UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 14 (2003), 26–36.

SOME CONSIDERATIONS IN CONNECTION WITH KUREPA'S FUNCTION

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In this paper we consider the functional equation for factorial sum and its particular solutions (KUREPA's function $K(z)$ [**3**] and function $K_1(z)$). We determine an extension of domain of functions $K(z)$ and $K_1(z)$ in the sense of CAUCHY's principal value at point [**2**]. In this paper we give an addendum to the proof of SLAVIĆ's representation of KUREPA's function $K(z)$ [**6**]. Also, we consider some representations of functions $K(z)$ and $K_1(z)$ via incomplete gamma function and we consider differential transcendency of previous functions too.

1. THE FUNCTIONAL EQUATION FOR FACTORIAL SUM AND ITS PARTICULAR SOLUTIONS

The main object of consideration in this paper is the functional equation for factorial sum

$$K(z) - K(z-1) = \Gamma(z), \tag{1}$$

with respect to the function $K : D \to \mathbb{C}$ with domain $D \subseteq \mathbb{C}\backslash\mathbb{Z}_0^-$, where Γ is the gamma function, $\mathbb{C}$ is the set of complex numbers and $\mathbb{Z}_0^-$ is the set of non-positive integer numbers. A solution of functional equation (1) over the set of natural numbers ($D = \mathbb{N}$) is the function of left factorial $!n$. ĐURO KUREPA introduced this function, in the paper [**3**], as sum of factorials $!n = 0! + 1! + 2! + \cdots + (n-1)!$. Let us use the standard notation

$$K(n) = \sum_{i=0}^{n-1} i!. \tag{2}$$

Sum (2) corresponds to the sequence $A003422$ in [**10**]. We call the functional equation (1) *the functional equation for factorial sum.* In consideration which follows we consider two particular solutions of the functional equation (1).

2000 Mathematics Subject Classification: 30E20, 11J91.
Research partially supported by the MNTRS, Serbia & Montenegro, Grant No. 1861.

1.1. The function $K(z)$. An analytical extension of the function (2) over the set of complex numbers is determined by integral [**5**]:

$$K(z) = \int_0^\infty e^{-t}\,\frac{t^z-1}{t-1}\,\mathrm{d}t, \tag{3}$$

which converges for $\operatorname{Re} z > 0$. For the function $K(z)$ we use the term Kurepa's function and it is a solution of the functional equation (1). Let us observe that since $K(z-1) = K(z) - \Gamma(z)$, it is possible to make analytical continuation of Kurepa's function $K(z)$ for $\operatorname{Re} z \le 0$. In that way, the Kurepa's function $K(z)$ is a meromorphic function with simple poles at $z=-1$ and $z=-n$ $(n \ge 3)$. At point $z=-2$ Kurepa's function has a removable singularity and $K(-2) \stackrel{\text{def}}{=} \lim_{z\to-2} K(z) = 1$. Kurepa's function has an essential singularity at point $z=\infty$. Kurepa's function has the following residues:

$$\begin{aligned} \operatorname*{res}_{z=-1} K(z) &= -1,\\ \operatorname*{res}_{z=-n} K(z) &= \sum_{k=2}^{n-1} \frac{(-1)^{k-1}}{k!} \qquad (n \ge 3). \end{aligned} \tag{4}$$

Previous results for Kurepa's function are presented according to [**5**] and [**6**].

1.2. The function $K_1(z)$. The functional equation (1), besides Kurepa's function $K(z)$, has another solution which is given by the following statement.

Theorem 1.1. *Let $D = \mathbb{C}\backslash\mathbb{Z}$. Then, series*

$$K_1(z) = \sum_{n=0}^{\infty} \Gamma(z-n) \tag{5}$$

absolutely converges and it is a solution of the functional equation (1) *over D.*

Proof. Set D has the following decomposition in two disjoint sets:

$$D_1 = \{z \in D \mid \operatorname{Re} z \notin \mathbb{Z}\} \tag{6}$$

and

$$D_2 = \{z \in D \mid \operatorname{Re} z \in \mathbb{Z} \;\wedge\; \operatorname{Im} z \neq 0\}. \tag{7}$$

We prove the statement by discussing the following two cases.

