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A CLASS OF REFLEXIVE CACTUSES WITH FOUR CYCLES

Zoran Radosavljević, Marija Rašajski

A simple graph is reflexive if its second largest eigenvalue λ_2 is less than or equal to 2. A graph is a cactus, or a treelike graph, if any pair of its cycles (circuits) has at most one common vertex. For a lot of cactuses the property $\lambda_2 \leq 2$ can be tested by identifying and deleting a single cut-vetex (Theorem 1). if this theorem cannot be applied to a connected reflexive cactus and if all its cycles do not form a bundle, such a graph has at most five cycles. On the same conditions, in this paper we find some classes of maximal reflexive cactuses with four cycles. The complete case of four cycles, together with that of five cycles, is being settled in [**10**].

1. INTRODUCTION

For a simple graph G (an undirected graph without loops and/or multiple edges) let $P_G(\lambda) = \det(\lambda I - A)$ be the characteristic polynomial of its $(0, 1)$ adjacency matrix. It is defined to be the characteristic polynomial of G and will be denoted as $P(\lambda)$ if it is clear which graph it is related to. Its roots are eigenvalues of G, while the family of these roots makes up the spectrum of G. Since the eigenvalues of a simple graph are real, we assume their non-increasing order: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, The largest eigenvalue $\lambda_1(G)$ is also called the index of G. (Recall, if G is connected, then $\lambda_1(G) > \lambda_2(G)$, while for a disconnected graph $\lambda_1(G) = \lambda_2(G)$ if these are the indices of two distinct components of G.) The results of this paper concern only connected graphs.

Graphs with the spectral property $\lambda_2 \leq 2$ are usually called reflexive graphs and, if $\lambda_2 \leq 2 \leq \lambda_1$ they are also known as hyperbolic graphs. Reflexive graphs correspond to some sets of vectors in the Lorentz space $R^{p,1}$ and are interesting since they have some application to the construction and classification of reflexion

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groups [**7**]. In particular, reflexive trees have been studied in [**5**] and [**6**], and a class of bicyclic reflexive graphs in [**11**] (see also [**8**] and [**3**]).

A cactus or a treelike graph is a graph in which any two cycles have at most one common vertex (i.e. are edge disjoint).

The interrelation between the spectra of a graph G and its induced subgraph H is expressed by the so-called interlacing theorem.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of a graph G and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ eigenvalues of its induced subgraph H. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ $(i = 1, 2, \ldots, m)$ hold.

Thus, e.g. if $m = n - 1$, it will be $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$, and also λ_1 μ_1 if G is connected.

According to this theorem, the graphic property $\lambda_2 \leq 2$ is hereditary (every induced subgraph preserves the property) and that is why the results of this paper are to be expressed through sets of maximal graphs.

In Section 2 we give some known results, to be useful devices in further investigations, including Theorem 1, whose non-applicability will be a permanent condition throughout further consideration. We also give some facts concerning the cyclic structure of reflexive cactuses whose cycles do not form a bundle (the complete discussion will be carried out in [**10**]). Section 3 contains the main result of the paper - a class of maximal reflexive cactuses with four cycles. The complete case of four cycles, together with that of five cycles (which is the maximum number of cycles on the assumed conditions) is being solved in [**10**]).

The terminology of the theory of graph spectra in this paper follows [**1**], while for other graph theoretic notions one can see [**4**].

2. SOME PRELIMINARY FACTS AND GENERAL RESULTS

The following facts will play important roles in getting the results of Section 3.

Lemma 1. ([**13**], see also [**1**], p.79) *The index of a graph G satisfies $\lambda_1(G) \leq 2$ $(\lambda_1(G) < 2)$ if and only if each component of G is a subgraph (a proper subgraph) of one of the graphs displayed in Fig. 1, all of which have $\lambda_1 = 2$.*

These graphs are known as Smith graphs.

The following formulae express an interrelation between the characteristic polynomial of a graph and a set of its subgraphs.

Lemma 2. ([**12**], see also [**1**], p.78) *Given a graph G, let $C(v)$ $\big(C(uv)\big)$ denote the set of all cycles containing a vertex v and an edge uv of G, respectively. Then*

$$(1) P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u \in \mathrm{Adj}(v)} P_{G-v-u}(\lambda) - 2 \sum_{C \in \mathcal{C}(v)} P_{G-V(C)}(\lambda),$$

$$(2) P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-v-u}(\lambda) - 2 \sum_{C \in \mathcal{C}(uv)} P_{G-V(C)}(\lambda),$$

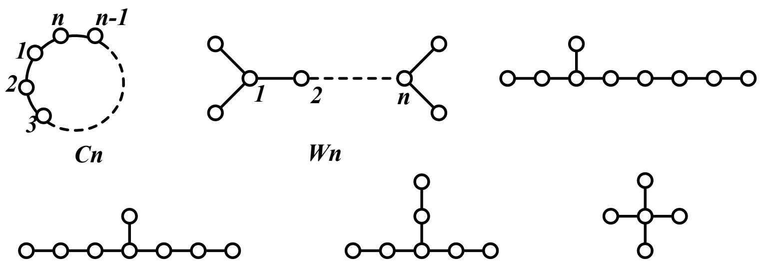


Figure 1.

where $Adj(v)$ denotes the set of neighbours of v, while $G - V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C.

These relations of A. Schwenk have some simple consequences (due to E. Heilbronner, see e.g. [**1**], p.59).

Corollary 1. *Let G be a graph obtained by joining a vertex v_1 of a graph G_1 to a vertex v_2 of a graph G_2 by an edge. Let $G_1'(G_2')$ be the induced subgraph of $G_1(G_2)$ obtained by deleting the vertex $v_1(v_2)$ from G_1 (resp. G_2).* Then

$$P_G(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) - P_{G_1'}(\lambda)P_{G_2'}(\lambda).$$

Corollary 2. *Let G be a graph with a pendent edge v_1v_2, v_1 being of degree 1. Then*

$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda),$$

where $G_1(G_2)$ is the graph obtained from G (resp. G_1) by deleting the vertex v_1 (resp. v_2).

The next lemma gives a list of values $P(2)$ for some types of graphs, which will appear to be a very suitable device for treating many particular cases in the coming investigation.

Lemma 3. [**11**] *Let $G_1, \ldots, G_{10}$ be the graphs displayed in Fig. 2. Then*

1. $P_{G_1}(2) = k + 2$;
2. $P_{G_2}(2) = 4$;
3. $P_{G_3}(2) = -klm + k + l + m + 2$;
4. $P_{G_4}(2) = 4\,(1 - kl)$;
5. $P_{G_5}(2) = -km$;
6. $P_{G_6}(2) = -m\,(2kl + k + l)$;
7. $P_{G_7}(2) = -4m$;
8. $P_{G_8}(2) = -m\,(3kl + 2k + 2l + 1)$;
9. $P_{G_9}(2) = klmn - (m + n)(2kl + k + l)$;
10. $P_{G_{10}}(2) = -(3k + 2)\,mn$.

If the removal of a cut-vertex (a cutpoint) of a graph G decomposes it into two components which are both Smith graphs, according to the interlacing theorem

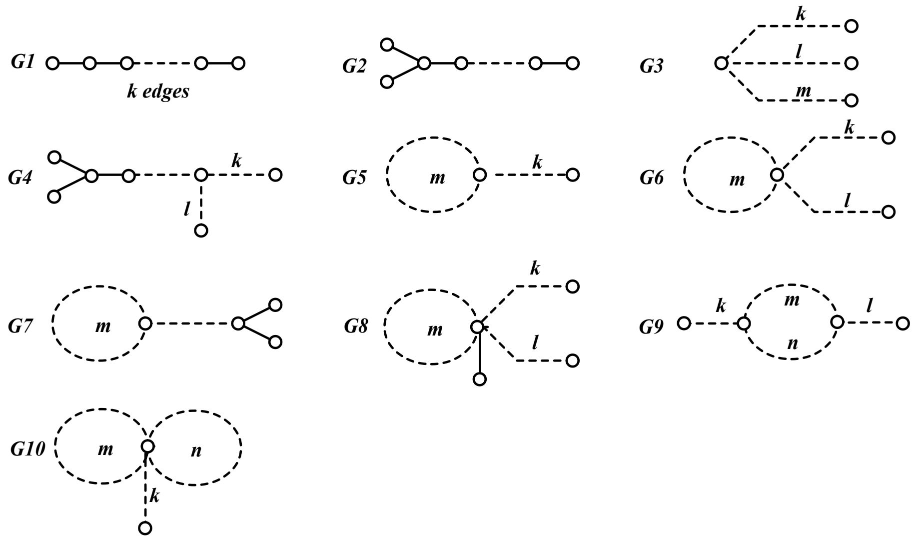


Figure 2.

