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GENERALISED MOMENTS FOR THE POISSON LAW

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For the ordinary Poisson probability law with parameter λ we consider generalised moments of the form $E(X^\rho L(X))$, $\rho \in R$, where $L(\cdot)$ is slowly varying function in Karamata sense. We are proving here that $E(X^\rho L(X)) \sim \lambda^\rho L(\lambda)$, $\lambda \to \infty$, and, as a consequence, obtain inversion formulae in terms of Laplace-Stieltjes transform.

Preliminaries. The Poisson probability law with parameter $\lambda > 0$ is defined for a random variable X as: $P\{X = k\} = \dfrac{\lambda^k}{k!}\, e^{-\lambda}$. Its ordinary moments of order m are defined as:

$$E(X^m) := \sum_{k=1}^{\infty} e^{-\lambda} k^m \frac{\lambda^k}{k!} \qquad (m \in \mathbf{N}).$$

There is a very complicated asymptotic formula for $E(X^m)$ when $m \to \infty$, $1-\delta < \lambda < 1+\delta$, $0 < \delta < 1$, (cf.[**3**] p.p. 294-5). Our task here is to reveal the behavior of generalised moments

$$E(X^\rho L(X)) := \sum k^\rho L(k)\, \frac{\lambda^k}{k!} e^{-\lambda} \qquad (\rho \in \mathbf{R}),$$

for large values of parameter λ. We take a slowly varying function $L(x)$ as defined for $x > 0$, positive, measurable and satisfying $\forall t > 0 : L(ty) \sim L(y)$, $(y \to \infty)$. Some examples of slowly varying functions are:

$$1, \quad \log^a x, \quad \log^b(\log x), \quad \exp(\log^c x); \quad a, b \in \mathbf{R}; \quad 0 < c < 1.$$

Topics of Karamata's theory of regular variation can be found in [**1**] and [**4**]. A tantamount of our results is a valuation of the Poisson distribution function

$$P(x) := \sum_{k \le x} \frac{\lambda^k}{k!}\, e^{-\lambda} \qquad (x, \lambda \in \mathbf{R}^+),$$

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(cf.[**5**]). Namely, in [**5**] we proved

$$P(\xi\lambda) = \begin{cases} O(1)\, e^{-\lambda g(\xi)} & (0 < \xi < 1); \\ 1 + O(1)\, e^{-\lambda g(\xi)} & (\xi > 1); \end{cases} \qquad (\lambda \to \infty), \tag{1}$$

where $g(\xi) := \xi \log \xi + 1 - \xi$ is convex for $\xi > 0$ and positive for $\xi \neq 1$.

Results. For generalised moments of the Poisson law, mentioned above, we have the following theorem:

Theorem 1. *For any* $\rho \in \mathbf{R}$,

$$E(X^\rho L(X)) \sim \lambda^\rho L(\lambda) \qquad (\lambda \to \infty).$$

For the proof we need two lemma's.

Lemma 1. *For any slowly varying* $L(\cdot)$ *defined as above, and any* $\rho \in \mathbf{R}$,

$$e^{-\lambda} \sum_{k \le \xi\lambda} k^\rho L(k) \frac{\lambda^k}{k!} \sim \begin{cases} o(\lambda^\rho L(\lambda)) & (0 < \xi < 1); \\ \lambda^\rho L(\lambda) & (xi > 1). \end{cases} \quad (\lambda \to \infty)$$

This lemma is proved in [**5**] using the estimation (1).

Lemma 2. *For* $\alpha > 0$ *and any slowly varying* $L(\cdot)$,

$$\sup_{t \ge y} (t^{-\alpha} L(t)) \sim y^{-\alpha} L(y) \qquad (y \to \infty).$$

This is a well-known fact (cf.[**1**] p. 23).

Proof of Theorem 1. We have that

$$E(X^\rho L(X)) := e^{-\lambda} \sum_k k^\rho L(k)], \frac{\lambda^k}{k!} = e^{-\lambda} \Big(\sum_{k < 3\lambda} + \sum_{k \ge 3\lambda} \Big) k^\rho L(k) \frac{\lambda^k}{k!} = S_1 + S_2.$$

According to Lemma 1, $S_1 \sim \lambda^\rho L(\lambda),\ \rho \in \mathbf{R}\ \ (\lambda \to \infty)$. Using Lemma 2 and the fact that for $k \ge 3\lambda$,

$$\frac{\lambda^k}{k!} \le \frac{(k/3)^k}{k!} = o(1)(e/3)^k \quad (k \to \infty)$$

we get

$$S_2 = o(1) \sup_{k \ge 3\lambda} (k^{-|\rho|-1} L(k)) \sum_{k \ge 3\lambda} k^{\rho+|\rho|+1} (e/3)^k = o(1) \lambda^{-|\rho|-1} L(\lambda) \quad (\lambda \to \infty)$$

since the last sum is tending to zero as a reminder of a convergent series. Therefore, Theorem 1 is proved.

We consider now a distribution function F with a support on $\mathbf{R}^+$ and its LAPLACE-STIELTJES transform ϕ defined for $s > 0$ as

$$\phi(s) := \int_0^\infty e^{-st} \mathrm{d}(F(t)).$$

Derivatives $\phi^{(k)},\ k \in N$, always exist and

$$(-1)^k \phi^{(k)}(s) = \int_0^\infty e^{-st} t^k \mathrm{d}F(t).$$

Our task now is to obtain an inversion formula which is a generalisation of the known one ($m = 0$, cf.[**2**], II p. 270).

Theorem 2. *For any fixed* $m \in \mathbf{N}$*,*

$$\sum_{k \le xy} (-1)^{k+m} \frac{y^{k+m}}{k^m k!} \phi^{(k+m)}(y) \to F(x) \quad (y \to \infty)$$

at any point of continuity of the distribution F*.*

Proof. Putting in Lemma 1, $L(\cdot) := 1,\ \rho = -m,\ \lambda = ty,\ \xi t = x,$ we get:

$$\sum_{k \le xy} \frac{y^{k+m}}{k^m k!} e^{-ty}\, t^{k+m} \to \begin{cases} 0, & t > x; \\ 1, & t < x. \end{cases} \qquad (y \to \infty) \tag{2}$$

Integrating (2) over $t \in \mathbf{R}^+$ with respect to the measure $\mathrm{d}F$, we obtain the assertion from Theorem 2.

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