1°. Let $z = x+iy \in D_1$. Let us denote $m = [x]$. For each $n \in \mathbb{N}_0$, such that $n \ge m+2$, it is true that

$$|\Gamma(z-n)| \le |\Gamma(x-n)| = \frac{\pi}{\big|\sin(\pi x)\,\Gamma\big(n+1-x\big)\big|} < \frac{\pi}{|\sin(\pi x)|}\cdot\frac{1}{\Gamma(n-m)}. \tag{8}$$

Thus, for $m_1 = \max\{m+2, 0\}$, the following is true

$$\sum_{n=m_1}^{+\infty} |\Gamma(z-n)| < \frac{\pi}{|\sin(\pi x)|} \cdot \sum_{n=m_1}^{\infty} \frac{1}{\Gamma(n-m)}, \tag{9}$$

which is sufficient to conclude that the statement is true over D_1.

2°. Let $z = x + iy \in D_2$. Let us denote by $m = \operatorname{Re} z$. For each $n \in \mathbb{N}_0$, such that $n \geq m+1$, it is true that

$$\begin{aligned} |\Gamma(z-n)| &= |\Gamma(iy-(n-m))| = \frac{\big|\Gamma(iy)\big|}{\Big|\prod\limits_{r=1}^{n-m}(-r+iy)\Big|} = \frac{\sqrt{\dfrac{\pi}{y\,\mathrm{sh}(\pi y)}}}{\prod\limits_{r=1}^{n-m}|-r+iy|} \\ &= \sqrt{\frac{\pi}{y\,\mathrm{sh}(\pi y)}} \cdot \frac{1}{\prod\limits_{r=1}^{n-m}\sqrt{r^2+y^2}} < \sqrt{\frac{\pi}{y\,\mathrm{sh}(\pi y)}} \cdot \frac{1}{\Gamma(n-m+1)}. \end{aligned} \tag{10}$$

Thus, for $m_2 = \max\{m+1, 0\}$, the following is true

$$\sum_{n=m_2}^{+\infty} |\Gamma(z-n)| < \sqrt{\frac{\pi}{y\,\mathrm{sh}(\pi y)}} \cdot \sum_{n=m_2}^{+\infty} \frac{1}{\Gamma(n-m+1)}, \tag{11}$$

which is sufficient to conclude that the statement is true over $D_2 = D \backslash D_1$. Let us note that it is easy to prove that the function $K_1(z)$ is a solution of the functional equation (1). $\square$

Remark. Function $K_1(z)$, defined by (5) over $\mathbb{C}$, has poles at integer points $z = m \in \mathbb{Z}$.

2. EXTENDING THE DOMAIN OF FUNCTIONS $K(z)$ AND $K_1(z)$ IN THE SENSE OF CAUCHY'S PRINCIPAL VALUE

Let us observe a possibility of extending the domain of the functions $K(z)$ and $K_1(z)$, in the sense of Cauchy's principal value, over the set of complex numbers. Namely, for a meromorphic function $f(z)$, on the basis of Cauchy's integral formula, we define *the principal value at point* a as follows [**2**]:

$$\underset{z=a}{\mathrm{p.v.}}\, f(z) = \lim_{\rho \to 0_+} \frac{1}{2\pi i} \oint\limits_{|z-a|=\rho} \frac{f(z)}{z-a}\,\mathrm{d}z. \tag{12}$$

It is easily proved that the principal value at pole $z = a$ exists as a finite complex number. For two meromorphic functions $f_1(z)$ and $f_2(z)$ additivity is true [**2**]:

$$\underset{z=a}{\mathrm{p.v.}} \Big(f_1(z) + f_2(z)\Big) = \underset{z=a}{\mathrm{p.v.}}\, f_1(z) + \underset{z=a}{\mathrm{p.v.}}\, f_2(z). \tag{13}$$

The main formula in this section is given by the following statement:

Lema 2.1. *For function $f(z)$ with simple pole at point $z = a$ the following is true*

$$\underset{z=a}{\text{p.v.}}\, f(z) = \lim_{\varepsilon\to 0} \frac{f(a-\varepsilon)+f(a+\varepsilon)}{2}. \tag{14}$$

