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LOG-CONVEX MATRIX FUNCTIONS

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Let f be a positive real-valued function defined on an interval $\mathbf{I} \subseteq \mathbf{R}$. The function f is said to be log-convex if $\log f$ is convex on I. In this note, we study an analogue of log-convexity for matrix functions and discuss the gamma function in this setting. The notion of log-convexity on the positive cone of positive continuous functions is also discussed. A criterion for log-convexity for each of the classes of matrix functions and the functions defined on the positive cone of positive continuous functions is obtained.

1. Introduction. Let f be a positive function defined on an interval $\mathbf{I} \subseteq \mathbf{R}$. Then f is called log-convex or multiplicatively convex if for $x, y \in \mathbf{I}$ and $0 \leq \lambda \leq 1$, the inequality

$$\log f\big(\lambda x + (1-\lambda)y\big) \leq \lambda \log f(x) + (1-\lambda)\log f(y), \tag{1}$$

or equivalently,

$$f\big(\lambda x + (1-\lambda)y\big) \leq \big(f(x)\big)^{\lambda}\big(f(y)\big)^{1-\lambda} \tag{2}$$

holds. For properties of such functions, the reader may refer to ROBERTS and VARBERG [**16**].

From now on $\mathbf{I}$ will denote the interval $(0, \infty)$ and we shall take our function $f : \mathbf{I} \to \mathbf{I}$ to be continuous. This mild restriction on f shall allow us to state an analogue of log-convexity (see (3) and (4) below) for matrix functions with the special choice of λ, namely, $\lambda = 1/2$. For an $n \times n$ positive definite hermitian matrix A, $f(A)$ is defined by familiar functional calculi. The above definition of log-convexity when extended to matrix functions could be independently described by any of the following two inequalities:

$$f\left(\frac{A+B}{2}\right) \leq f(A)\#f(B). \tag{3}$$

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$$\log f\left(\frac{A+B}{2}\right) \le \frac{\log f(A) + \log f(B)}{2}, \tag{4}$$

where A and B are positive definite hermitian matrices of order n, $\#$ denotes geometric mean. We shall call a function $f : \mathbf{I} \to \mathbf{I}$, satisfying (3) (resp. (4)) for $n \times n$ positive definite hermitian matrices A and B, multiplicatively matrix (resp. log matrix) convex on $\mathbf{I}$ of order n. Note that the class of multiplicatively matrix convex (resp. log matrix convex) functions of order 1 in the sence of (3) (resp. (4)) is precisely the class of log-convex functions. It is also clear, using the matrix monotonicity of log function (see Ando [**2**]), that (3) implies (4) in the case of commuting matrices. In section 2, we study the inequality (3). It is shown that the class of functions satisfying (3) is a convex cone.

A typical example of usual log convex function is the Gamma function. Matrix valued Gamma function has been studied by a variety of authors including K. J. Heuvers, D. Moak [**11**] and K. I. Gross, W. J. III. Holman [**9**]. Whereas the authors in [**11**] seek solutions of the functional equation $f(z+1) = zf(z)$ for matrix valued functions. K. I. Gross and W. J. III. Holman [**9**] study the properties of the matrix valued Gamma function generalising its usual integral representation. For a detailed study of matrix valued special functions, the reader may refer to [**17**]. We seek to characterise log-convex matrix functions defined on commuting matrices satisfying the functional equation $f(x+1) = xf(x)$ and the normalising condition $f(1) = 1$. Though restricted in scope, the treatment is satisfying as it establishes a complete analogue of the treatment in [**3**].

In section 3, the inequality (4) is studied. Here we provide a characterisation of log-convex functions in terms of Frechet derivatives. In the final section log-convex functions on the Banach space of continuous functions on a compact Housdorff space are studied and an easily verifiable criterion of log-convexity in the above said space is given.

