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# ON FRACTIONAL DERIVATIVES OF SOME FUNCTIONS OF EXPONENTIAL TYPE

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In this paper a new proof of the well known fact that the derivative of $e^{\lambda x}$ of order $\alpha \in \mathbf{R}$ is equal to $\lambda^\alpha e^{\lambda x}$ is given. It enables to conclude that $\sin^{(\alpha)}(x) = \sin(x + \alpha\pi/2)$ and $\cos^{(\alpha)}(x) = \cos(x + \alpha\pi/2)$ which is initial assumption (axiom) for the classical theory of fractional derivatives. Namely we use a new method for calculation of fractional derivatives of functions of exponential type.

## 1. PRELIMINARIES

Several authors have considered and introduced different methods for calculating of fractional derivatives of a given function (see [**2**]).

In this paper will be considered summation of series, more precisely summation of "divergent" series which helps for calculating of fractional derivatives of functions of exponential type and also will be calculated some fractional derivatives via introduced method. Namely, for a given series $\sum\limits_{i=0}^{+\infty} a_i$, we consider the formal potential series $\sum\limits_{i=0}^{+\infty} a_i x^i$ and look for a differential equations which it satisfies, even if the radius of convergence of the potential series is 0. If $f$ is the solution of the corresponding differential equation, then we take that $f(x) = \sum\limits_{i=0}^{+\infty} a_i x^i$ for each $x$, and we put $\sum\limits_{i=0}^{+\infty} a_i = f(1)$. The method of calculating of the fractional derivatives is the following. If $f$ is developed in the form

$$f(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^i}{i!}\,, \tag{1.1}$$

then for any $\alpha \in \mathbf{R}$, the derivation of order $\alpha$ is defined to be

$$f^{(\alpha)}(x) = \sum_{i=-\infty}^{+\infty} a_i \frac{x^{i-\alpha}}{(i-\alpha)!}\,, \tag{1.2}$$

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where $x! = \Gamma(x+1)$. Note that $(-1)! = (-2)! = \cdots = \pm\infty$ and hence $x^i/i! = 0$ for $i = -1, -2, \ldots$, but these zero summands of $f$ have important role for the derivatives of order $\alpha$ because $x^{i-\alpha}/(i-\alpha)! \neq 0$ if $\alpha$ is non-integer number. The interpretation of the coefficients $a_{-i}$, $i \in \mathbf{Z}^+$ is the following. The coefficient $a_{-1}$ is equal to $g(0)$, where $g'(x) = f(x)$, i.e. it is the integral constant of $\int f\,\mathrm{d}x$. Analogously $a_{-2}$ is equal to the integral constant of $\int(\int f\,\mathrm{d}x)\,\mathrm{d}x$, and so on. Thus the calculation of the fractional derivative of $f$ requires knowledge of all coefficients of integration, but not only the analytical mapping $f : \mathbf{R} \to \mathbf{R}$. Note that if $f$ is given by the right side of (1.2), then its derivatives of any order is defined analogously. It happens very often that the left side of (1.2) diverges, and then we apply the method of summation of the divergent series presented previously.

The paper belongs to the theory of non-standard analysis, but on the other hand it is convenient for obtaining results. This approach has not appeared until now in the literature and it is a subject of our consideration. The analytical functions should be treated as given series but not classically according to the set theory of functions. This new approach can find application in solving the differential equations with fractional derivatives.

Using this new approach in this paper we verify some formulas for fractional derivatives of functions of exponential type.

## 2. FRACTIONAL DERIVATIVE OF $e^{\lambda x}$ AND ITS IMPLICATIONS

First we prove the following lemma.