Proof. Let $z = a$ be a simple pole of the function $f(z)$. Then, there exist $\rho > 0$ such that function $f(z)$ has the following representation: $f(z) = \dfrac{g(z)}{z-a}$, for some regular function $g(z)$ over $|z-a| < \rho$. Hence, using Cauchy's integral formula

$$\underset{z=a}{\text{p.v.}}\, f(z) = \lim_{\rho\to 0} \frac{1}{2\pi i} \oint\limits_{|z-a|=\rho} \frac{f(z)}{z-a}\,\mathrm{d}z = \lim_{\rho\to 0} \frac{1}{2\pi i} \oint\limits_{|z-a|=\rho} \frac{g(z)}{(z-a)^2}\,\mathrm{d}z = g'(a). \tag{15}$$

On the other hand

$$\lim_{\varepsilon\to 0} \frac{f(a-\varepsilon)+f(a+\varepsilon)}{2} = \lim_{\varepsilon\to 0} \frac{g(a+\varepsilon)-g(a-\varepsilon)}{2\varepsilon} = g'(a). \tag{16}$$

Previous two equalities are sufficient that we conclude that (14) is true. □

Corollary 2.2. *For gamma function $\Gamma(z)$ it is true*

$$\underset{z=-n}{\text{p.v.}}\, \Gamma(z) = \lim_{\varepsilon\to 0} \frac{\Gamma(-n-\varepsilon)+\Gamma(-n+\varepsilon)}{2} = (-1)^n \frac{\Gamma'(n+1)}{\Gamma(n+1)^2} \quad (n\in\mathbb{N}_0). \tag{17}$$

Proof. On the basis of equality $\Gamma(-z) = \dfrac{-\pi}{\Gamma(z+1)\sin\pi z}$, $(z \notin \mathbb{Z})$, for real $\varepsilon > 0$, is true

$$\frac{\Gamma(-n-\varepsilon)+\Gamma(-n+\varepsilon)}{2} = (-1)^{n-1} \frac{\pi\varepsilon}{\sin\pi\varepsilon} \cdot \frac{\dfrac{1}{\Gamma(n+1+\varepsilon)} - \dfrac{1}{\Gamma(n+1-\varepsilon)}}{2\varepsilon}. \tag{18}$$

Hence, using real limit $\varepsilon \to 0_+$, we obtain result (17) from [**2**] in an easier way. □

Remark 2.3. For $n \in \mathbb{N}_0$ it is true [**2**]:

$$\frac{\Gamma'(n+1)}{\Gamma(n+1)^2} = \frac{-\gamma + 1 + \frac{1}{2} + \ldots + \frac{1}{n}}{n!}, \tag{19}$$

where γ is Euler's constant.

Extension of the domain of the functions $K(z)$ and $K_1(z)$, in the sense of Cauchy's principal value, is given by the following two theorems.

Theorem 2.4. *For Kurepa's function $K(z)$ it is true*

$$\underset{z=-n}{\text{p.v.}}\, K(z) = -\sum_{i=0}^{n-1} \underset{z=-i}{\text{p.v.}}\, \Gamma(z) = \sum_{i=0}^{n-1} (-1)^{i+1} \frac{\Gamma'(i+1)}{\Gamma(i+1)^2} \quad (n\in\mathbb{N}). \tag{20}$$

Proof. If equality

$$K(z) = K(z+n) - \big(\Gamma(z+1) + \cdots + \Gamma(z+n)\big), \tag{21}$$

we consider at the point $z = -n$ in the sense of CAUCHY's principal value, according to (17), the equality (20) follows. Let us remark that $\mathop{\text{p.v.}}\limits_{z=-2} K(z) = K(-2) = 1$. □

Lemma 2.5. *Let's define* $L_1 = -\sum\limits_{n=0}^{+\infty} \mathop{\text{p.v.}}\limits_{z=-n} \Gamma(z)$, *then*

$$L_1 = \sum_{n=0}^{+\infty} (-1)^{n+1} \frac{\Gamma'(n+1)}{\Gamma(n+1)^2} \approx 0.697\,174\,883\,. \tag{22}$$

Proof. The previous series converges on the basis of the remark 2.3.