2. In this section, we shall consider the inequality (3), namely,

$$f\left(\frac{A+B}{2}\right) \le f(A)\#f(B),$$

where A and B are positive definite hermitian matrices and f is a positive continuous function defined on $\mathbf{I}$. Since geometric mean is less than or equal to the arithmetic mean, Ando [**1**], it follows that if f satisfies (3), it is mid-matrix convex and hence matrix convex, using continuity of f, Kwong [**14**]. That the class of functions satisfying (3) is strictly contained in the class of matrix convex functions follows on observing that the function $f(x) = x$, $x \in (0, \infty)$, is matrix convex of order n for every positive integer n but it does not satisfy inequality (3) even in the case $n = 1$. Our first proposition shows that the class of functions satisfying (3) is fairly rich. Indeed, we have the following proposition:

Proposition 2.1. *Let $f : \mathbf{I} \to \mathbf{I}$ be operator concave. Then $1/f$ satisfies the inequality* (3).

Proof. For A, B positive definite hermitian matrices, we have

$$f\left(\frac{A+B}{2}\right) \geq \frac{f(A)+f(B)}{2} \geq f(A)\#f(B),$$

using [**1**, Corollary I.2.4]. Consequently,

$$\left(f\left(\frac{A+B}{2}\right)\right)^{-1} \leq \big(f(A)\#f(B)\big)^{-1} = \big(f(A)\big)^{-1}\#\big(f(B)\big)^{-1},$$

using that $f(x) = -x^{-1}$ is matrix monotone on $\mathbf{I}$ of order n for every positive integer n and [**1**, Corollary I.2.1 (vii)] respectively.

Theorem 2.2. (i) *Let $f, g : \mathbf{I} \to \mathbf{I}$ satisfy inequality* (3) *and $\alpha \geq 0$, then $f+g$ and αf satisfy* (3).

(ii) *Let $\{f_n\}_{n\geq 1}$, where $f_n : \mathbf{I} \to \mathbf{I}$, be a sequence of functions satisfying* (3) *and let $f_n \to f$, and f is a positive function, then f satisfies the inequality* (3).

(iii) *Let g satisfy* (3) *and f be matrix monotone and positive linear, then $f \circ g$ satisfies* (3).

Proof. (i) For A, B positive definite hermitian matrices, we have

$$\begin{aligned}(f+g)\left(\frac{A+B}{2}\right) = f\left(\frac{A+B}{2}\right) + g\left(\frac{A+B}{2}\right) &\leq \big(f(A)\#f(B)\big) + \big(g(A)\#g(B)\big)\\ &\leq (f+g)(A)\#(f+g)(B),\end{aligned}$$

using [**11**, Theorem 3.5 (I‘)].

That αf, $\alpha \geq 0$, satisfies (3) whenever f does, follows on using [**1**, Corollary I.2.1 (ii)].

(ii) For A, B positive definite hermitian matrices,

$$f_n\left(\frac{A+B}{2}\right) \leq f_n(A)\#f_n(B) \qquad (n = 1, 2, \ldots)$$

holds. On taking limits as $n \to \infty$, we obtain the desired result.

(iii) For A, B positive definite hermitian matrices,

$$\begin{aligned}f \circ g\left(\frac{A+B}{2}\right) &= f\left(g\left(\frac{A+B}{2}\right)\right) \leq f\big(g(A)\#g(B)\big) \leq f\big(g(A)\big)\#f\big(g(B)\big)\\ &= f \circ g(A)\#f \circ g(B);\end{aligned}$$

the last two inequalities follow since f is matrix monotone and positive linear.

Theorem 2.3. *Let $f, g : \mathbf{I} \to \mathbf{I}$ be functions satisfying the inequality* (3). *Let $h(A) = f(A) * g(A)$, where A is a positive definite hermitian matrix, be the Hadamard product of $f(A)$ and $g(A)$. Then h satisfies the inequality* (3).

Proof. For A, B positive definite hermitian matrices, we have

$$\begin{aligned} h\left(\frac{A+B}{2}\right) &= f\left(\frac{A+B}{2}\right) * g\left(\frac{A+B}{2}\right) \le \big(f(A)\#f(B)\big) * \big(g(A)\#g(B)\big) \\ &\le \big(f(A) * g(A)\big)\#\big(f(B) * g(B)\big) = h(A)\#h(B), \end{aligned}$$

using [**2**, Corollary 8.1] and [**4**, Theorem 4.1].

