Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 12 (2001), 48–51.

ON A CONVERSE OF JENSEN'S INEQUALITY

S. S. Dragomir

The main aim of this paper is to point out a refinement of the reverse of Jensen's inequality obtained in 1994 by Dragomir and Ionescu.

1. INTRODUCTION

In 1994, S. S. Dragomir and N. M. Ionescu [**2**], obtained the following reverse of Jensen's inequality for convex functions:

$$0 \le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i x_i f(x_i) - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f'(x_i), \tag{1.1}$$

provided that $f : I \subseteq \mathbf{R} \to \mathbf{R}$ is differentiable convex on $\mathring{I}$ ($\mathring{I}$ is the interior of the interval I), $x_i \in \mathring{I}$, $p_i > 0$ $(i = 1, \dots, n)$ and $\sum_{i=1}^{n} p_i = 1$. If f is strictly convex on $\mathring{I}$, then the case of equality holds in (1.1) iff $x_1 = \cdots = x_n$.

For some applications of this result see for example the recent papers [**3**] and [**4**].

The main aim of this paper is to point out a refinement of the reverse of Jensen's inequality stated in (1.1).

THE RESULTS

The following inequality holds.

Lemma 1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable convex function on $\mathring{I}$, $x_i \in \mathring{I}$, $p_i > 0$ $(i = 1, \dots, n$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$\sum_{i=1}^{n} p_i f(x_i) \le \sum_{i=1}^{n} p_i x_i f'(x_i) + \inf_{x \in \mathring{I}} \left(f(x) - x \sum_{i=1}^{n} p_i f'(x_i) \right). \tag{2.1}$$

2000 Mathematics Subject Classification: Primary 26D15; Secondary 26D10.
Keywords and Phrases: Jensen's inequality, Convex functions.

Proof. As f is differentiable convex on $\mathring{I}$, then for all $x, y \in \mathring{I}$ we have the inequality:

$$f(x) - f(y) \geq f'(y)(x-y). \tag{2.2}$$

If we choose in (2.2) $y = x_i\,(i = 1, \ldots, n)$, multiply with $p_i > 0$ and sum over i from 1 up to n, we get

$$f(x) - \sum_{i=1}^{n} p_i f(x_i) \geq x \sum_{i=1}^{n} p_i f'(x_i) - \sum_{i=1}^{n} p_i x_i f'(x_i)$$

which is clearly equivalent to

$$f(x) - x \sum_{i=1}^{n} p_i f'(x_i) + \sum_{i=1}^{n} p_i x_i f'(x_i) \geq \sum_{i=1}^{n} p_i f(x_i) \tag{2.3}$$

for all $x \in \mathring{I}$.

Taking the infimum over $x \in \mathring{I}$, we deduce (2.1). □

The following result concerning a refinement of the DRAGOMIR-IONESCU (1.1) holds.

Theorem 1. *With the assumptions of Lemma* 1 *for* f, x_i, p_i, *we have the inequality:*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\leq \inf_{x \in \mathring{I}} \left(f(x) - x \sum_{i=1}^{n} p_i f'(x_i) \right) + \sum_{i=1}^{n} p_i x_i f'(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ &\leq \sum_{i=1}^{n} p_i x_i f'(x_i) - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f'(x_i). \end{aligned} \tag{2.4}$$

Proof. The second inequality in (2.4) follows by the first one in (2.1).

It is obvious that

$$\inf_{x \in \mathring{I}} \left(f(x) - x \sum_{i=1}^{n} p_i f'(x_i) \right) \leq f(\bar{x}) - \bar{x} \sum_{i=1}^{n} p_i f'(x_i),$$

where $\bar{x} := \sum_{i=1}^{n} p_i x_i \in \mathring{I}$, and then the last part of (2.4) is proved. □

For applications we may use the following result.

Lemma 2. *Let* $f : I \subseteq \mathbf{R} \to \mathbf{R}$ *be a differentiable, strictly convex function on* $\mathring{I}$, $x_i \in \mathring{I}$, $p_i > 0$ *and* $\sum_{i=1}^{n} p_i = 1$. *Then we have the inequality*

$$\sum_{i=1}^{n} p_i f(x_i) \leq \sum_{i=1}^{n} p_i x_i f'(x_i) + f\left((f')^{-1}\left(\sum_{i=1}^{n} p_i f'(x_i) \right) \right) \tag{2.5}$$

$$-(f')^{-1}\Big(\sum_{i=1}^{n} p_i f'(x_i)\Big)\cdot \sum_{i=1}^{n} p_i f'(x_i),$$

where $(f')^{-1}$ *denotes the inverse function of the derivative* f' *defined on* $f'(\mathring{I})$.

The case of equality holds in (2.5) iff $x_1 = \cdots = x_n$.

