UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 12 (2001), 38–40.

INEQUALITIES FOR PARTS OF THE HARMONIC SERIES

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In this short note we prove some inequalities for sums of the certain subsequences of the set $\{1,\ 1/2,\ 1/3,\ldots\}$.

PRELIMINARIES

In [**3**] there were published five inequalities of type

$$f(N;a,b)-\frac{1}{aN+a+b}<\sum_{k=0}^{N}\frac{1}{ak+b}<f(N;a,b)-\frac{1}{2aN+a+b},$$

namely the cases $(a,b)\in\{(2,1),\ (3,1),\ (3,2),\ (4,1),\ (4,3)\}$. (Note throughout the quoted paper the misprint n instead N.)

It is the goal of this note to improve and extend the above inequalities to more general pairs $(a,b)\in\mathbf{N}^2$ of coefficients satisfying $a\geq 2$ and $1\leq b\leq a-1$.

In order to achieve this we recall the following three facts.

As usually let $\Psi(x)=\dfrac{\Gamma'(x)}{\Gamma(x)}$ be the derivative of $\log\Gamma(x)$. Then

(A) $\displaystyle\sum_{k=0}^{N}\frac{1}{k+z}+\Psi(z)=\Psi(z+N+1)$, where $N\geq 0$ is entire and $z\in\mathbf{C}$, $z\neq 0$, $-1,\ -2,\ldots$ ([**2**], p. 774).

(B) $\displaystyle\Psi\Big(\frac{b}{a}\Big)=-C-\log(2a)-\frac{\pi}{2}\cot\frac{b\pi}{a}+2\sum_{j=1}^{[(a-1)/2]}\cos\frac{bj\pi}{a}\log\sin\frac{j\pi}{a}$, where a and b are natural numbers satysfying $a\geq 2$ and $1\leq b\leq a-1$ and C denotes EULER's constant ([**2**], p. 775).

2000 Mathematics Subject Classification: 26D15
Keywords and Phrases: Inequalities, Harmonic series

(C) Inequality of J. Sándor ([**1**], p. 453):

$$\log\left(x-\frac{1}{2}\right) < \log\left(x-\frac{1}{2}+\frac{1}{16x}\right),$$

where $x > 1$.

RESULT

We are now in the position to prove the announced result.

Let $F(a,b) = \dfrac{1}{a}\left(C + \dfrac{\pi}{2}\cot\dfrac{b\pi}{a} - 2\displaystyle\sum_{j=1}^{[(a-1)/2]}\cos\dfrac{bj\pi}{a}\log\sin\dfrac{j\pi}{a}\right)$. Then we have the following

Theorem. *Let a and b be natural numbers such that $a \geq 2$ and $1 \leq b \leq a-1$. Then for all nonnegative entire numbers N the inequality*

$$\frac{1}{a}\log(2aN+a+2b) + F(a,b) < \sum_{k=0}^{N}\frac{1}{ak+b} < \frac{1}{a}\log\left(2aN+a+2b+\frac{a^2}{8(aN+a+b)}\right) + F(a,b)$$

is valid.

Proof. Putting in (A) $z = b/a$, we get

$$a\sum_{k=0}^{N}\frac{1}{ak+b} + \Psi\left(\frac{b}{a}\right) = \Psi\left(\frac{b}{a}+N+1\right). \tag{1}$$

Furthermore (C) yields

$$\log(2aN+a+2b) - \log(2a) < \Psi\left(\frac{b}{a}+N+1\right) < \log\left(2aN+a+2b+\frac{a^2}{8(aN+a+b)}\right) - \log(2a). \tag{2}$$

Hence (1), (2) and (B) readily lead to the stated inequality. □

Remark. Because of

$$\log\left(2aN+a+2b+\frac{a^2}{8(aN+a+b)}\right) < \log(2aN+a+2b) + \log\left(1+\frac{a^2}{2aN\cdot 8aN}\right) < \log(2aN+a+2b) + \frac{1}{16N^2},$$

we can replace the right-hand side of the double inequality by the weaker but maybe more appealing expression $\dfrac{1}{a}\log(2aN+a+2b) + F(a,b) + \dfrac{1}{16aN^2}$.

As an immediate consequence of the proved theorem we note the following asymptotic formula.

Corollary. $\sum_{k=0}^{N} \frac{1}{ak+b} = \frac{1}{a} \log(2aN + a + 2b) + F(a,b) + O(1/N^2)$, as $N \to \infty$.

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(Received February 15, 2000)