For $x \ge 0$, the gamma function Γ has been characterised as one which satisfies the functional equation $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(1) = 1$ and is log-convex. For an account of this characterisation, the reader may refer to Artin [**3**]. In what follows, we give a characterisation of the gamma function for commuting matrices of order n, $n \in \mathbf{N}$, is arbitrary. The proof is a suitable adaption of the one given in Artin's text [**3**]. We first show that

$$\text{(i)}\ \ \Gamma(A+I) = A\Gamma(A), \quad \text{(ii)}\ \ \Gamma(I) = I, \quad \text{(iii)}\ \ \Gamma\left(\frac{A+B}{2}\right) \le \Gamma(A)\#\Gamma(B),$$

where A, B are positive definite hermitian matrices of order n satisfying $AB = BA$ and I denotes the identity matrix. Indeed, if $A = \sum_{i=1}^{n} \lambda_i E_i$ is the spectral resolution of A, where for $i = 1, 2, \ldots, n$, λ_i are the eigen values of A and E_i are the corresponding projections, then $A + I = \sum_{i=1}^{n} (\lambda_i + 1)E_i$. Consequently,

$$\Gamma(A+I) = \sum_{i=1}^{n} \Gamma(\lambda_i + 1)E_i = \sum_{i=1}^{n} \lambda_i\Gamma(\lambda_i)E_i = \Big(\sum_{i=1}^{n} \lambda_i E_i\Big)\Big(\sum_{i=1}^{n} \Gamma(\lambda_i)E_i\Big) = A\Gamma(A).$$

That $\Gamma(I) = I$ is obvious. We next assume that A and B commute. Then $B = \sum_{i=1}^{n} \mu_i Ei$ [**12**, Theorem 3.2.4.2]. Consequently,

$$\begin{aligned} \Gamma\left(\frac{A+B}{2}\right) &= \sum_{i=1}^{n} \Gamma\left(\frac{\lambda_i + \mu_i}{2}\right) E_i \le \sum_{i=1}^{n} \big(\Gamma(\lambda_i)\big)^{1/2}\big(\Gamma(\mu_i)\big)^{1/2} E_i \\ &= \left(\sum_{i=1}^{n} \big(\Gamma(\lambda_i)\big)^{1/2} E_i\right)\left(\sum_{i=1}^{n} \big(\Gamma(\mu_i)\big)^{1/2} E_i\right) = \Gamma(A)\#\Gamma(B). \end{aligned}$$

Theorem 2.4. *If a function f satisfies the following three conditions*:

(i) *The domain of definition of* f *is* $\mathbf{I}$ *and* f *satisfies the inequality* (3) *for commuting* A *and* B,

(ii) $f(A+I) = Af(A)$, *where* A *is a positive definite hermitian matrix of order* n,

(iii) $f(I) = I$, *where* I *denotes the identity matrix,*

then

$$\log f(A) = \lim_{n\to\infty} \left(A \log(nI) + \log(n!I) - \sum_{k=0}^{n} \log(A+kI) \right).$$

Proof. For an f satisfying the hypothesis,

$$f(nI) = f\big((n-1)I + I\big) = (n-1) f\big((n-1)I\big) = \cdots = (n-1)!I,$$

using (ii) and (iii) of the hypothesis. Assume that $0 < A \le I$ and n is an integer ≥ 2. Using monotonicity of the log function [**2**], it follows, on using (i) of the hypothesis, that

$$\log f\left(\frac{A+B}{2}\right) \le \frac{\log f(A) + \log f(B)}{2},$$

since $AB = BA$. Since $(n-1)I \le nI \le A + nI \le (n+1)I$, and $\log f$ is convex, we have

$$\begin{aligned} -\Big(\log f\big((n-1)I\big) - \log f(nI)\Big) &\le A^{-1/2}\Big(\log f(A+nI) - \log f(nI)\Big)A^{-1/2} \\ &\le \log f\big((n+1)I\big) - \log f(nI), \end{aligned}$$

using [**5**, Theorem 3.2]. Consequently,

$$\log\big((n-1)I\big) \le A^{-1/2}\big(\log f(A+nI) - \log f(nI)\big)A^{-1/2} \le \log(nI),$$

or

$$A\log\big((n-1)I\big) + \log\big((n-1)!I\big) \le \log f(A+nI) \le A\log(nI) + \log\big((n-1)!I\big).$$