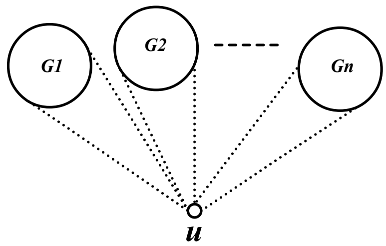


Figure 3.

we get $\lambda_2(G) = 2$. The general answer what will happen in case of an arbitrary number of components among which there are SMITH graphs is contained in the following theorem. Let us call a graph G positive, null or negative if $\lambda_1(G) > 2$, $\lambda_1(G) = 2$ and $\lambda_1(G) < 2$, respectively.

Theorem 1. [**11**] *Let G be a graph of the form depicted in Fig.* 3, *u being a cutpoint.*

1. *If at least two components of $G - u$ are supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then $\lambda_2(G) > 2$.*

2. *If at least two components of $G - u$ are Smith graphs, and the rest are subgraphs of Smith graphs, then $\lambda_2(G) = 2$.*

3. *If at most one component of $G - u$ is a Smith graph, and the rest are proper subgraphs of Smith graphs, then $\lambda_2(G) < 2$.*

We see that this theorem gives no response if we have one positive and all other negative components. That is why in further investigations, considering cer-

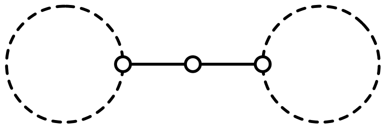


Figure 4.

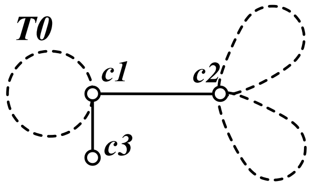


Figure 5.

tain class of graphs, we always assume that Theorem 1 is not applicable, and are engaged in solving only those cases which are not concerned by this theorem.

If all cycles of a cactus have the common vertex, we say that they form a bundle. The problem of finding all maximal reflexive cactuses whose cycles make a bundle appears to be a hard one, and anyhow it is to be regarded as a separate problem. That is why the condition that the cycles of a considered cactus do not form a bundle is another permanent assumption throughout this paper.

Theorem 1 completely covers the case of a cactus having cycles which are connected with the unique walk whose length is 2 (Fig. 4).

The case which takes place when two cycles are connected by the bridge has been completely solved in [**11**]. It turned out that those graphs are bicyclic, with only one exception; the tricyclic graph T_0 (Fig. 5) is a maximal reflexive cactus and it has a crucial role in some steps of further research.

If between cycles of a cactus there are no bridges (recall, it goes without saying that we do not consider bundles), there must be a cycle which touches at least two other cycles (at two distinct vertices). If these two vertices of touch are not adjacent, the only possibility is that of Fig. 6(a) (otherwise, Theorem 1 gives a clear answer), and if they are adjacent, we have the graph at Fig. 6(b).

In the case (a) the possibility of adding new cycles is being exhausted in the way which gives rise to the graphs Q_1 and Q_2 of Fig. 7. It is interesting that all

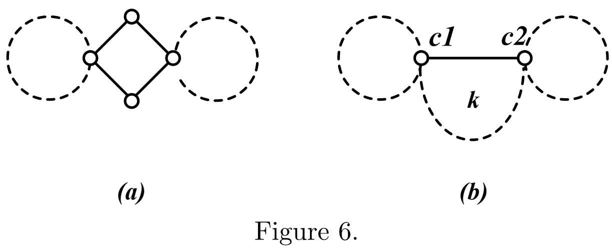


Figure 6.

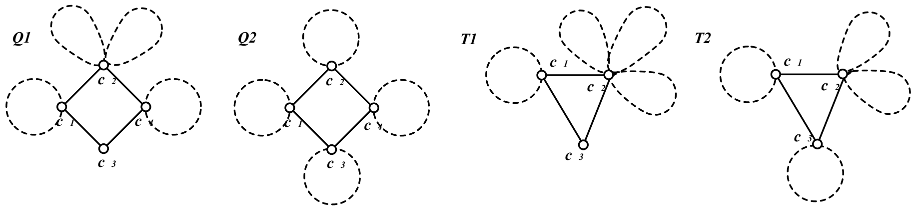


Figure 7.

graphs within the range between the graph of Fig. 4 and the graphs Q_1 and Q_2 have $\lambda_2 = 2$ and that Q_1 and Q_2 are maximal for that property.

In the case (b), if we add a new cycle leaned, say, on the vertex c_2, applying Lemma 2.(1) to c_2 we get $k = 2$

$\big(P(2), < 0$ i.e. $\lambda_2 < 2\big)$ and

$k = 3$ $(\lambda_2 = 2)$, which, after some additional investigation, gives graphs T_1 and T_2 of Fig. 7, both of them being maximal reflexive graphs $(\lambda_2 = 2)$, and again the graphs Q_1 and Q_2.

The complete discussion concerning this result will be given in [**10**], together with the entire list of maximal reflexive cactuses with four cycles (on the assumed conditions).

Theorem 2. *A cactus to which Theorem* 1 *cannot be applied and whose cycles do not make a bundle has at most five cycles. The only such graphs with five cycles, which are all maximal, i.e. cannot be extended at any vertex, are the four families of graphs* Q_1, Q_2, T_1 *and* T_2.

3. MAXIMAL REFLEXIVE CACTUSES WITH FOUR CYCLES

Graphs with five cycles of Fig. 7 are the apparent starting point for producing various classes of maximal reflexive cactuses with four cycles. Let the central quadrangles of the graphs Q_1 and Q_2 and triangles of T_1 and T_2 be called central cycles, and let us call the remaining cycles (of arbitrary lengths) of these graphs their outer cycles. If we extend a graph by introducing an additional edge incident to some vertex v, we will say that v is loaded by this edge. The procedure which leads to finding maximal reflexive cactuses with four cycles evidently includes starting from a graph obtained by removing a cycle from some of the graphs of Fig. 7, and then extending it by new acyclic additional parts up to the moment when it becomes maximal for the property $\lambda_2 \leq 2$. In this paper we will find all such maximal graphs on one additional condition, namely, that at least one vertex distinct from c-vertices (vertices of central cycles) of such a graph is loaded. The remaining cases will be solved in [**10**].

Graphs Q_1 and Q_2 generate two starting graphs with four cycles (Fig. 8).

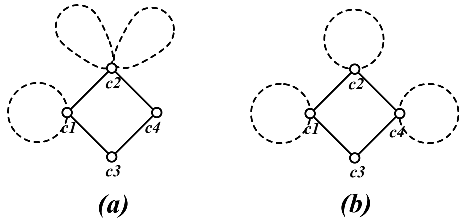


Figure 8.

Proposition 1. *Any extension of the graph of Fig.* 8(a) *at any vertex of its outer cycles different from* c_1 *and* c_2 *implies* $\lambda_2 > 2$.

Proof. After deleting the vertex c_4 we get just the graph T_0 of Fig. 5 which is maximal for the property $\lambda_2 \leq 2$ in the class of cactuses with the bridge between the cycles, and therefore cannot be extended within this class, i.e. can be extended only backwards (by c_4).