**Lemma 2.1.** *For all $\alpha > -1$ the following identity*

$$e\Big((-\alpha)! + \alpha\int_0^1 e^{-1/t}t^{\alpha-1}\,\mathrm{d}t\Big) = 1 + \frac{1}{1-\alpha} + \frac{1}{(1-\alpha)(2-\alpha)} + \frac{1}{(1-\alpha)(2-\alpha)(3-\alpha)} + \cdots$$

*is true, where $(-\alpha)! = \Gamma(1-\alpha)$.*

**Proof.** Let $F(z) = \int_z^{+\infty} e^{-t}t^{-\alpha}\,\mathrm{d}t$, where $\alpha > 0$ and $\operatorname{Re} z > 0$. In [**3**], p.288–289, is proved the following identity

$$(1)\qquad F(z) = e^{-z}\Bigg(\frac{1}{z^\alpha} - \frac{\alpha}{z^{\alpha+1}} + \frac{\alpha(\alpha+1)}{z^{\alpha+2}}$$

$$-\cdots + (-1)^n\,\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{z^{\alpha+n}}\Bigg) + (-1)^{n+1}\alpha(\alpha+1)\cdots(\alpha+n)\int_z^{+\infty}\frac{e^{-t}}{t^{\alpha+n+1}}\,\mathrm{d}t.$$

We have:

$$(-\alpha)! = \Gamma(-\alpha+1) = \int_0^{+\infty} e^{-t}t^{-\alpha}\,\mathrm{d}t = \int_0^1 e^{-t}t^{-\alpha}\,\mathrm{d}t + \int_1^{+\infty} e^{-t}t^{-\alpha}\,\mathrm{d}t.$$

For the first integral we obtain

$$\int_0^1 e^{-t}t^{-\alpha}\,\mathrm{d}t = \int_0^1 \left(\sum_{n=0}^{+\infty}(-1)^n\frac{t^{n-\alpha}}{n!}\right)\mathrm{d}t \tag{2}$$

$$= \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\int_0^1 t^{n-\alpha}\,\mathrm{d}t = \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\,\frac{1}{n-\alpha+1}\,.$$

The second integral is equal to $F(1)$, i.e.

$$\int_1^{+\infty} e^{-t}t^{-\alpha}\,\mathrm{d}t = e^{-1}\big(1-\alpha+\alpha(\alpha+1)-\cdots+(-1)^n\alpha(\alpha+1)\cdots(\alpha+n-1)\big) \tag{3}$$
$$+(-1)^{n+1}\alpha(\alpha+1)\cdots(\alpha+n)\int_1^{+\infty}\frac{e^{-t}}{t^{\alpha+n+1}}\,\mathrm{d}t.$$

On the other hand according to (1) we obtain,

$$\int_0^1 e^{-1/t}t^{\alpha-1}\,\mathrm{d}t = \int_1^{+\infty}\frac{e^{-t}}{t^{\alpha+1}}\,\mathrm{d}t = e^{-1}\Big(1-(\alpha+1)+(\alpha+1)(\alpha+2)-\cdots \tag{4}$$
$$+(-1)^{n-1}(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)\Big)$$
$$+(-1)^n(\alpha+1)(\alpha+2)\cdots(\alpha+n)\int_1^{+\infty}\frac{e^{-t}}{t^{\alpha+n+1}}\,\mathrm{d}t.$$

Now, from (2), (3) and (4) we have:

$$(-\alpha)!\,e + e\alpha\int_0^1 e^{-1/t}t^{\alpha-1}\,\mathrm{d}t = e\Bigg(\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\,\frac{1}{n-\alpha+1} + e^{-1}\Big(1-\alpha+\alpha(\alpha+1)-\cdots$$
$$+(-1)^n\alpha(\alpha+1)\cdots(\alpha+n-1)\Big) + (-1)^{n+1}\alpha(\alpha+1)\cdots(\alpha+n)\int_1^{+\infty}\frac{e^{-t}}{t^{\alpha+n+1}}\,\mathrm{d}t$$
$$+e^{-1}\Big(\alpha-\alpha(\alpha+1)+\alpha(\alpha+1)(\alpha+2)-\cdots+(-1)^{n-1}\alpha(\alpha+1)\cdots(\alpha+n-1)\Big)$$
$$+(-1)^n\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n)\int_1^{+\infty}\frac{e^{-t}}{t^{\alpha+n+1}}\,\mathrm{d}t\Bigg)$$
$$= e\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\,\frac{1}{n-\alpha+1}+1 = \sum_{n=0}^{+\infty}\frac{1}{n!}\,\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\,\frac{1}{n-\alpha+1}+1.$$