Theorem 2.6. *For function* $K_1(z)$ *it is true*

$$\mathop{\text{p.v.}}\limits_{z=n} K_1(z) = \mathop{\text{p.v.}}\limits_{z=n} K(z) - L_1 \quad (n \in \mathbb{Z}). \tag{23}$$

Proof. For $n \geq 0$ it is true

$$\mathop{\text{p.v.}}\limits_{z=n} K_1(z) = \sum_{i=0}^{+\infty} \mathop{\text{p.v.}}\limits_{z=n-i} \Gamma(z) = \sum_{i=0}^{+\infty} \mathop{\text{p.v.}}\limits_{z=-i} \Gamma(z) + \sum_{i=1}^{n} \Gamma(i) = K(n) - L_1. \tag{24}$$

For $n < 0$ it is true

$$\begin{aligned} \mathop{\text{p.v.}}\limits_{z=n} K_1(z) &= \sum_{i=0}^{+\infty} \mathop{\text{p.v.}}\limits_{z=n-i} \Gamma(z) = \sum_{i=0}^{+\infty} \mathop{\text{p.v.}}\limits_{z=-i} \Gamma(z) - \sum_{i=0}^{(-n)-1} \mathop{\text{p.v.}}\limits_{z=-i} \Gamma(z) \\ &= \Bigg(- \sum_{i=0}^{(-n)-1} \mathop{\text{p.v.}}\limits_{z=-i} \Gamma(z) \Bigg) - L_1 = \mathop{\text{p.v.}}\limits_{z=n} K(z) - L_1. \end{aligned} \tag{25}$$

□

3. SLAVIĆ'S FORMULA FOR KUREPA'S FUNCTION

In this section we give an addendum to the proof of SLAVIĆ's representation of KUREPA's function $K(z)$ [**6**], by the following two statements:

Lemma 3.1 *Function*

$$F(z) = \sum_{n=1}^{+\infty} \Bigg(\sum_{k=1}^{+\infty} \frac{(-1)^{n+k-1}(n+k+1)}{(n+k)!} \, z^k \Bigg), \tag{26}$$

is entire, whereas following is true

$$F(z) = \sum_{k=1}^{+\infty} \Bigg(\sum_{n=1}^{+\infty} \frac{(-1)^{n+k-1}(n+k+1)}{(n+k)!} \, z^k \Bigg) = e^{-z} - 1. \tag{27}$$

Proof. For $z=0$ the equality (27) is true. Let us introduce a sequence of functions

$$f_n(z)=\sum_{k=1}^{+\infty}\frac{(-1)^{n+k-1}(n+k+1)}{(n+k)!}\,z^k, \tag{28}$$

for $z\in\mathbb{C}$ $(n\in\mathbb{N})$. Previous series converge over $\mathbb{C}$ because, for $z\neq 0$, it is true that

$$f_n(z)=\sum_{j=0}^{n}(-1)^j\left(\frac{j}{j!}+\frac{1}{j!}\right)z^{j-n}+e^{-z}(z^{-n+1}-z^{-n}). \tag{29}$$

Let us mention that the previous equality is easily checked by the following substitution $e^{-z}=\sum_{k=0}^{+\infty}\frac{(-z)^k}{k!}$ at the right side of equality of formula (29). Let $\rho>0$ be fixed. Over the set $D=\{z\in\mathbb{C}\,|\,0<|z|<\rho\}$ let us an auxiliary function $g(z)=(z-1)e^{-z}:D\to\mathbb{C}$. If we denote by $R_n(.)$ the remainder of n-th order of Maclaurin's expansion, then for $z\in D$ the following representation is true

$$f_n(z)=\frac{R_n\big(g(z)\big)}{z^n}\,. \tag{30}$$

Over $E=(0,\rho)$ let us form an auxiliary function $h(t)=(t+1)e^t:E\to\mathbb{R}^+$. Then, for $|z|<\rho$ it is true

$$|f_n(z)|\le\frac{e^{\rho}(n+2+\rho)}{n!}\,\rho. \tag{31}$$

Indeed, previous inequality is true for $z=0$. For $z\in D$ and $t=|z|\in E$ exists $c\in(0,t)$ such that

$$|f_n(z)|=\left|\frac{R_n(g(z))}{z^n}\right|\le\frac{R_n(h(t))}{t^n}=\frac{h^{(n+1)}(c)}{(n+1)!}\,t\le\frac{e^{\rho}(n+2+\rho)}{n!}\,\rho. \tag{32}$$