Proposition 2. *A cactus with four cycles, with the same cyclic structure as that of Fig.* 8(b), *whose at least one vertex of its outer cycles different from c-vertices is loaded, is reflexive if and only if it is an induced subgraph of some of the* 48 *(families of) graphs* $H_1 - H_{48}$ *displayed in Fig.* 9.

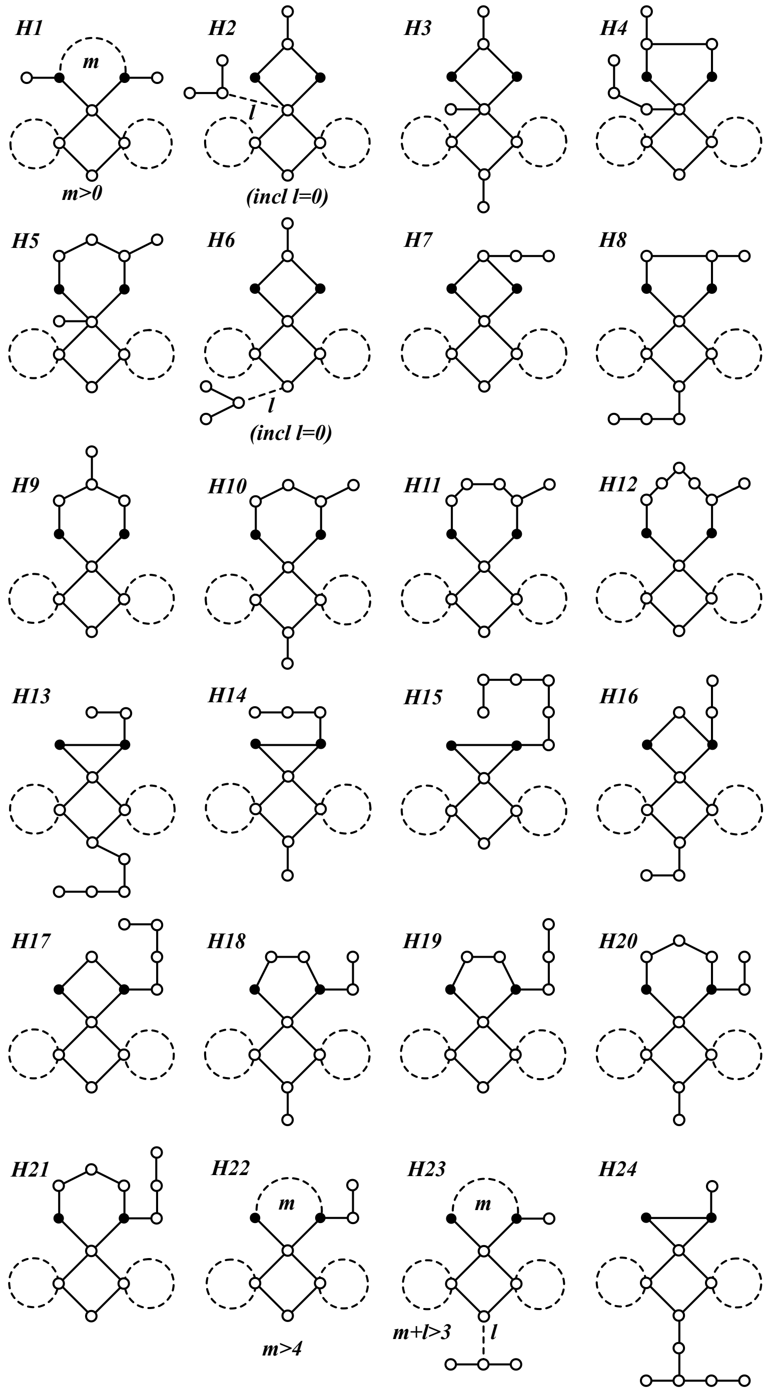
Proof. The only outer cycle whose vertices are to be loaded is the middle one (which contains c_2); otherwise, by removing c_3 and applying Theorem 1 to c_2, we would get $\lambda_2 > 2$. Of course, the vertex c_3 also can be loaded.

The rest of the proof is based on the results of the Propositions 4.2, 4.3 and 4.4 of [**11**], because all possibilities of extension of the starting graph are bounded by the maximal graphs of these propositions. The resulting sets of maximal graphs (in the class of bicyclic graphs with a bridge between the cycles) of these three propositions are, respectively, $A_1 - A_{14}$ of Fig. 20, $B_1 - B_{11}$ of Fig. 11 and $C_1 - C_{41}$ of Fig. 9 of [**11**] (not depicted here). For example, the graph A_2 of Proposition 4.2. (the only graph of this proposition to be considered, see Fig. 20) points out the analogous maximal possible extension of the starting graph (Fig. 10).

One should only verify whether the graph (b) satisfies $\lambda_2 \leq 2$ and, if not, to find the maximal one by reducing it. But in all cases except for the graphs $C_{11} - C_{15}$ of Proposition 4.3 it just happens that the analogous graph satisfies the condition of being reflexive, while $C_{11} - C_{15}$ apparently have to be excluded, since otherwise one of the c-vertices incident with outer cycles of arbitrary lengths (c_1 or c_4) would be loaded.

Remark. In Proposition 4.3. of [**11**] three resulting graphs were missed. Those are the graphs $B_9 - B_{11}$ at Fig.11. Also, the graph A_{10} of Fig. 20 was misdrawn at Fig. 5 in [**11**].

In what follows, as suggested by Fig. 9-11, let the vertices of outer cycles adjacent to c-vertices be denoted and called black vertices, and let the rest be white vertices.


H1
m
m>0
H2
l
(incl l=0)
H3
H4
H5
H6
l
(incl l=0)
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H8
H9
H10
H11
H12
H13
H14
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m
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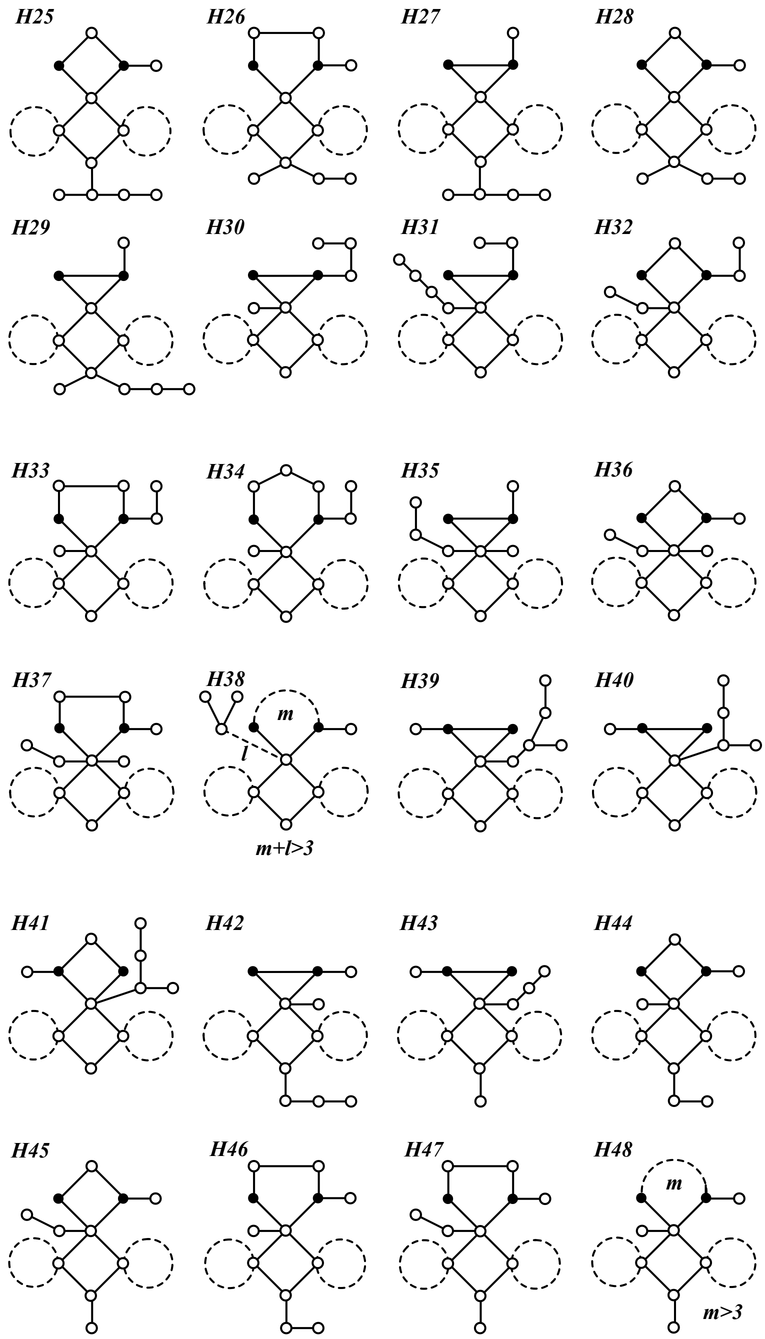
m
m+l>3
l
H24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H37
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m
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Figure 9.