Let $a_s = \dfrac{1}{s!}$ , $b_s = \dfrac{(-1)^s}{s!}\,\dfrac{1}{s-\alpha+1}$. Then $\sum\limits_{s=0}^{+\infty} a_s \cdot \sum\limits_{s=0}^{+\infty} b_s = \sum\limits_{s=0}^{+\infty} c_s$, where

$$c_s = \sum_{k=0}^{s} a_{s-k}b_k = \sum_{k=0}^{s} \frac{1}{(s-k)!}\,\frac{(-1)^k}{k!}\,\frac{1}{k-\alpha+1} = \frac{1}{s!}\sum_{k=0}^{s}(-1)^k\binom{s}{k}\frac{1}{k-\alpha+1}.$$

Then we shall apply the following identity, obtained by D. S. Mitrinović and J. D. Kečkić (see [**1**], p.146)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{ak+b} = \frac{n!\,a^n}{b(a+b)(2a+b)\cdots(na+b)}\,.$$

The left hand side of this identity is, in fact, equal to

$$(1/b)F(-n, b/a; 1+b/a; 1),$$

and so it is an immediate consequence of the Chu–Vandermonde summation theorem for the finite Gauss hypergeometric series $F(-n,b;c;1)$ with $b$ replaced by $b/a$ and $c = 1 + b/a$.

If $a = 1$ and $b = 1-\alpha$, we obtain:

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k}\frac{1}{k-\alpha+1} = \frac{s!}{(1-\alpha)(2-\alpha)(3-\alpha)\cdots(s+1-\alpha)}\,.$$

Thus,

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{1}{n!}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}\,\frac{1}{n-\alpha+1}+1 &= \frac{1}{n!}\sum_{n=0}^{+\infty}\frac{n!}{(1-\alpha)(2-\alpha)\cdots(n+1-\alpha)}+1\\ &= 1+\frac{1}{1-\alpha}+\frac{1}{(1-\alpha)(2-\alpha)}+\cdots,\end{aligned}$$

i.e. our identity is finally proved. □

For other proved identities of this type, see ([**4**], [**5**]).

Now we are ready to prove the main theorem. We convenient that everywhere further $e^x$ will denotes the expansion $\sum\limits_{k=-\infty}^{+\infty}\dfrac{x^k}{k!}$ .

**Theorem 2.1.** *The $\alpha$-th derivative of $e^{\lambda x}$ is equal to $\lambda^\alpha e^{\lambda x}$, i.e.*

$$(e^{\lambda x})^{(\alpha)} = \lambda^\alpha e^{\lambda x}.$$

**Proof.** It is sufficient to prove the theorem for $\alpha \notin \mathbf{Z}$, because for $\alpha \in \mathbf{Z}$, the theorem is obvious. So, assume that $\alpha \notin \mathbf{Z}$.

Using the expansion

$$e^{\lambda x} = \cdots + \lambda^{-2}\,\frac{x^{-2}}{(-2)!} + \lambda^{-1}\,\frac{x^{-1}}{(-1)!} + \lambda^{0}\,\frac{x^{0}}{0!} + \lambda^{1}\,\frac{x^{1}}{1!} + \lambda^{2}\,\frac{x^{2}}{2!} + \cdots$$

we obtain

$$\begin{aligned}(e^{\lambda x})^{(\alpha)} = \cdots + \lambda^{-2}\,\frac{x^{-2-\alpha}}{(-2-\alpha)!} + \lambda^{-1}\frac{x^{-1-\alpha}}{(-1-\alpha)!}&\\ + \lambda^{0}\,\frac{x^{-\alpha}}{(-\alpha)!} + \lambda^{1}\frac{x^{1-\alpha}}{(1-\alpha)!} + \lambda^{2}\,\frac{x^{2-\alpha}}{(2-\alpha)!} + \cdots&\end{aligned}$$