For function

$$F(z)=\sum_{n=1}^{+\infty}f_n(z), \tag{33}$$

it is possible, for $|z|<\rho$, to apply Weierstrass's double series theorem [**8**] (page 83). Indeed, on the basis of (28), the functions $f_n(z)$ are regular for $|z|<\rho$. On the basis of (31), the series $\sum_{n=1}^{+\infty}f_n(z)$ is uniformly convergent for $|z|\le r<\rho$, for every $r<\rho$. Then on the basis of the Weierstrass's theorem, for $|z|<\rho$, the following is true

$$F(z)=\sum_{k=1}^{+\infty}\left(\sum_{n=1}^{+\infty}\frac{(-1)^{n+k-1}(n+k+1)}{(n+k)!}\,z^k\right)=e^{-z}-1, \tag{34}$$

because

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+k-1}(n+k+1)}{(n+k)!} = \frac{(-1)^k}{k!}. \tag{35}$$

The previous series is a telescoping series. Let us note that $\rho > 0$ can be arbitrarily large positive number. Hence, the equality (27) is true for all $z \in \mathbb{C}$; i.e. the function $F(z)$ is entire. $\square$

REMARK 3.2. For $|z| > 1$ the following equalities are true:

$$\sum_{n=1}^{+\infty} e^{-z}(z^{-n+1} - z^{-n}) = e^{-z} \tag{36}$$

and

$$\sum_{n=1}^{+\infty} \Big(\sum_{j=0}^{n} (-1)^j \frac{j}{j!} z^{j-n} \Big) = -1 - \sum_{n=1}^{+\infty} \Big(\sum_{j=0}^{n} (-1)^j \frac{1}{j!} z^{j-n} \Big). \tag{37}$$

Hence for $|z| > 1$, on the basis of (29), it also follows that (27) is true.

Lemma 3.3. *For $z \in \mathbb{C}$ it is true*

$$(z-1) \sum_{n=1}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^{k+n}}{(k+n)!} z^k = e^{-z} - e^{-1}. \tag{38}$$

Proof. On the basis of the lemma 3.1. it is true that

$$\begin{aligned}
(z-1) \sum_{n=1}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^{k+n}}{(k+n)!} z^k &= \sum_{n=1}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^{k+n}}{(k+n)!} (z^{k+1} - z^k) \\
&= \sum_{n=1}^{+\infty} \Big(-\frac{(-1)^n}{n!} + \sum_{k=1}^{+\infty} \Big(\frac{(-1)^{k+n-1}}{(k+n-1)!} - \frac{(-1)^{k+n}}{(k+n)!} \Big) z^k \Big) \\
&\underset{(27)}{=} \sum_{k=1}^{+\infty} \Big(\sum_{n=1}^{+\infty} \frac{(-1)^{n+k-1}(n+k+1)}{(n+k)!} \Big) z^k - \sum_{n=1}^{+\infty} \frac{(-1)^n}{n!} \\
&\underset{(27)}{=} e^{-z} - e^{-1}.
\end{aligned} \tag{39}$$

$\square$

On the basis of the previous lemmas we give an addendum to the proof of SLAVIĆ's formula for KUREPA's [**6**]:

Theorem 3.4. *For Kurepa's function $K(z)$ the following representation is true*

$$K(z) = \frac{\mathrm{Ei}(1)}{e} - \frac{\pi}{e} \operatorname{ctg} \pi z + \sum_{n=0}^{+\infty} \Gamma(z-n), \tag{40}$$

where the values in the previous formula, in integer points z, are determined in the sense of Cauchy's principal value.