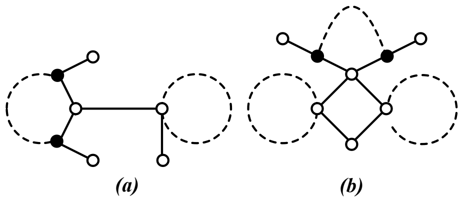


Figure 10.

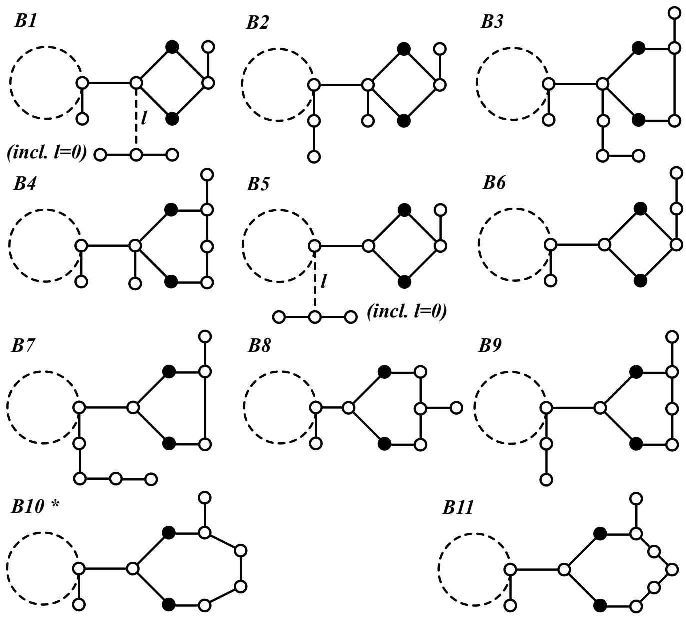


Figure 11.

Graphs T_1 and T_2 generate the next two starting graphs for families of reflexive cactuses with four cycles (Fig. 12).

Proposition 3. *Let G be a cactus with the same cyclic structure as that of Fig.* 12(a). *If at least two vertices of its outer cycles, different from c-vertices, are loaded, i.e. are of degree at least three, G is reflexive if and only if it is an induced subgraph of some of the* 9 *(families of) graphs $I_1 - I_9$ displayed at Fig.* 13.

Proof. First, no vertex of the left outer cycle (leaned on c_1) can be loaded: if we add a pendent edge at c_1 and apply Lemma 2.1. and Lemma 3, we get $P(2) = mn_1n_2 > 0$, i.e. $\lambda_2 > 2$, and the like happens if we load some other vertex of this cycle. Also, a white vertex cannot be of degree four, because Theorem 1, when applied to c_2, gives $\lambda_2 > 2$ and the same reason says that there cannot be two loaded white vertices on the same cycle. The assumption that there is a loaded white vertex at each of two cycles, after applying Lemma 2.1 and Lemma 3 to c_2, gives $P(2) = 0$ in the case of the graph I_1.

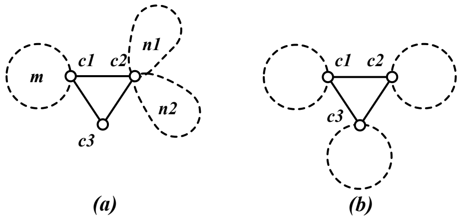


Figure 12.

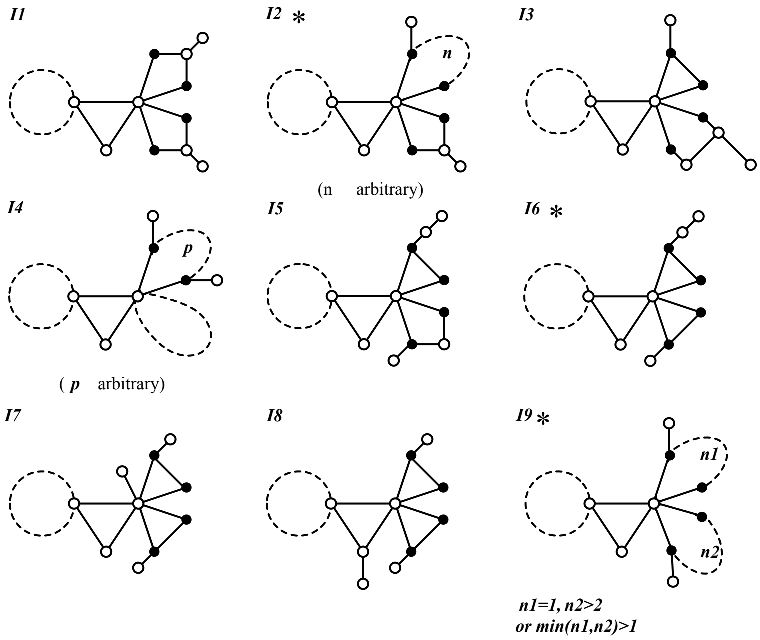


Figure 13.

A black vertex also cannot be of degree four ($P(2) > 0$ after applying the same Lemmas as before). The same holds if we have a white and a black vertex of degree 3 on the same cycle ($P(2) = mn\,(k+4)\,l > 0$, see Fig.14(a)), then in the similar way if there are two black vertices of one cycle and a white vertex of another cycle of degree 3, and if three black vertices are loaded. If there is a loaded white vertex on one cycle and such a black vertex on another cycle (Fig. 14(b)) we see that $P(2) = m\,\big((n+2)(kl-1)+2kl-k-l-4\big)$. For $k = l = 1$ n is not bounded, while $k = 2, l = 1$ gives $n = 1$ ($\lambda_2 = 2$). Since in the first case such a graph cannot be extended, it is a member of the solution (the graph I_2, $\lambda_2 < 2$), while in the second case we have I_3 ($\lambda_2 = 2$).

If two black vertices of the same cycle are loaded we always get $P(2) = 0$ (the graph I_4). Now, if we suppose that one black vertex of each cycle is of degree 3, let us consider the graph I_9 of Fig.13. We see that $P(2) = -m\,(n_1+n_2+6) < 0$ and,

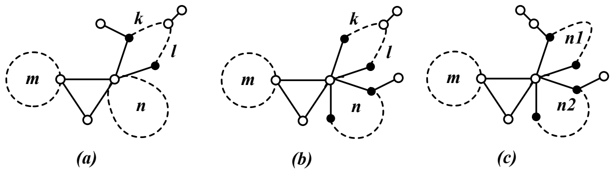


Figure 14.

in order to attain maximal graphs, we can try to extend it by adding new pendent edges at the vertices of degree one or at c-vertices. In the first case (Fig. 14(c)) we find $P(2) = m\,\big(n_1(n_2+2)-4\big)$ getting possibilities $(1,1)$ and $(1,2)$ for (n_1,n_2). Neither of these graphs can be extended any more, and they are the graphs I_5 and I_6 of the solution. In the second case we have $P(2) = m\,(n_1n_2+n_1+n_2-3)$ (for both c_2 and c_3) and $n_1 = n_2 = 1$ (the graphs I_7 and I_8). Thus, the graph I_9 is a solution if $n_1 = 1$, $n_2 \geq 3$ (and vice versa) or if $\min(n_1,n_2) \geq 2$.