and hence the theorem will be proved if we prove the identity

$$\begin{aligned}\cdots + \lambda^{-2}\,\frac{x^{-2-\alpha}}{(-2-\alpha)!} + \lambda^{-1}\,\frac{x^{-1-\alpha}}{(-1-\alpha)!} + \lambda^{0}\,\frac{x^{-\alpha}}{(-\alpha)!} + \lambda^{1}\frac{x^{1-\alpha}}{(1-\alpha)!}&\\ + \lambda^{2}\,\frac{x^{2-\alpha}}{(2-\alpha)!} + \cdots = \lambda^{\alpha} e^{\lambda x}.&\end{aligned}$$

Multiplying this equality by $(-\alpha)!\,x^{\alpha}$ we get

$$\begin{aligned}\cdots + (\lambda x)^{-2}\,\frac{(-\alpha)!}{(-2-\alpha)!} + (\lambda x)^{-1}\,\frac{(-\alpha)!}{(-1-\alpha)!} + (\lambda x)^{0}\,\frac{(-\alpha)!}{(-\alpha)!}&\\ + (\lambda x)^{1}\,\frac{(-\alpha)!}{(1-\alpha)!} + (\lambda x)^{2}\,\frac{(-\alpha)!}{(2-\alpha)!} + \cdots = (-\alpha)!\,(\lambda x)^{\alpha} e^{\lambda x}&\end{aligned}$$

and using the identity $(x+n)! = (x+n)(x+n-1)\cdots(x+1)\,x!$ for any $x$ and positive integer $n$, we obtain the following equivalent equality

$$\begin{aligned}\cdots - z^{-3}\alpha(\alpha+1)(\alpha+2) + z^{-2}\alpha(\alpha+1) - z^{-1}\alpha + 1 + z\,\frac{1}{1-\alpha}&\\ +z^{2}\,\frac{1}{(2-\alpha)(1-\alpha)} + z^{3}\,\frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots = (-\alpha)!\,z^{\alpha} e^{z},&\end{aligned}$$

where $z = \lambda x$. By multiplying this equality by $z^{-\alpha}$ we should prove the following equivalent equality

$$\begin{aligned}\cdots - z^{-3-\alpha}\alpha(\alpha+1)(\alpha+2) + z^{-2-\alpha}\alpha(\alpha+1) - z^{-1-\alpha}\alpha + z^{-\alpha} + z^{1-\alpha}\,\frac{1}{1-\alpha}&\\ +z^{2-\alpha}\,\frac{1}{(2-\alpha)(1-\alpha)} + z^{3-\alpha}\,\frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots = (-\alpha)!\,e^{z}.&\end{aligned}$$

Note that the left side "L" of the previous equality satisfies $\mathrm{d}L/\mathrm{d}z = L$, it follows that $L = C \times e^{z}$. Hence it is sufficient to prove that $C = (-\alpha)!$. Namely, it is sufficient to prove the previous equality for $z = 1$, i.e.

$$\begin{aligned}&\big(\cdots - \alpha(\alpha+1)(\alpha+2) + \alpha(\alpha+1) - \alpha\big)\\ &+\Big(1 + \frac{1}{1-\alpha} + \frac{1}{(2-\alpha)(1-\alpha)} + \frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)} + \cdots\Big) = (-\alpha)!\,e.\end{aligned}$$

