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AN INEQUALITY INVOLVING PRIME NUMBERS

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From EUCLID's proof of the existence infinitely many prime numbers one can deduce the inequality

$$p_1 p_2 \cdots p_n > p_{n+1},$$

where p_k is the k-th prime number.

Using elementary methods, BONSE proves in [**1**] that

$$p_1 p_2 \cdots p_n > p_{n+1}^2 \text{ for } n \geq 4,$$

and

$$p_1 p_2 \cdots p_n > p_{n+1}^3 \text{ for } n \geq 5.$$

Stronger results of the same nature have been obtained by J. SANDÓR in [**2**]. For example

$$p_1 p_2 \cdots p_n > p_{n+5}^2 + p_{[n/2]}^2 \text{ for } n \geq 24.$$

Without the restrictions imposed by the use of elementary methods the precise determination of the margin from which the inequality holds, L. PÓSA [**3**] proves the following result:

For all $k > 1$ there is an n_k such that

$$p_1 p_2 \cdots p_n > p_{n+1}^k \text{ for all } n \geq n_k.$$

The aim of the present note is to improve this inequality. We recall two results due to ROSSER and SCHOENFELD [**5**]:

(1) $$p_n \leq n\left(\log n + \log\log n - \frac{1}{2}\right) \text{ for } n \geq 20$$

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$$\pi(x) > \frac{x}{\log x} + \frac{x}{2\log^2 x} \quad \text{for} \quad n \geq 59, \tag{2}$$

where we denoted by $\pi(x)$ the number of prime numbers not exceeding x. We will also use the following result due to G. Robin [**4**]:

$$\theta(p_n) > n\left(\log n + \log\log n - 1 + \frac{\log\log n - a}{\log n}\right) \quad \text{for} \quad n \geq 3, \tag{3}$$

where $a = 2.1454$ and $\theta(x) = \sum\limits_{p \leq x} \log p$, the sum being taken after primes p.

All these results allow as to prove the following

Theorem. *For $n \geq 2$*

$$p_1 p_2 \cdots p_n > p_{n+1}^{n-\pi(n)}.$$

We begin by proving the following

Lemma. *For $n \geq 59$ we have*

$$\log p_{n+1} < \log n + \log\log n + \frac{\log\log n - 0.4}{\log n}.$$

Proof. It is well known that $\log x \leq x - 1$ for $x > 0$, from which we get for $x = 1 + 1/n$ that

$$\log(n+1) < \log n + \frac{1}{n}$$

and then

$$\log\big(\log(n+1)\big) < \log\left(\log n + \frac{1}{n}\right) = \log\log n + \log\left(1 + \frac{1}{n\log n}\right) < \log\log n + \frac{1}{n\log n}.$$

We apply the inequality (1) and for $n \geq 19$ we get

$$\begin{aligned} \log p_{n+1} &< \log(n+1) + \log\left(\log(n+1) + \log\log(n+1) - \frac{1}{2}\right) \\ &< \log n + \frac{1}{n} + \log\log n + \log\left(1 + \frac{1 + \log\log n}{\log n} + \frac{1}{n\log^2 n} - \frac{1}{2\log n}\right) \\ &< \log n + \frac{1}{n} + \log\log n + \frac{\log\log n}{\log n} + \frac{1}{n\log n} + \frac{1}{n\log^2 n} - \frac{1}{2\log n}. \end{aligned}$$

It remains to show that

$$\frac{\log n + 1}{n} + \log\log n + \frac{1}{n\log n} - \frac{1}{2} < \log\log n - 0.4,$$

that is

$$\frac{\log n + 1}{n} + \frac{1}{n\log n} < 0.1,$$

which holds for $n \geq 59$.

Proof of the theorem. For $n \geq 59$ we use (2) and the Lemma. We have

$$\big(n-\pi(n)\big)\log p_{n+1} < n\left(1-\frac{1}{\log n}-\frac{1}{2\log^2 n}\right)\left(\log n+\log\log n+\frac{\log\log n-0.4}{\log n}\right).$$

In order to prove the theorem it is enough to show, using (3), that

$$\left(1-\frac{1}{y}-\frac{1}{2y^2}\right)\left(y+\log y\,\frac{\log y-0.4}{y}\right) < y+\log y-1+\frac{\log y-a}{y},$$

where $y=\log n>\log 59$. This last inequality is equivalent to

$$a-0.9<\left(1+\frac{1}{2y}\right)\left(\log y+\frac{\log y-0.4}{y}\right),$$

which is true, since $a-0.9<1.3$ and $\log y>\log\log 59>1.4$. The theorem is thus proved for $n\geq 59$. It may be checked directly that the assertion in the statement also holds for $2\leq n\leq 58$.

From the above result we immediately obtain an improvement of L. Pósa's inequality.

Corollary. *For any integer $k, k\geq 1$, and $n\geq 2k$ the following inequality holds*:

$$p_1p_2\cdots p_n > p_{n+1}^k.$$

Proof. The function $f:\mathbf{N}^*\mapsto\mathbf{N}$, $f(n)=n-\pi(n)$ in increasing. For $n\geq 2k$, $f(n)\geq f(2k)=2k-\pi(2k)\geq k$, since $\pi(2k)\leq k$ for $k\in\mathbf{N}^*$.

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