In all cases we have passed through (when $\lambda_2 < 2$, and also if $\lambda_2 = 2$), the fact that a graph cannot be extended any more can be verified by appropriate application of Lemma 1, or Corollary 2 (with respect to added pendent edge), or with a little aid of a computer, which is actually very often much more suitable.

The resulting maximal graphs for which $\lambda_2 < 2$ are marked by asterisk.

Proposition 4. *Let G be a cactus with the same cyclic structure as that of Fig.* 12(a). *If one of its white vertices and none of its black vertices is loaded, G is reflexive if and only if it is an induced subgraph of some of the* 11 *(families of) graphs $J_1 - J_{11}$ displayed at* Fig. 15.

Proof. By assumption, we should discuss the situation of Fig. 16(a). Applying appropriate Lemmas one gets $P(2) = mn\,(2kl-k-l-4)$, i.e. $P(2) = 0$ for $(k,l) = (2,2)$ and $(k,l) = (1,5)$ (the resulting maximal reflexive graphs J_1 and J_2 cannot be extended any more). For $k = 1$ and $l = 1,2,3,4$ we need some further discussion.

If $l = 1$, the graph can be extended by the new edge leaned at the vertex of degree one (J_3), and also by adding two pendent edges at c_2 and c_3 (J_4). If we want to load only one of the two possible c-vertices, let us have a look at Fig. 16(b). Since we always get $P(2) < 0$, the added path can be of arbitrary length. But if we introduce a new pendent edge as in Fig. 16(c), we see that $\lambda_2 = 2$ for $q = 1$, and the same holds if we choose c_2 instead of c_3. Thus we get J_5 and J_6. For $l = 2$ the extension is possible up to the graphs J_7 and J_8 and $l = 3$ gives rise to J_9 and J_{10}. In all considered cases the obtained graphs are maximal and have $\lambda_2 = 2$. Finally, if $l = 4$, we get the graph J_{11}, which has $\lambda_2 < 2$, but also cannot be extended in any way.

Proposition 5. *Let G be a cactus with the cyclic structure as in Fig.* 12(a). *If it has one black vertex and no white vertex loaded, G is reflexive if and only if it is an induced subgraph of some of the* 36 *(families of) graphs $K_1 - K_{36}$ of* Fig.17.

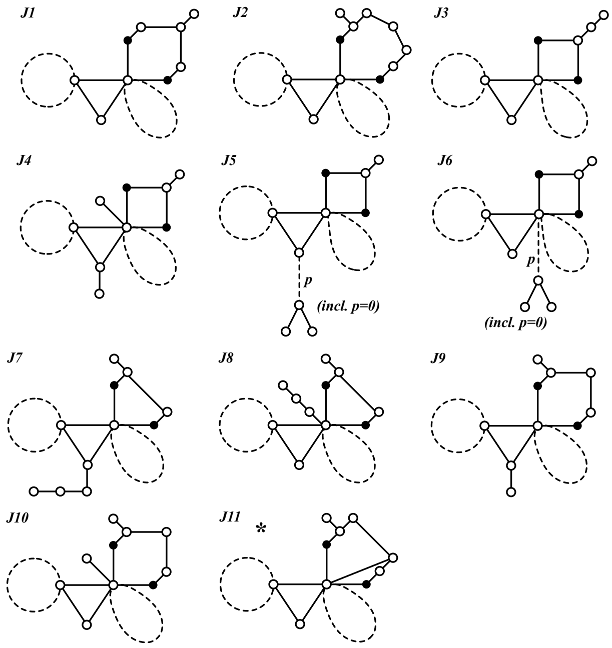


Figure 15.

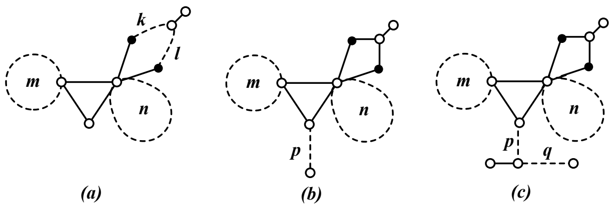
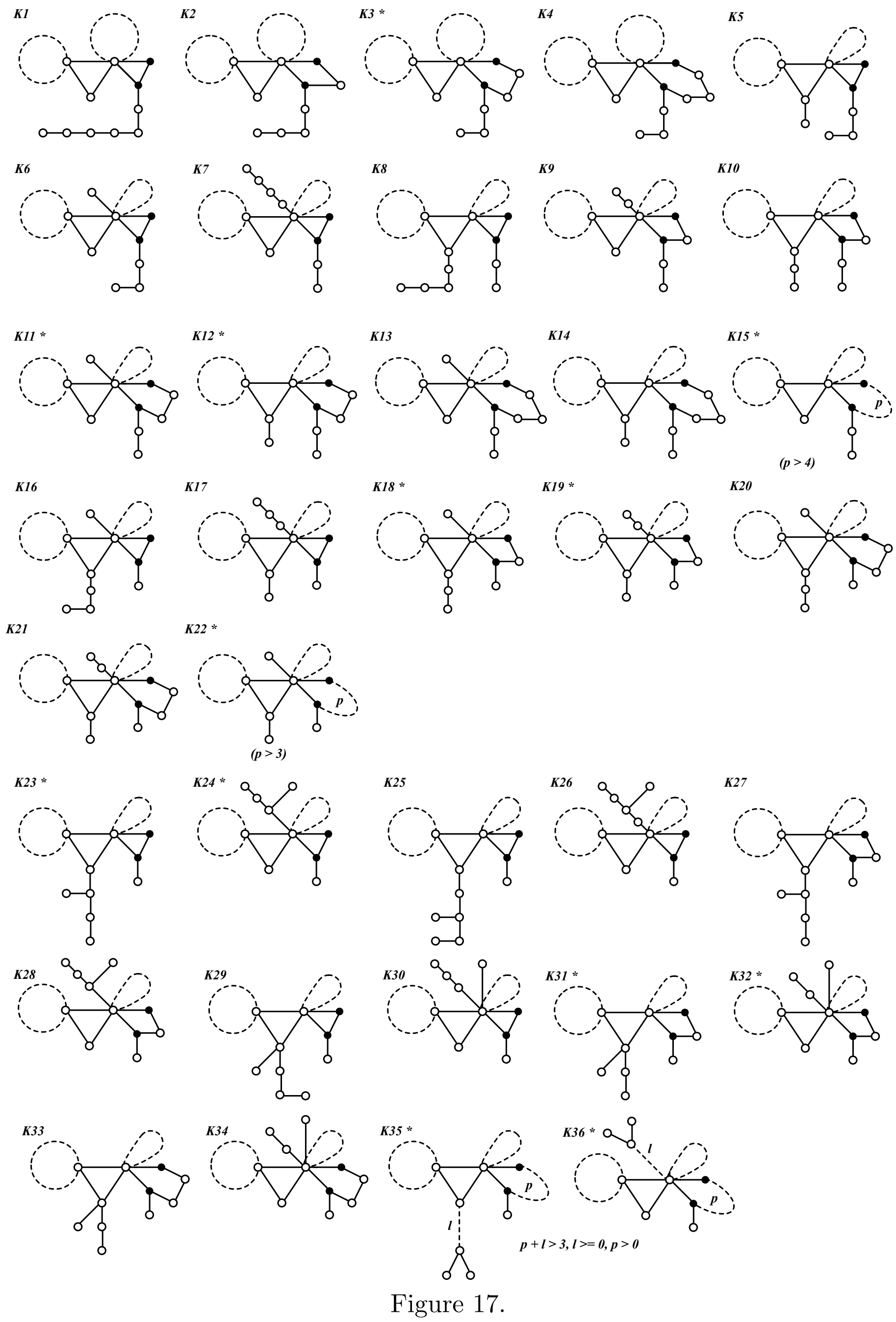


Figure 16.