Since

$$f(A+nI) = \big(A + (n-1)I\big)\big(A + (n-2)I\big)\cdots(A+I)Af(A),$$

the above inequality yields

$$\begin{aligned} &A\log\big((n-1)I\big) + \log\big((n-1)!I\big) - \sum_{k=0}^{n-1}\log(A+kI) \\ &\le \log f(A) \\ &\le A\log(nI) + \log\big((n-1)!I\big) - \sum_{k=0}^{n-1}\log(A+kI) \\ &= A\log(nI) + \log(n!I) + \log(A+nI) - \sum_{k=0}^{n}\log(A+kI) - \log(nI). \end{aligned}$$

Since the above inequality holds for all $n \geq 2$, we can replace n by $(n+1)$ on the left side. Thus

$$\begin{aligned} &A \log(nI) + \log(n!I) - \sum_{k=0}^{n} \log(A + kI) \\ &\quad \leq \log f(A) \\ &\quad \leq A \log(nI) + \log(n!I) - \sum_{k=0}^{n} \log(A + kI) + \log(I + A/n). \end{aligned}$$

Since $\log(I + A/n) \to 0$ as $n \to \infty$, we obtain

$$\log f(A) = \lim_{n\to\infty} \big(A \log(nI) + \log(n!I) - \sum_{k=0}^{n} \log(A + kI)\big).$$

3. We next turn our attention to the inequality (4), i.e.,

$$\log f\left(\frac{A+B}{2}\right) \leq \frac{\log f(A) + \log f(B)}{2},$$

where A and B are positive definite hermitian matrices of order n. In this case, we have the following theorem, whose proof is easy and is, therefore, not included.

Theorem 3.1. *The class of functions $f : \mathbf{I} \to \mathbf{I}$ satisfying* (4) *is closed under multiplication and taking of limits, provided the limits exist and are positive.*

Let $\mathcal{X}$ and $\mathcal{Y}$ be real Banach spaces. Let f be a map from an open subset E of the space $\mathcal{X}$ into the space $\mathcal{Y}$. We say that f is differentiable at $u \in E$ if there exists a linear map $\mathrm{D}f(u)$ from $\mathcal{X}$ to $\mathcal{Y}$ satisfying

$$\|f(u + x) - f(u) - \mathrm{D}f(u)(x)\| = o(\|x\|)$$

for all x. The linear map is called the derivative of f at u. We have

$$\mathrm{D}f(u)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(u + tx) \qquad (x \in \mathcal{X}).$$

If f is differentiable at all $u \in E$, we get a map $u \to \mathrm{D}f(u)$ from E in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$, the bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. The derivative of this map at u, if it exists, is called the second derivative of f at u and is denoted by $\mathrm{D}^2 f(u)$. Observe that $\mathrm{D}^2 f(u)$ is an element of $\mathcal{B}\big(\mathcal{X}, \mathcal{B}(\mathcal{X}, \mathcal{Y})\big)$. This latter space can be identified with the space of bounded bilinear maps from $\mathcal{X}$ into $\mathcal{Y}$ equiped with the norm

$$\|\phi\| = \inf\big\{\alpha : \|\phi(x_1, x_2)\| \leq \alpha \|x_1\|\,\|x_2\|\big\}.$$

In case $\mathcal{X} = \mathcal{Y} = \mathcal{B}(\mathcal{H})$, bounded linear maps on a Hilbert space $\mathcal{H}$ and $f(A) = A^{-1}$, where A is in the set of invertible operators,

$$\mathrm{D}f(A)(B) = -A^{-1}BA^{-1}$$

and

$$\mathrm{D}^2 f(A)(B_1, B_2) = A^{-1}B_1A^{-1}B_2A^{-1} + A^{-1}B_2A^{-1}B_1A^{-1}$$

for all B, B_1, B_2 in $\mathcal{B}(\mathcal{H})$.