Proof. Some parts of this proof are reproduced according to [**6**]. For $-(n+1) < \operatorname{Re} z < -n$ and $n = 0, 1, 2, \ldots$ the following formula is true [**1**]:

$$\Gamma(z) = \int\limits_0^{+\infty} \left(e^{-t} - \sum_{m=0}^{n} \frac{(-t)^m}{m!} \right) t^{z-1} \,\mathrm{d}t. \tag{41}$$

Hence, for $0 < \operatorname{Re} z < 1$ and $n = 1, 2, \ldots$ the following formula is true

$$\Gamma(z-n) = \int\limits_0^{+\infty} \left(e^{-t} - \sum_{m=0}^{n-1} \frac{(-t)^m}{m!} \right) t^{z-n-1} \,\mathrm{d}t. \tag{42}$$

Further we observe the following difference

$$\begin{aligned} K(z) - \sum_{n=0}^{+\infty} \Gamma(z-n) = & \int\limits_0^{+\infty} e^{-t}\, \frac{t^z - 1}{t-1} \,\mathrm{d}t - \int\limits_0^{+\infty} e^{-t} t^{z-1} \,\mathrm{d}t \\ & - \sum_{n=1}^{\infty} \int\limits_0^{+\infty} \left(e^{-t} - \sum_{m=0}^{n-1} \frac{(-t)^m}{m!} \right) t^{z-n-1} \,\mathrm{d}t. \end{aligned} \tag{43}$$

In this part of the proof we give an addendum using lemma 3.3. For $0 < \operatorname{Re} z < 1$ the following derivation is true

$$\begin{aligned} K(z) - \sum_{n=0}^{+\infty} \Gamma(z-n) &= \int\limits_0^{+\infty} e^{-t}\, \frac{t^{z-1} - 1}{t-1} \,\mathrm{d}t - \int\limits_0^{+\infty} \sum_{n=1}^{+\infty} \left(e^{-t} - \sum_{m=0}^{n-1} \frac{(-t)^m}{m!} \right) t^{z-n-1} \,\mathrm{d}t \\ &= \int\limits_0^{+\infty} \left(\frac{e^{-t}}{1-t} + e^{-t}\, \frac{t^{z-1}}{t-1} - \sum_{n=1}^{+\infty} \sum_{m=n}^{+\infty} \frac{(-t)^m}{m!}\, t^{z-n-1} \right) \mathrm{d}t \\ &= \int\limits_0^{+\infty} \left(\frac{e^{-t}}{1-t} + \frac{t^{z-1}}{t-1} \left(e^{-t} - (t-1) \sum_{n=1}^{+\infty} \sum_{m=n}^{+\infty} \frac{(-1)^m}{m!}\, t^{m-n} \right) \right) \mathrm{d}t \\ &= \int\limits_0^{+\infty} \left(\frac{e^{-t}}{1-t} + \frac{t^{z-1}}{t-1} \left(e^{-t} - (t-1) \sum_{n=1}^{+\infty} \sum_{k=0}^{+\infty} \frac{(-1)^{k+n}}{(k+n)!}\, t^{k} \right) \right) \mathrm{d}t \\ &\underset{(38)}{=} \int\limits_0^{+\infty} \left(\frac{e^{-t}}{1-t} + \frac{1}{e}\, \frac{t^{z-1}}{t-1} \right) \mathrm{d}t, \end{aligned} \tag{44}$$

Integral at the right side of equality (44), which converges in ordinary sense, will be substituted by two integrals which converge in the sense of the Cauchy's principal value. Namely, using the function of exponential integral [**1**]:

$$\operatorname{Ei}(x) = \text{p.v.} \int\limits_{-\infty}^{x} \frac{e^t}{t} \,\mathrm{d}t, \tag{45}$$

and using the formulas 3.352-6 and 3.311-8 from [**4**]:

$$\text{p.v.} \int\limits_0^{+\infty} \frac{e^{-t}}{1-t}\,\mathrm{d}t = \frac{\mathrm{Ei}\,(1)}{e} \tag{46}$$

and

$$\text{p.v.} \int\limits_0^{+\infty} \frac{t^{z-1}}{1-t}\,\mathrm{d}t = \pi \operatorname{ctg} \pi z, \tag{47}$$

we conclude that formula (40) is true for $0 < \operatorname{Re} z < 1$. According to Riemann's theorem we conclude that Slavić's formula (40) is true for each complex z. Namely, formula (40), in integer points z, is true in the sense of Cauchy's principal value on the basis of the theorem 2.6. □

Corrollary 3.5. *Function $K_1(z)$ is a meromorphic function with simple poles in integer points $z = m$ ($m \in \mathbb{Z}$) and with residue values*

$$\operatorname*{res}_{z=m} K_1(z) = \frac{1}{e} + \operatorname*{res}_{z=m} K(z) \quad (m \in \mathbb{Z}). \tag{48}$$

At the point $z = \infty$ function $K_1(z)$ has an essential singularity.