Proof. Let us start with the fact that, when leaning two graphs G_1 and G_2 on the vertices c_1 and c_2, respectively, of the starting graph of Fig. 12(a), we get the same value $P(2)$ as if we interchange them, i.e. lean G_1 on c_2 and G_2 on c_1. It can easily be proved applying Lemma 1, combined with Corollary 1, to the c-vertices, but the proof needs some more designations and we will omit it.

Suppose now that a black vertex is loaded by a path of length k (Fig. 18(a)). Since $P(2) = mn\big(p\,(k-2)-4\big)$, the maximum possible value is $k = 6$ (if $p = 1$), and the following possibilities arise: $p = 1$ if $k = 5$ or 6, $p \leq 2$ if $k = 4$, $p \leq 4$ if $k = 3$ and for $k \leq 2$ p is not bounded. On the other hand we already proved that the loaded black vertex cannot be of degree greater than 3, and in the similar way one can prove that the path cannot spread at any vertex. Taking care of these facts and other possibilities of extension (c_2 and c_3), for $k \leq 3$ we come to the graphs $K_1 - K_6$. If $k = 2$, look at the graph at Fig. 18(b) having paths leaned on c_2 and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K24 *
K25
K26
K27
K28
K29
K30
K31 *
K32 *
K33
K34
K35 *
p
l
K36 *
l
p
p + l > 3, l >= 0, p > 0

Figure 17.

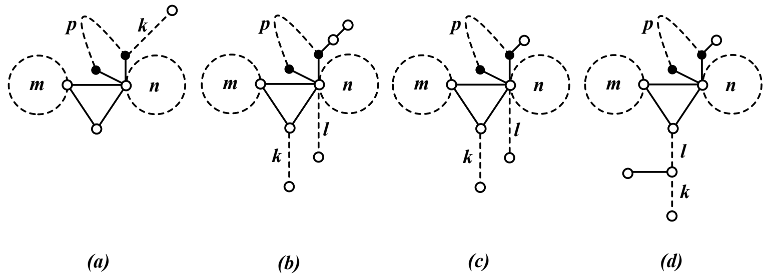

Figure 18.

c_3. Here we obtain $P(2) = mn\left(p\,(2kl + k + l) + 4\,(kl - 1)\right)$ and see that, since $p \geq 1$, one number of the pair (k, l) must be 0. Having checked out all possibilities of extension (spreading of paths), we get graphs $K_7 - K_{14}$ for $p = 1, 2, 3, 4$, while for $p > 4$ $(k, l) = (0, 0)$ and the maximal graph is K_{15} $(\lambda_2 < 2)$.

If a black vertex is loaded only by a single pendent edge, let us consider the graph of Fig. 18(c). Since now $P(2) = mn\left(p(kl - 1) + 2kl - k - l - 4\right)$, we see that $k, l \neq 0$ is possible (if $\min(k, l) = 0$, the extension would give rise to graphs we already obtained or to those to be obtained in the last part of the proof). The possible combinations of p, k and l produce the graphs $K_{16} - K_{22}$; of course, if $k = l = 1$, p is not bounded (K_{22}).

Finally, let us inspect the graph of Fig. 18(d) and suppose $l \geq 1$. Now $P(2) = mn\left((p + lt)(k - 1) + k - 5\right)$, and if $k = 1$, $P(2) < 0$ for any p. If $p = 1$, $(k, l) = (2, 1)$ (the graphs K_{23} and K_{24}) or $(k, l) = (2, 2)$ (K_{25} and K_{26}). If $p = 2$, $(k, l) = (2, 1)$ (the graphs K_{27} and K_{28}), while $l = 2$ gives only $k = 1$. Thus, if $p + l \geq 4$, we get $k = 1$ and the graphs K_{35} and K_{36}. Now, if we suppose $l = 0$ (c_3 is of degree 4,) in the similar manner we find the graphs $K_{29} - K_{34}$, and again K_{35} and K_{36} for $p \geq 4$.

In further investigation the starting graph will be the graph of Fig. 12(b).

Proposition 6. *Let G be a cactus with the same cyclic structure as that of Fig.* 12(b). *If at least two vertices of its outer cycles different from c-vertices are loaded, G is reflexive if and only if it is an induced subgraph of some of the* 12 *(families of) graphs $L_1 - L_{12}$ displayed at Fig.* 19.

Proof. Now every c-vertex has its incident outer cycle, and pairwise bounds are defined by the result of the Proposition 4.2 of [**11**] (maximal graphs $A_1 - A_{14}$ of Fig. 20) if both outer cycles have loaded vertices, and also by Propositions 4.3 and 4.4 of [**11**] (maximal graphs $B_1 - B_{11}$ of Fig. 11 and $C_1 - C_{41}$ if one outer cycle is without loaded vertices, in which case, of course, the result does not depend on its length).

Three white loaded vertices have the bound defined by A_1 and give rise to a maximal graph with four cycles, denoted by L_1. If we have two white vertices and one black vertex loaded, the corresponding bounds are determined by A_1 and A_{12}, and, indeed, generate the maximal reflexive graph L_2. The case when two

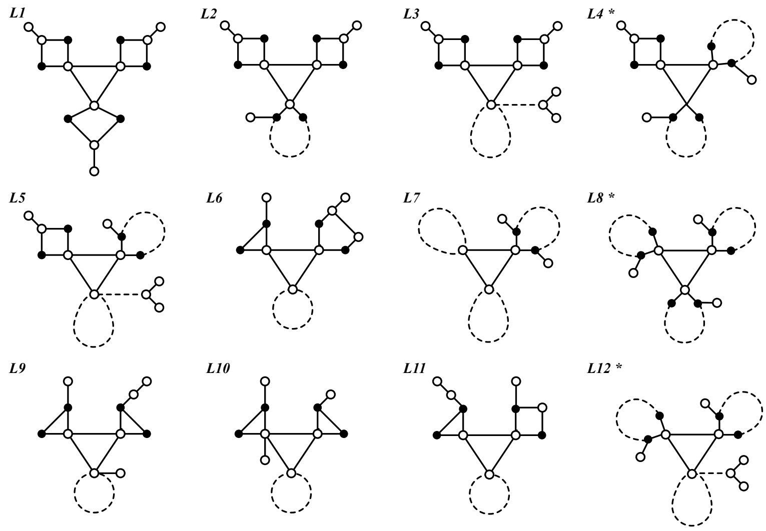

Figure 19.