The following analogue of the standard calculus results shall be used in the sequel. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be BANACH spaces, let g be a map from $\mathcal{X}$ to $\mathcal{Y}$, and f a map from $\mathcal{Y}$ to $\mathcal{Z}$. Let $\phi = f \circ g$. Then for all $x, x_1, x_2 \in \mathcal{X}$,

$$\mathrm{D}\phi(x)(x_1) = \Big(\mathrm{D}f\big(g(x)\big) \circ \mathrm{D}g(x)\Big)(x_1),$$

$$\mathrm{D}^2\phi(x)(x_1, x_2) = \mathrm{D}^2 f\big(g(x)\big)\big(\mathrm{D}g(x)(x_1), \mathrm{D}g(x)(x_2)\big) + \mathrm{D}f\big(g(x)\big)\big(\mathrm{D}^2 g(x)(x_1, x_2)\big).$$

For the above definitions, results and other related material, the reader may refer to FLETT [**8**].

Let $f : (0,\infty) \to (0,\infty)$ and A be a positive definite Hermitian matrix with spectral resolution $A = \sum\limits_{i=1}^{n} \mu_i E_i$. Then $f(A) = \sum\limits_{i=1}^{n} f(\mu_i)E_i = \sum_{i=1}^{n} a_i E_i$, where $a_i = f(\mu_i)$, $i = 1, 2, \dots, n$ and $\big(\lambda - f(A)\big)^{-1} = \sum\limits_{i=1}^{n} (\lambda - a_i)^{-1} E_i$. We shall use the symbols X, Y, Z for $\mathrm{D}f(A)(B_1)$, $\mathrm{D}f(A)(B_2)$ and $\mathrm{D}^2 f(A)(B_1, B_2)$ respectively.

Proposition 3.2. (i) $$\int_{-\infty}^{0} \big(\lambda - f(A)\big)^{-1} Z \big(\lambda - f(A)\big)^{-1} \mathrm{d}\lambda = \big(f(A)\big)^{-1/2} Z \big(f(A)\big)^{-1/2} + \sum_{i \neq j} \left(\frac{\log a_j - \log a_i}{a_j - a_i} - \frac{1}{\sqrt{a_i a_j}} \right) E_i Z E_j.$$

(ii) $$\begin{aligned}
&\int_{-\infty}^{0} \big(\lambda - f(A)\big)^{-1} X \big(\lambda - f(A)\big)^{-1} Y \big(\lambda - f(A)\big)^{-1} \mathrm{d}\lambda \\
&= -\frac{1}{2} \big(f(A)\big)^{-1/2} X \big(f(A)\big)^{-1} Y \big(f(A)\big)^{-1/2} \\
&+ \sum_{i=j\neq k} \left(\frac{\log a_k - \log a_i}{(a_k - a_i)^2} - \frac{1}{a_i(a_k - a_i)} + \frac{1}{2a_i^{3/2} a_k} \right) E_i X E_i Y E_k \\
&+ \sum_{i=k\neq j} \left(\frac{\log a_j - \log a_i}{(a_j - a_i)^2} - \frac{1}{a_i(a_j - a_i)} + \frac{1}{2a_i a_j} \right) E_i X E_j Y E_i \\
&+ \sum_{i\neq j=k} \left(\frac{\log a_i - \log a_j}{(a_i - a_j)^2} - \frac{1}{a_j(a_i - a_j)} + \frac{1}{2a_i^{1/2} a_j^{3/2}} \right) E_i X E_j Y E_j \\
&+ \sum_{i\neq j\neq k} \Bigg(\frac{\log a_i}{(a_i - a_j)(a_i - a_k)} + \frac{\log a_j}{(a_j - a_i)(a_j - a_k)} \\
&\qquad + \frac{\log a_k}{(a_k - a_i)(a_k - a_j)} + \frac{1}{2a_i^{1/2} a_j a_k^{1/2}} \Bigg) E_i X E_j Y E_k.
\end{aligned}$$

