frac{1}{2^{m_1 - 1}} \, \frac{1}{2^{m_2 - 1}} \cdots \frac{1}{2^{m_n - 1}}\,. \tag{8}$$

Theorem 3.2. *Let*

$$f(z) = z^k \prod_{i=1}^{m-k} (z - R^2 \alpha_i) \qquad (\alpha_i \ne 0, \ |\alpha_i| \le 1/R)$$

and

$$g(z) = z^r \prod_{j=1}^{n-r} (z - R^2 \beta_j) \qquad (\beta_j \ne 0, \ |\beta_j| \le 1/R)$$

be polynomials on $|z| \le R$ $(R > 1)$. Then

$$\mathcal{M}_{fg} \ge \delta \, \mathcal{M}_f \mathcal{M}_g, \quad \textit{for } \delta = \frac{1}{2^{m-k}} \, \frac{1}{2^{n-r}}\,. \tag{9}$$

Proof. Consider $h(z) = z^s \prod\limits_{k=1}^{w-s} (z - z_k)$ on $|z| \le R$. The following inequalities are easily found:

$$\mathcal{M}_h \ge R^w, \ \mathcal{M}_h \le 2^{w-s} R^w \ \text{ and } \ \mathcal{M}_f \le 2^{m-k} R^k, \ \mathcal{M}_g \le 2^{n-r} R^n.$$

Let us form now the following polynomials on the circle $|z| \le R$:

$$F_2(z) = z^k \prod_{i=1}^{p-1} (z - R^2 \alpha_i) \prod_{j=p}^{m-k} \left(z - \frac{1}{\overline{\alpha}_j}\right), \quad G_2(z) = z^r \prod_{i=1}^{q-1} (z - R^2 \beta_i) \prod_{j=q}^{n-r} \left(z - \frac{1}{\overline{\beta}_j}\right).$$

Taking $\mathcal{A}_2 = \overline{\alpha}_p \cdots \overline{\alpha}_{m-k}$, $\mathcal{B}_2 = \overline{\beta}_q \cdots \overline{\beta}_{n-r}$, we can write

$$f(z) = \mathcal{A}_2 \, F_2(z) \prod_{\mu=p}^{m-k} \left(\frac{R^2 \alpha_\mu - z}{1 - \overline{\alpha}_\mu z} \right), \quad g(z) = \mathcal{B}_2 \, G_2(z) \prod_{\eta=q}^{n-r} \left(\frac{R^2 \beta_\eta - z}{1 - \overline{\beta}_\eta z} \right).$$

From these equalities the following is deduced:

$$\mathcal{M}_f = |\mathcal{A}_2| \, \mathcal{M}_{F_2} \, R^{m-k-p}, \quad \mathcal{M}_g = |\mathcal{B}_2| \, \mathcal{M}_{G_2} \, R^{n-r-q},$$

$$\mathcal{M}_{fg} = |\mathcal{A}_2| \, |\mathcal{B}_2| \, \mathcal{M}_{F_2 G_2} \, R^{m+n-k-p-r-q}$$

and

$$\frac{\mathcal{M}_{fg}}{\mathcal{M}_f \mathcal{M}_g} = \frac{\mathcal{M}_{F_2G_2}}{\mathcal{M}_{F_2}\mathcal{M}_{G_2}}. \tag{10}$$

But, on the other hand we have $\mathcal{M}_{F_2} \le 2^{m-k} R^m$, $\mathcal{M}_{G_2} \le 2^{n-r} R^n$ and $\mathcal{M}_{F_2G_2} \ge R^{m+n}$. Thus (9) is obtained from (10).

Corollary 3.2. *Let $f_1(z)$, $f_2(z), \dots, f_n(z)$ be polynomials of degrees m_1, m_2, $\dots, m_n$, and suppose that each one accepts $z = 0$ as r_i $(i = 1, 2, \dots, n)$ multiple root, respectively. When $\varepsilon_2 = \dfrac{1}{2^{m_1-r_1}} \dfrac{1}{2^{m_2-r_2}} \cdots \dfrac{1}{2^{m_n-r_n}}$, then*

$$\mathcal{M}_{f_1 f_2 \cdots f_n} \ge \varepsilon_2 \, \mathcal{M}_{f_1}\mathcal{M}_{f_2} \cdots \mathcal{M}_{f_n}. \tag{11}$$

Result. For $\varepsilon_2 = 1$ it is necessary and sufficient that

$$f_1(z) = z^{m_1},\ f_2(z) = z^{m_2}, \dots, f_n(z) = z^{m_n}.$$

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