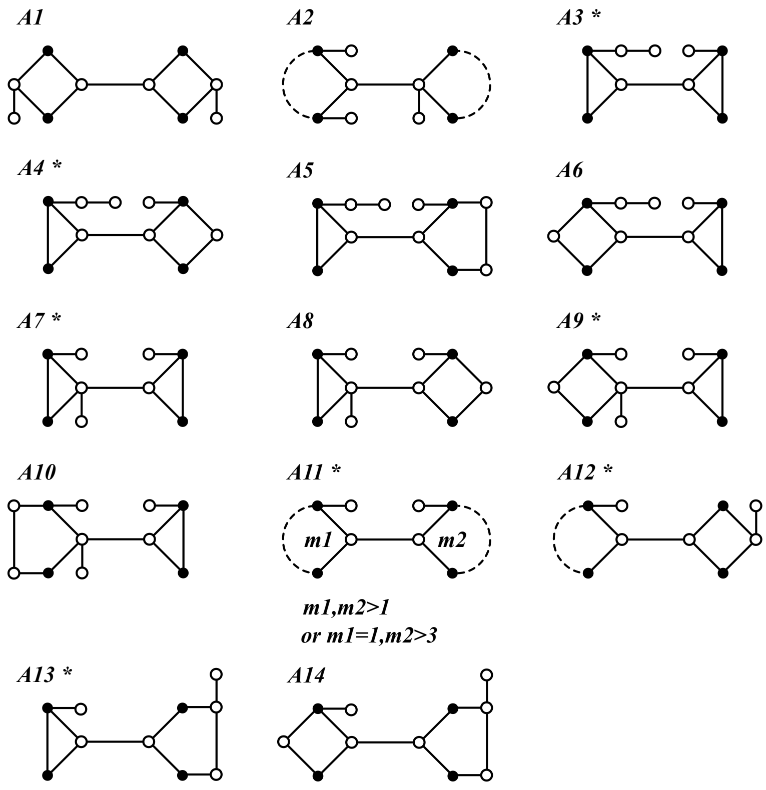

Figure 20.

white vertices and no black ones are loaded is bounded by A_1 and B_5 and again the upper bound is at the same time the resulting maximal graph L_3. If there are two black loaded vertices and only one white, we have some more possibilities: the relation white-black is defined by $A_{12} - A_{14}$, and black-black by $A_3 - A_{11}$. But only the combination A_{11} with A_{12} gives a maximal graph (L_4), and all the rest is not possible. If we have one white and one black vertex loaded, i.e. one outer cycle has no loaded vertices (except the c-vertex), the graph A_{12}, combined with the bound B_5 for the white vertex and C_{16} for the black one, gives L_5, while A_{13} generates only L_6 and A_{14} gives nothing.

Now, consider the case when two loaded black vertices are on the same cycle. The graph A_2 shows that we can have at most one pendent edge at each of the remaining two c-vertices. Such a graph (the upper bound) has $P(2) > 0$ and we can easily make sure (e.g. by Corollary 1) that after removing one pendent edge we also do not have a reflexive graph. Also, leaning two paths of lengths k and l on c-vertices (instead of single edges) we find $P(2) = 0$ only for $k = l = 1$, which proves that the only maximal graph whose two black loaded vertices are on the same cycle is L_7.

Suppose now that three black vertices are loaded (one on each cycle). The case generated by A_{11} gives rise to L_8. All other bounds ($A_3 - A_{10}$) combined with each other, give $P(2) > 2$, and one can easily verify that actually no c-vertex can be loaded by a new pendent edge and no black vertex can be loaded by more than a single pendent edge.

Finally, let us consider the case of only two loaded black vertices. The case of two triangles, bounded by A_3 and A_7 (and also by the corresponding graphs of the set $C_1 - C_{41}$ of Proposition 4.4) gives rise to L_9 and L_{10}, while A_4, A_6 and A_8 (quadrangle - triangle), after all necessary verifications, generate only L_{11} and the case pentagon - triangle (A_5, A_{10}) gives nothing. The bound defined by A_{11}, combined with C_{16}, gives rise to L_{12} and by applying of appropriate Lemmas one can easily get evidence that L_{12} is a maximal reflexive graph for arbitrary lengths of its outer cycles.

Proposition 7. *Let G be a cactus with the same cyclic structure as that of Fig.* 12(b). *If exactly one white vertex and no black vertex of its outer cycles is loaded, G is reflexive if and only if it is an induced subgraph of some of the* 12 *(families of) graphs $M_1 - M_{12}$ depicted at Fig.* 21.

Proof. Let us start with the graph of Fig. 22(a), which, besides a white vertex of degree 3, has all c-vertices loaded, each by an additional pendent edge. But since $P(2) = mn\,(5kl + k + l - 3) > 0$ for $k, l \geq 1$, we see that at most two c-vertices can be loaded with new edges. Also, if we assume that a white vertex is loaded by the path of length 2 and even if all c-vertices are of degree 4, we get the maximal graph M_1.

Now, let us consider the graph of Fig. 22(b). We find that

$$P(2) = mn\left((p+q)(kl-1) + 2kl - k - l - 4\right).$$

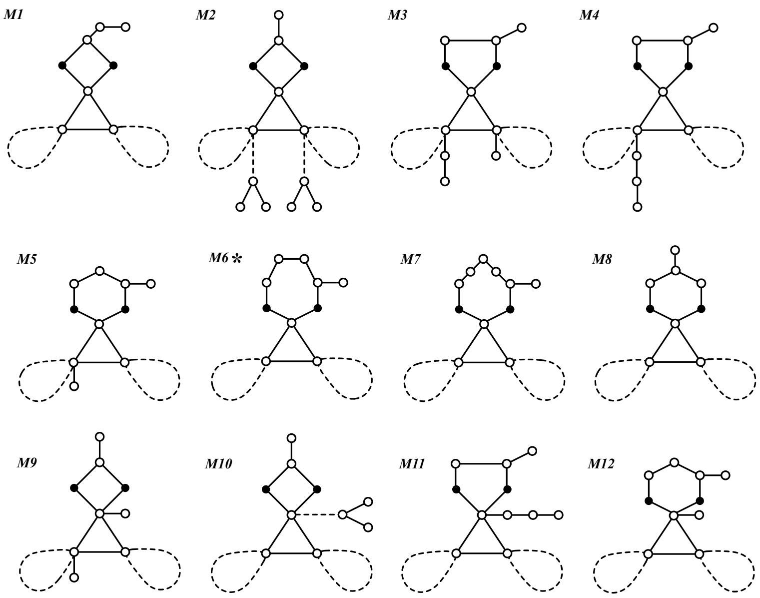

Figure 21.

For $k = l = 1$ the lengths p and q of two added paths are arbitrary, but the graph B_5 of Fig. 11 points to the upper bounds if these two paths are replaced by some trees (which means that the paths cannot spread in any other way), and one can verify that this bound really generates the maximal graph M_2. The rest of the proof in this case includes number analysis of corresponding expressions and checking out that the obtained graphs cannot be extended any more. Thus, for $k = 2$, $l = 1$ we get $P(2) = mn\,(p + q - 3)$, implying $p = 2$, $q = 1$ and $p = 3$, $q = 0$ and giving rise to M_3 and M_4. In the same way, if (k, l) is $(3, 1), (4, 1), (5, 1)$ and $(2, 2)$, we get M_5, M_6, M_7 and M_8 respectively.

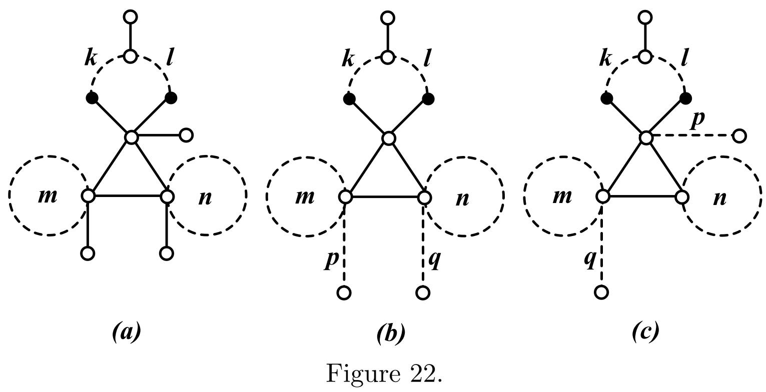

Figure 22.