Proof. (i) $\int_{-\infty}^{0} \big(\lambda - f(A)\big)^{-1} Z \big(\lambda - f(A)\big)^{-1} \mathrm{d}\lambda = \int_{-\infty}^{0} \sum_{i,j} \frac{\mathrm{d}\lambda}{(\lambda - a_i)(\lambda - a_j)} E_i Z E_j$

$$
\begin{aligned}
&= \sum_i \int_{-\infty}^{0} \frac{\mathrm{d}\lambda}{(\lambda - a_i)^2} E_i Z E_i + \sum_{i \neq j} \int_{-\infty}^{0} \frac{\mathrm{d}\lambda}{(\lambda - a_i)(\lambda - a_j)} E_i Z E_j \\
&= \sum_i \frac{1}{a_i} E_i Z E_i + \sum_{i \neq j} \frac{\log a_i - \log a_j}{a_i - a_j} E_i Z E_j \\
&= \Big(\sum_i a_i^{-1/2} E_i\Big) Z \Big(\sum_i a_i^{-1/2} E_i\Big) + \sum_{i \neq j} \Big(\frac{\log a_i - \log a_j}{a_i - a_j} - a_i^{-1/2} a_j^{-1/2}\Big) E_i Z E_j \\
&= \big(f(A)\big)^{-1/2} Z \big(f(A)\big)^{-1/2} + \sum_{i \neq j} \Big(\frac{\log a_i - \log a_j}{a_i - a_j} - \frac{1}{\sqrt{a_i a_j}}\Big) E_i Z E_j.
\end{aligned}
$$

(ii) $\int_{-\infty}^{0} \big(\lambda - f(A)\big)^{-1} X \big(\lambda - f(A)\big)^{-1} Y \big(\lambda - f(A)\big)^{-1} \mathrm{d}\lambda$

$$
\begin{aligned}
&= \int_{-\infty}^{0} \sum_{i,j,k} \frac{\mathrm{d}\lambda}{(\lambda - a_i)(\lambda - a_j)(\lambda - a_k)} E_i X E_j Y E_k \\
&= \sum_i \int_{-\infty}^{0} \frac{\mathrm{d}\lambda}{(\lambda - a_i)^3} E_i X E_i Y E_i + \sum_{i = j \neq k} \int_{-\infty}^{0} \frac{\mathrm{d}\lambda}{(\lambda - a_i)^2 (\lambda - a_k)} E_i X E_i Y E_k \\
&+ \sum_{i = k \neq j} \int_{-\infty}^{0} \frac{\mathrm{d}\lambda}{(\lambda - a_i)^2 (\lambda - a_j)} E_i X E_j Y E_i + \sum_{i \neq j = k} \int_{-\infty}^{0} \frac{\mathrm{d}\lambda}{(\lambda - a_i)(\lambda - a_j)^2} E_i X E_j Y E_j \\
&+ \sum_{i \neq j \neq k} \int_{-\infty}^{0} \frac{\mathrm{d}\lambda}{(\lambda - a_i)(\lambda - a_j)(\lambda - a_k)} E_i X E_j Y E_k.
\end{aligned}
$$

Now

$$
\int_{-\infty}^{0} \frac{\mathrm{d}\lambda}{(\lambda - a_i)^3} = -\frac{1}{2a_i^2},
$$

$$
\begin{aligned}
\int_{-\infty}^{0} \frac{\mathrm{d}\lambda}{(\lambda - a_i)^2 (\lambda - a_k)} &= \int_{-\infty}^{0} \frac{1}{(a_i - a_k)^2} \Big(-\frac{1}{\lambda - a_i} + \frac{a_i - a_k}{(\lambda - a_i)^2} + \frac{1}{\lambda - a_k}\Big) \mathrm{d}\lambda \\
&= \frac{\log a_k - \log a_i}{(a_k - a_i)^2} - \frac{1}{a_i (a_k - a_i)}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{-\infty}^{0} \frac{\mathrm{d}d\lambda}{(\lambda - a_i)(\lambda - a_j)(\lambda - a_k)} = \int_{-\infty}^{0} \Big(&\frac{1}{(a_i - a_j)(a_i - a_k)} \frac{1}{(\lambda - a_i)} \\
&+ \frac{1}{(a_j - a_i)(a_j - a_k)} \frac{1}{(\lambda - a_j)} + \frac{1}{(a_k - a_i)(a_k - a_j)} \frac{1}{(\lambda - a_k)}\Big) \mathrm{d}\lambda.
\end{aligned}
$$

