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ANTIREGULAR GRAPHS ARE UNIVERSAL FOR TREES

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A graph on n vertices is *antiregular* if its vertex degrees take on $n-1$ different values. For every $n \geq 2$ there is a unique connected antiregular graph on n vertices. Call it A_n. (The unique disconnected antiregular graph on n vertices is A_n^c.) The main result of this note is that every tree on n vertices is isomorphic to a subgraph of A_n.

1. ANTIREGULAR GRAPHS

Let $G = (V, E)$ be a graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Denote by $d_G(v)$ the degree of vertex v, so that $n - 1 \geq d_G(v) \geq 0$. If $d_G(v_1) = d_G(v_2) = \cdots = d_G(v_n)$, then G is *regular.* At the other extreme are graphs whose vertex degrees are as different from each other as possible.

If $n \geq 2$, then vertex v has degree $n-1$ if and only if it is a *dominating vertex*, adjacent to every other vertex, which precludes the existence of an *isolated vertex* of degree 0. Since no graph can have both a dominating vertex and an isolated vertex, some two vertices of G have the same degree. Following [**11**], we say that G is *antiregular* if its vertex degrees attain $n - 1$ different values, and adopt the convention that K_1, the (unique) graph on 1 vertex, is antiregular.

Let $d(G) = (d_1, d_2, \ldots, d_n)$ be the vertex degrees of G arranged in non-increasing order, $d_1 \geq d_2 \geq \cdots \geq d_n$. Because $d(G^c) = (n - 1 - d_n, n - 1 - d_{n-1}, \ldots, n - 1 - d_1)$, G is antiregular if and only if its complement is antiregular. Moreover, G has a dominating vertex if and only if G^c has an isolated vertex. Apart from K_1, antiregular graphs come in natural pairs, one of which is connected and the other of which is not.

Theorem 1. [**1**] *Suppose $n \geq 2$. Then, up to isomorphism, there is a unique connected antiregular graph on n vertices, and its repeated vertex degree is $\lfloor n/2 \rfloor$.*

Proof sketch. The unique connected graph on 2 vertices is the complete graph K_2 having two vertices of degree 1. Let G is a connected antiregular graph on $n \geq 2$

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vertices. Then $d(G) = (n-1,\ n-2, \ldots)$. If $d_G(u) = n-1$ and $d_G(w) = n-2$, then $G - u$ is an antiregular graph on $n-1$ vertices which is connected because w is a dominating vertex. The result follows by induction. □

Definition. *Define by A_n the unique connected antiregular graph on n vertices.*

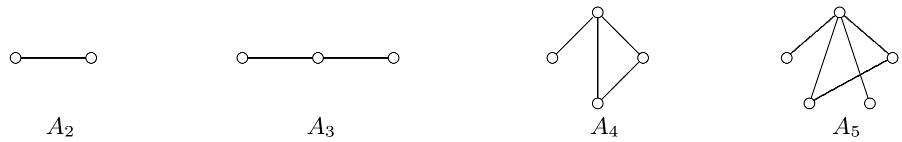

Figure 1

2. UNIVERSAL GRAPHS

Graph G on n vertices is *universal for trees* if every tree on n vertices is isomorphic to a subgraph of G. (See, e.g., [**2**]–[**7**], [**9**]–[**10**], [**12**], and [**14**]–[**15**].)

Theorem 2. *The connected antiregular graph A_n is universal for trees.*

Proof. Recall that a forest is a graph without cycles, i.e., a graph of whose connected components is a tree. We will prove the theorem by showing that every forest on n vertices is isomorphic to a subgraph of A_n.

If $G = (V, E)$ and $H = (W, F)$ are graphs on disjoint sets of vertices V and W, their *union* is $G + H = (V \cup W,\ E \cup F)$. The *join* of G and H is $G \vee H = (G^c + H^c)^c$, the graph obtained from $G + H$ by adding new edges joining each vertex of G to every vertex of H.

Because $A_1 = K_1$ and $A_2 = K_2$, every graph on n vertices is isomorphic to a subgraph of A_n, $n \leq 2$. So, suppose $n \geq 3$. Because $A_n + K_1$ is a disconnected antiregular graph on $n+1$ vertices, it must be the complement of A_{n+1}, i.e.,

$$A_{n+1} = (A_n + K)^c = A_n^c \vee K_1 = (A_{n-1} + K_1) \vee K_1.$$

Let F be a forest on $n+1$ vertices. Suppose u is a pendant (degree 1) vertex of F with unique neighbour v. Then $F' = F - u - v$ is a forest on $n-1$ vertices which, by induction, is isomorphic to a subgraph of A_{n-1}. Because it is isomorphic to a subgraph of the tree $(F' + u) \vee v$, F is isomorphic to a subgraph of $(A_{n-1} + u) \vee v = A_{n+1}$. □

3. CONCLUDING REMARKS

Antiregular graphs have many other interesting properties. They are, for example, *threshold graphs.* (See, e.g., [**13**].) If G is a threshold graph, then both G and G^c are *chordal* [**8**]. Thus, A_n is a *perfect* graph. Its line graph is hamiltonian. Its chromatic and matching numbers are $\chi(A_n) = \lfloor n/2 \rfloor + 1$ and $\mu(A_n) = \lfloor n/2 \rfloor$, respectively. If $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are the eigenvalues of its adjacency matrix, then either $\gamma_r = 0 = \gamma_{n-r+1}$, or they have opposite signs, $1 \leq r \leq n$, i.e., while A_n is not bipartite (for $n \geq 4$) it has *bipartite character.* Finally, the Laplacian eigenvalues of A_n consist of all but one of the integers 0, 1, 2, ..., n. The "missing eigenvalue" is $\lambda = \lfloor (n+1)/2 \rfloor$.

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