4. SOME REPRESENTATIONS OF FUNCTIONS $K(z)$ AND $K_1(z)$ VIA INCOMPLETE GAMMA FUNCTION

In this section we give some representations of functions $K(z)$ and $K_1(z)$ via gamma and incomplete gamma functions, where the later ones are defined by integrals

$$\gamma(a,z) = \int\limits_0^z e^{-t} t^{\alpha-1}\,\mathrm{d}t, \quad \Gamma(a,z) = \int\limits_z^\infty e^{-t} t^{\alpha-1}\,\mathrm{d}t. \tag{49}$$

Parameters α and z are complex numbers and t^α takes its principal value. Let us remark that the value $\gamma(\alpha, z)$ exists for $\operatorname{Re} \alpha > 0$ and the value $\Gamma(\alpha, z)$ exists for $|\arg z| < \pi$. Then, we have: $\gamma(a,z) + \Gamma(a,z) = \Gamma(a)$. Analytical continuation can be obtained on the basis of representation of the γ function using series.

Маричев in [**7**] (page 109, formula 7.116) proved that for $\operatorname{Re} z > -1$ the following formula is true

$$\text{p.v.} \int\limits_0^\infty e^{-t} \frac{t^z}{t-1}\,\mathrm{d}t = \frac{\pi}{e} \operatorname{ctg} \pi z - \Gamma(z)\,{}_1\mathrm{F}_1(1, 1-z, -1), \tag{50}$$

where

$$
{}_1\mathrm{F}_1(a,b,z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\,\frac{z^k}{k!} \tag{51}
$$

is KUMMER's hypergeometric function and where $(x)_n = x(x+1)\dots(x+n-1)$, $(x)_0 = 1$ is POCHHAMMER's symbol. By converting KUMMER's hypergeometric function we get the formula

$$
{}_1\mathrm{F}_1(1,1-z,-1)=\frac{(-1)^z}{e}\big(\Gamma(1-z)+z\Gamma(-z,-1)\big). \tag{52}
$$

From (50) and (52) for $\operatorname{Re} z > -1$ the following formula is true

$$
\text{p.v.}\int_0^{\infty} e^{-t}\,\frac{t^z}{t-1}\,\mathrm{d}t=\frac{(-1)^z\Gamma(z+1)\Gamma(-z,-1)}{e}+\frac{i\pi}{e}\,. \tag{53}
$$

Hence, as one consequence we get representations of functions $K(z)$ and $K_1(z)$ via an incomplete gamma function.

Theorem 4.1. *For functions $K(z)$ and $K_1(z)$ the following representations are true*:

$$
K(z)=\frac{\mathrm{Ei}\,(1)+i\pi}{e}+\frac{(-1)^z\Gamma(1+z)\Gamma(-z,-1)}{e} \tag{54}
$$

and

$$
K_1(z)=\frac{(-1)^z\pi}{e\sin\pi z}+\frac{(-1)^z\Gamma(1+z)\Gamma(-z,-1)}{e}, \tag{55}
$$

where the values in the previous formulas, in integer points z, are determined in the sense of Cauchy's principal value.

5. DIFFERENTIAL TRANSCENDENCY OF FUNCTIONS $K(z)$ AND $K_1(z)$

In this section we provide one statement about differential transcendency of some solutions of functional equations (1). Namely, using the method of MIJAJLOVIĆ [**9**] we can conclude that the following statement is true:

Theorem 5.1. *Let $\mathcal{M}$ be a differential field of the meromorphic functions $f=f(z)$ over the connected open set $D \subseteq \mathbb{C}\backslash\mathbb{Z}_0^-$. If $g=g(z)\in\mathcal{M}$ is one solution of a functional equation* (1), *then g is not a solution of any algebraic-differential equation over the field of rational functions $\mathbb{C}(z)$.*

Corollary 5.2. *Especially the functions $K(z)$ and $K_1(z)$ are not solutions of any algebraic-differential equation over the field of rational functions $\mathbb{C}(z)$.*

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(Received September 12, 2003)