Observe that

$$\int_{-\infty}^{0} \frac{1}{(a_i-a_j)(a_i-a_k)}\left(\frac{1}{\lambda-a_i}-\frac{\lambda}{\lambda^2+1}\right)\mathrm{d}\lambda=\frac{\log a_i}{(a_i-a_j)(a_i-a_k)}.$$

It then follows that

$$\int_{-\infty}^{0} \frac{d\lambda}{(\lambda-a_i)(\lambda-a_j)(\lambda-a_k)}=\frac{\log a_i}{(a_i-a_j)(a_i-a_k)}+\frac{\log a_j}{(a_j-a_i)(a_j-a_k)}+\frac{\log a_k}{(a_k-a_i)(a_k-a_j)}.$$

Hence

$$\begin{aligned}
&\int_{-\infty}^{0} \big(\lambda-\big(f(A)\big)^{-1}X\big(\lambda-f(A)\big)^{-1}Y\big(\lambda-f(A)\big)^{-1}\mathrm{d}\lambda=-\frac{1}{2a_i^2}E_iXE_iYE_i\\
&\quad+\sum_{i=j\neq k}\left(\frac{\log a_k-\log a_i}{(a_k-a_i)^2}-\frac{1}{a_i(a_k-a_i)}\right)E_iXE_iYE_k\\
&\quad+\sum_{i=k\neq j}\left(\frac{\log a_j-\log a_i}{(a_j-a_i)^2}-\frac{1}{a_i(a_j-a_i)}\right)E_iXE_jYE_i\\
&\quad+\sum_{i\neq j=k}\left(\frac{\log a_i-\log a_j}{(a_i-a_j)^2}-\frac{1}{a_j(a_i-a_j)}\right)E_iXE_jYE_j\\
&\quad+\sum_{i\neq j\neq k}\left(\frac{\log a_i}{(a_i-a_j)(a_i-a_k)}+\frac{\log a_j}{(a_j-a_i)(a_j-a_k)}\right.\\
&\qquad\qquad\left.+\frac{\log a_k}{(a_k-a_i)(a_k-a_j)}\right)E_iXE_jYE_k\\
&\quad=-\frac{1}{2}\big(f(A)\big)^{-1/2}X\big(f(A)\big)^{-1}Y\big(f(A)\big)^{-1/2}\\
&\quad\sum_{i=j\neq k}\left(\frac{\log a_k-\log a_i}{(a_k-a_i)^2}-\frac{1}{a_i(a_k-a_i)}+\frac{1}{2a_i^{3/2}a_k}\right)E_iXE_iYE_k\\
&\quad+\sum_{i=k\neq j}\left(\frac{\log a_j-\log a_i}{(a_j-a_i)^2}-\frac{1}{a_i(a_j-a_i)}+\frac{1}{2a_ia_j}\right)E_iXE_jYE_i\\
&\quad+\sum_{i\neq j=k}\left(\frac{\log a_i-\log a_j}{(a_i-a_j)^2}-\frac{1}{a_j(a_i-a_j)}+\frac{1}{2a_i^{1/2}a_j^{3/2}}\right)E_iXE_jYE_j\\
&\quad+\sum_{i\neq j\neq k}\left(\frac{\log a_i}{(a_i-a_j)(a_i-a_k)}+\frac{\log a_j}{(a_j-a_i)(a_j-a_k)}\right.\\
&\qquad\qquad\left.+\frac{\log a_k}{(a_k-a_i)(a_k-a_j)}+\frac{1}{2a_i^{1/2}a_ja_k^{1/2}}\right)E_iXE_jYE_k.
\end{aligned}$$