We will prove this identity for $\alpha > -1$. Note that the series

$$1+\frac{1}{1-\alpha}+\frac{1}{(2-\alpha)(1-\alpha)}+\frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)}+\cdots$$

is convergent for any $\alpha \notin \{1,2,3,\cdots\}$, but the series

$$-\alpha+\alpha(\alpha+1)-\alpha(\alpha+1)(\alpha+2)+\cdots$$

is divergent and first we should sum it. Thus we consider the function

$$f(z)=-\alpha z^{1+\alpha}+\alpha(\alpha+1)z^{2+\alpha}-\alpha(\alpha+1)(\alpha+2)z^{3+\alpha}+\cdots$$

and we should find $f(1)$. Hence $f$ satisfies the following differential equation

$$\begin{aligned} f'(z) &= -\alpha(\alpha+1)z^{\alpha}+\alpha(\alpha+1)(\alpha+2)z^{1+\alpha} \\ &\quad -\alpha(\alpha+1)(\alpha+2)(\alpha+3)z^{2+\alpha}+\cdots=-\frac{f(z)+\alpha z^{1+\alpha}}{z^2}, \end{aligned}$$

i.e.

$$f'(z)=-\frac{f(z)}{z^2}-\alpha z^{\alpha-1}.$$

Moreover, $f(0)=0$ because $\alpha > -1$. Hence the required function is

$$f(z)=-e^{1/z}\alpha\int_0^z e^{-1/t}t^{\alpha-1}\,\mathrm{d}t$$

and $f(1)=-e\alpha\int_0^1 e^{-1/t}t^{\alpha-1}\,\mathrm{d}t$. Thus the proof of theorem for $\alpha > -1$ is finished if we prove that

$$\begin{aligned} -e\alpha\int_0^1 e^{-1/t}t^{\alpha-1}\,\mathrm{d}t+1+\frac{1}{1-\alpha}+\frac{1}{(2-\alpha)(1-\alpha)} \\ +\frac{1}{(3-\alpha)(2-\alpha)(1-\alpha)}+\cdots=(-\alpha)!\,e. \end{aligned}$$

But this equality is true according to lemma 2.1 and the proof of the theorem for $\alpha > -1$ is finished.

Now, suppose that $\alpha \le -1$. Let $k$ be any integer smaller than $\alpha+1$. Then $\alpha-k \ge -1$ and using that

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\circ\frac{\mathrm{d}^{\beta}}{\mathrm{d}x^{\beta}}=\frac{\mathrm{d}^{\alpha+\beta}}{\mathrm{d}x^{\alpha+\beta}}\,,$$

we obtain

$$\frac{\mathrm{d}^{\alpha}e^x}{\mathrm{d}x^{\alpha}}=\frac{\mathrm{d}^{\alpha-k}}{\mathrm{d}x^{\alpha-k}}\circ\frac{\mathrm{d}^{k}}{\mathrm{d}x^{k}}\,e^x=\frac{\mathrm{d}^{\alpha-k}}{\mathrm{d}x^{\alpha-k}}\,e^x=e^x.$$

□

As a consequence of the previous theorem we obtain the following two corollaries.

**Corollary 2.1.** *If the functions* $\sin(x)$ *and* $\cos(x)$ *are defined by*

$$\cos(x) = \sum_{k=-\infty}^{+\infty} a_k \frac{x^k}{k!}, \quad \sin(x) = \sum_{k=-\infty}^{+\infty} b_k \frac{x^k}{k!},$$

*where* $a_{2k+1} = 0$, $a_{2k} = (-1)^k$, $b_{2k+1} = (-1)^k$ *and* $b_{2k} = 0$, $(k \in \mathbf{Z})$, *then*

$$\sin^{(\alpha)}(x) = \sin\Big(x + \frac{\alpha\pi}{2}\Big), \quad \cos^{(\alpha)}(x) = \cos\Big(x + \frac{\alpha\pi}{2}\Big).$$

**Proof.** According to the definitions of cos and sin it follows that $\cos(x)+i\sin(x) = e^{ix}$. According to the theorem 2.1 we get

$$\cos^{(\alpha)}(x) + i\sin^{(\alpha)}(x) = i^\alpha e^{ix}.$$

Hence we obtain

$$\cos^{(\alpha)}(x) = \mathrm{Re}\,(i^\alpha e^{ix}) = \cos\Big(x + \frac{\alpha\pi}{2}\Big)$$

and

$$\sin^{(\alpha)}(x) = \mathrm{Im}\,(i^\alpha e^{ix}) = \sin\Big(x + \frac{\alpha\pi}{2}\Big). \qquad \square$$

Note that the previous corollary is of special interest, because the classical theory of fractional derivatives starts just from the conclusions of the previous corollary, but without the assumption for sin and cos, which means by considering these functions as maps from **R** to **R**.