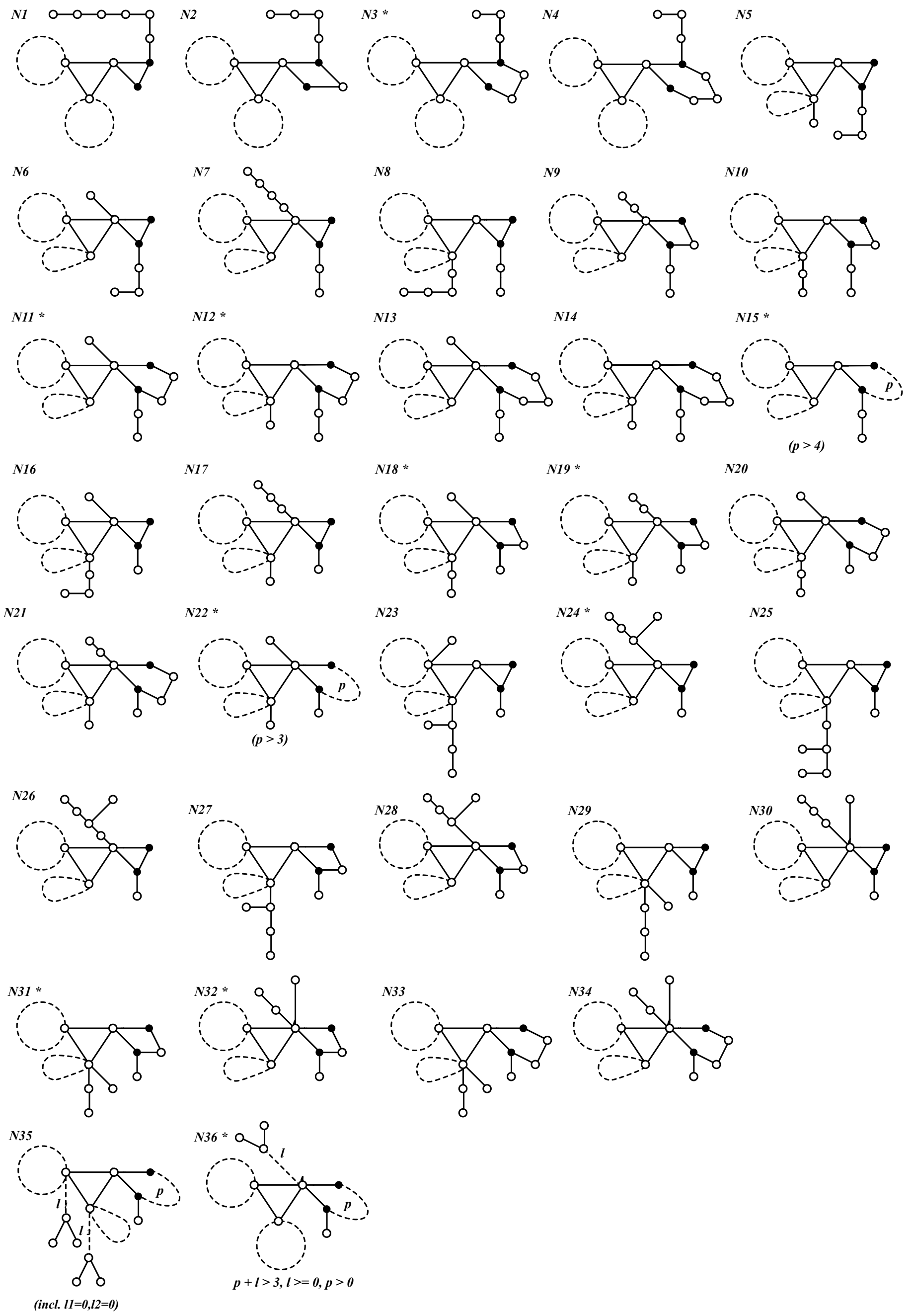
N1
N2
N3 *
N4
N5
N6
N7
N8
N9
N10
N11 *
N12 *
N13
N14
N15 *
p
(p > 4)
N16
N17
N18 *
N19 *
N20
N21
N22 *
p
(p > 3)
N23
N24 *
N25
N26
N27
N28
N29
N30
N31 *
N32 *
N33
N34
N35
p
l
l
(incl. l1=0,l2=0)
N36 *
l
p
p + l > 3, l >= 0, p > 0

Figure 23.

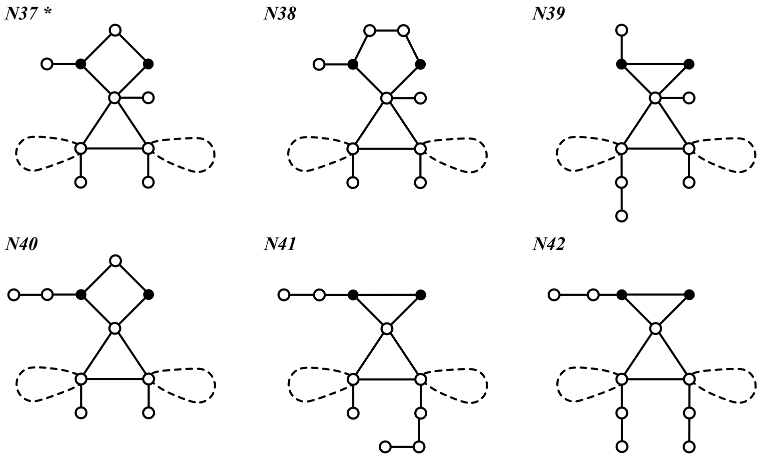

Figure 24.

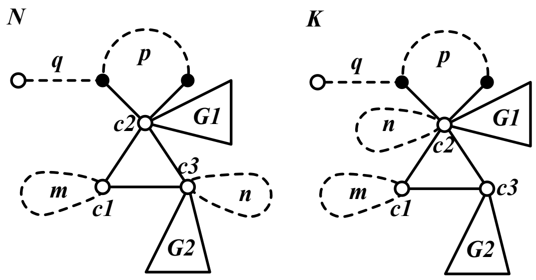

Figure 25.

At last, let us assume the situation as in Fig. 22(c). Now

$$P(2) = mn\left((p+q+2)(kl-k-l-3)+(p+1)(q+1)(k+l+2)\right),$$

where $k = l = 1$ implies $p = q = 1$ and we have M_9. Since $k, l \geq 1$, in all remaining cases $p = 0$ (the graphs we already obtained) or $q = 0$, which for $(k, l) = (1, 1)$ allows arbitrary p and gives rise to M_{10}, for $(2, 1)$ and $(3, 1)$ M_{11} and M_{12}, respectively, and for $(4, 1), (5, 1)$ and $(2, 2)$ we get three graphs which were already obtained.

Proposition 8. *Let G be a cactus with the cyclic structure as in Fig.* 12(b). *If it has exactly one black vertex and no white vertex loaded, G is reflexive if and only if it is an induced subgraph of some of the* 42 *(families of) graphs $N_1 - N_{42}$ of Figures* 23 *and* 24.[4]

Proof. Consider graphs N and K of Fig. 25, where G_1 and G_2 are arbitrary graphs with no cycles at vertices c_2 and c_3. Applying Lemma 2 to K and N, we get the following useful result: $P_K(2) = P_N(2)$.

Since Proposition 5 covers all possibilities for families of the K graphs, the discussion on values of parameters p, q, k, l is valid for corresponding families of the

N graphs, too. This observation and appropriate verifications on the computer give the graphs $N_1 - N_{36}$ of Fig. 23.

Also, we get three more graphs $N_{37}-N_{39}$ by loading all three vertices c_1, c_2, c_3, and graphs $N_{40} - N_{42}$ by loading vertices c_1 and c_3. The proof is similar to that in previous cases.

Thus, we have come to the final result.

Theorem 3. (The main result) *A cactus with four cycles, to which Theorem* 1 *cannot be applied, whose cycles do not form a bundle and which, besides the c-vertices, has at least one vertex of its outer cycles loaded, is reflexive if and only if it is an induced subgraph of some of the (families of) graphs $H_1 - H_{48}$, $I_1 - I_9$, $J_1 - J_{11}$, $K_1 - K_{36}$, $L_1 - L_{12}$, $M_1 - M_{12}$ and $N_1 - N_{42}$ of Propositions 2-8.*

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Faculty of Electrical Engineering,
Bulevar Kralja Aleksandra 73,
Beograd,
Serbia and Montenegro

(Received February 2, 2004)

E–mail: rasajski@etf.bg.ac.yu