Proof. Define the function $g : \mathring{I} \to \mathbf{R}$, $g(x) = f(x) - x\sum_{i=1}^{n} p_i f'(x_i)$. Obviously, g is differentiable on $\mathring{I}$ and

$$g'(x) = f'(x) - \sum_{i=1}^{n} p_i f'(x_i). \tag{2.6}$$

The equation $g'(x) = 0$, $x \in \mathring{I}$ is equivalent to

$$f'(x) = \sum_{i=1}^{n} p_i f'(x_i) \tag{2.7}$$

and since $\sum_{i=1}^{n} p_i f'(x_i) \in f'(\mathring{I})$, f' is one-to-one, being strictly increasing on $\mathring{I}$, it follows that the equation (2.7) has a unique solution $x_0 \in \mathring{I}$ which is given by

$$x_0 := (f')^{-1}\Big(\sum_{i=1}^{n} p_i f'(x_)\Big) \in \mathring{I}, \tag{2.8}$$

where $(f')^{-1}$ is the inverse function of the derivative f' defined on $f'(\mathring{I})$.

Taking into account that $g'(x) < 0$ if $x < x_0$, $x \in \mathring{I}$ and $g'(x) > 0$ if $x > x_0$, $x \in \mathring{I}$, it follows that

$$\inf_{x\in\mathring{I}} g(x) = g(x_0) = f\Big((f')^{-1}\Big(\sum_{i=1}^{n} p_i f'(x_i)\Big)\Big) - (f')^{-1}\Big(\sum_{i=1}^{n} p_i f'(x_i)\Big)\cdot\sum_{i=1}^{n} p_i f'(x_i).$$

Using (1.1) we deduce (2.5).

The case of equality follows by the strict convexity of $\mathring{I}$ and we omit the details. □

We can now state the following refinement of the Dragomir-Ionescu result (1.1).

Theorem 2. *Let* f, x_i, p_i *be as in Lemma* 2. *Then we have the inequality*

$$\begin{aligned}0 &\le \sum_{i=1}^{n} p_i f(x_i) - f\Big(\sum_{i=1}^{n} p_i x_i\Big)\\ &\le \sum_{i=1}^{n} p_i x_i f'(x_i) + f\Big((f')^{-1}\Big(\sum_{i=1}^{n} p_i f'(x_i)\Big)\Big)\\ &\qquad -(f')^{-1}\Big(\sum_{i=1}^{n} p_i f'(x_i)\Big)\cdot\sum_{i=1}^{n} p_i f'(x_i) - f\Big(\sum_{i=1}^{n} p_i x_i\Big)\\ &\le \sum_{i=1}^{n} p_i x_i f'(x_i) - \sum_{i=1}^{n} p_i f'(x_i)\sum_{i=1}^{n} p_i x_i.\end{aligned} \tag{2.9}$$

The equality holds in (2.9) *iff* $x_1 = \cdots = x_n$.

The proof is obvious by Theorem 1 Lemma 2.

REMARK. We note that with the assumptions in Lemma 2, we have the double inequality

$$f\Big(\sum_{i=1}^{n} p_i x_i\Big) \le \sum_{i=1}^{n} p_i f(x_i) \le \sum_{i=1}^{n} p_i x_i f'(x_i) + f\bigg((f')^{-1}\Big(\sum_{i=1}^{n} p_i f'(x_i)\Big)\bigg) - \sum_{i=1}^{n} p_i f'(x_i)\cdot (f')^{-1}\Big(\sum_{i=1}^{n} p_i f'(x_i)\Big), \tag{2.10}$$

with equality iff $x_1 = \cdots = x_n$.

If g is differentiable and strictly concave, then

$$g\Big(\sum_{i=1}^{n} p_i x_i\Big) \ge \sum_{i=1}^{n} p_i g(x_i) \ge \sum_{i=1}^{n} p_i x_i g'(x_i) + g\bigg(-(g')^{-1}\Big(-\sum_{i=1}^{n} p_i g'(x_i)\Big)\bigg) + \sum_{i=1}^{n} p_i g'(x_i)\cdot (g')^{-1}\Big(-\sum_{i=1}^{n} p_i g'(x_i)\Big), \tag{2.11}$$

with equality iff $x_1 = \cdots = x_n$.

The proof of (2.11) follows by (2.10) choosing $f = -g$.

REFERENCES

1. P. S. BULLEN, D. S. MITRINOVIĆ, P. M. VASIĆ: *Means and their Inequalities.* Kluwer Academic Publishers, 1988.
2. S. S. DRAGOMIR, N. M. IONESCU: *Some converse of Jensen's inequality and applications.* Anal. Num. Theor. Approx., **23** (1994), 71–78.
3. S. S. DRAGOMIR, C. J. GOH: *A counterpart of Jensen's discrete inequality for differentiable convex mappings and applications in information theory.* Math. Comput. Modelling, **24**(2) (1996), 1–11.
4. S. S. DRAGOMIR, C. J. GOH: *A counterpart of Hölders inequality.* Mitt. Math. Ges. Hamburg, **16** (1997), 99–106.

School of Communications and Informatics,
Victoria University of Technology,
PO Box 14428,
Melbourne City MC,
Victoria 8001, Australia.
sever@matilda.vu.edu.au
http:rgmia.vu.edu.au/SSDragomirWeb.html

(Received January 20, 2001)