**Corollary 2.2.** *Assume that the functions* sin *and* cos *are defined as in corollary* 2.1. *Then*

$$\big(e^x\cos(x)\big)^{(\alpha)} = 2^{\alpha/2}e^x\cos\Big(x + \frac{\alpha\pi}{4}\Big), \quad \big(e^x\sin(x)\big)^{(\alpha)} = 2^{\alpha/2}e^x\sin\Big(x + \frac{\alpha\pi}{4}\Big).$$

The proof is analogous to the proof of corollary 2.1. These equalities can be proved directly as follows. We develop the function $e^x$ in the form

$$\sum_{n=-\infty}^{+\infty} \frac{x^{\alpha+n}}{(\alpha+n)!}$$

and the functions cos and sin develop like in Corollary 2.1. Then the Cauchy products $e^x\cos x$ and $e^x\sin x$ are given by

$$e^x\cos x = \sum_{n=-\infty}^{+\infty} p_n \frac{x^{\alpha+n}}{(\alpha+n)!}, \quad e^x\sin x = \sum_{n=-\infty}^{+\infty} q_n \frac{x^{\alpha+n}}{(\alpha+n)!},$$

where $p_n = 2^{(n+\alpha)/2}\cos\big((n+\alpha)\pi/4\big)$ and $q_n = 2^{(n+\alpha)/2}\sin\big((n+\alpha)\pi/4\big)$. Hence $(e^x\cos x)^{(\alpha)}$ and $(e^x\sin x)^{(\alpha)}$ are obtained as TAYLOR's series and we obtain

$$(e^x\cos x)^{(\alpha)} = \sum_{n=-\infty}^{+\infty} p_n\,\frac{x^n}{n!} = \sum_{n=-\infty}^{+\infty} 2^{(n+\alpha)/2}\cos\frac{(n+\alpha)\pi}{4}\,\frac{x^n}{n!} = 2^{\alpha/2}e^x\cos\Big(x+\frac{\alpha\pi}{4}\Big)$$

and

$$(e^x\sin x)^{(\alpha)} = \sum_{n=-\infty}^{+\infty} q_n\,\frac{x^n}{n!} = \sum_{n=-\infty}^{+\infty} 2^{(n+\alpha)/2}\sin\frac{(n+\alpha)\pi}{4}\,\frac{x^n}{n!} = 2^{\alpha/2}e^x\sin\Big(x+\frac{\alpha\pi}{4}\Big).$$

REMARK. If the functions $\sin x$ and $\cos x$ are not defined as in the Corollary 2.1, and if $e^x$ is not defined as $\sum\limits_{k=-\infty}^{+\infty} x^k/k!$, then the Theorem 2.1 and the Corollaries 2.1 and 2.2 are not true. For example, if we use the expansions

$$e^x = \sum_{k=0}^{+\infty}\frac{x^k}{k!}\,,\quad \sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots,\quad \cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots,$$

then

$\lim\limits_{x\to 0}(\sin x)^{(1/2)} = 0$, while $\lim\limits_{x\to 0}\sin\Big(x+\dfrac{\pi}{4}\Big) = \sqrt{2}/2$,
$\lim\limits_{x\to 0}(\cos x)^{(1/2)} = \pm\infty$, while $\lim\limits_{x\to 0}\cos\Big(x+\dfrac{\pi}{4}\Big) = \sqrt{2}/2$, and
$\lim\limits_{x\to 0}(e^x)^{(1/2)} = \pm\infty$, while $\lim\limits_{x\to 0} e^x = 1$.

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