Theorem 3.3. *Let $f : (0,\infty) \to (0,\infty)$ be a twice continuously differentiable function. Then f is log matrix convex function of order n iff*

$$
\begin{aligned}
&\big(f(A)\big)^{-1/2} Z \big(f(A)\big)^{-1/2} - \frac{1}{2}\big(f(A)\big)^{-1/2} X \big(f(A)\big)^{-1} Y \big(f(A)\big)^{-1/2} \\
&-\frac{1}{2}\big(f(A)\big)^{-1/2} Y \big(f(A)\big)^{-1} X \big(f(A)\big)^{-1/2} \\
&+\sum_{i=j\neq k}\left(\frac{\log a_k - \log a_i}{(a_k-a_i)^2} - \frac{1}{a_i(a_k-a_i)} + \frac{1}{2a_i^{3/2}a_k}\right)(E_iXE_iYE_k + E_iYE_iXE_k) \\
&+\sum_{i=k\neq j}\left(\frac{\log a_j - \log a_i}{(a_j-a_i)^2} - \frac{1}{a_i(a_j-a_i)} + \frac{1}{2a_ia_j}\right)(E_iXE_jYE_i + E_iYE_jXE_i) \\
&+\sum_{i\neq j=k}\left(\frac{\log a_i - \log a_j}{(a_i-a_j)^2} - \frac{1}{a_i(a_i-a_j)} + \frac{1}{2a_i^{1/2}a_j^{3/2}}\right)E_iXE_jYE_j + E_iYE_jXE_j) \\
&+\sum_{i\neq j\neq k}\left(\frac{\log a_i}{(a_i-a_j)(a_i-a_k)} + \frac{\log a_j}{(a_j-a_k)(a_j-a_i)} + \frac{\log a_k}{(a_k-a_i)(a_k-a_j)}\right. \\
&\qquad\qquad\qquad\left. + \frac{1}{2a_i^{1/2}a_ja_k^{1/2}}\right)(E_iXE_jYE_k + E_iYE_jXE_k)
\end{aligned}
$$

is positive definite matrix for all positive definite A and for all B_1 and B_2.

Proof. Observe that for $x > 0$,

$$\log x = \int_{-\infty}^{0}\left(\frac{1}{\lambda - x} - \frac{\lambda}{\lambda^2+1}\right)\mathrm{d}\lambda,$$

(see page 27, [**7**]). Consequently,

$$\log f(A) = \int_{-\infty}^{0}\left(\frac{1}{\lambda I - f(A)} - \frac{\lambda}{\lambda^2+1}\,I\right)\mathrm{d}\lambda,$$

where A is a positive definite Hermitian matrix of order n. Since f is twice differentiable, $\log f(A)$ is twice Frechet differentiable [**6**, Theorem 3.1]. Moreover,

$$\mathrm{D}\log f(A)(B) = \int_{-\infty}^{0}\big(\lambda I - f(A)\big)^{-1}\mathrm{D}f(A)(B)\big(\lambda I - f(A)\big)^{-1}\mathrm{d}\lambda$$

for all $B \in \mathcal{B}(\mathcal{H})$, and

$$
\begin{aligned}
\mathrm{D}^2\log f(A)(B_1,B_2) \;=\; &\int_{-\infty}^{0}\big(\lambda I - f(A)\big)^{-1}\mathrm{D}^2 f(A)(B_1,B_2)\big(\lambda I - f(A)\big)^{-1}\mathrm{d}\lambda \\
&+\int_{-\infty}^{0}\big(\lambda I - f(A)\big)^{-1}\Big(\mathrm{D}f(A)(B_2)\big(\lambda I - f(A)\big)^{-1}Df(A)(B_1) \\
&+\mathrm{D}f(A)(B_1)\big(\lambda I - f(A)\big)^{-1}\mathrm{D}f(A)(B_2)\Big)\big(\lambda I - f(A)\big)^{-1}\mathrm{d}\lambda
\end{aligned}
$$

for all $B_1, B_2 \in \mathcal{B}(\mathcal{H})$.

Since f is log matrix convex iff $\log f$ is matrix convex, the result follows on using Proposition 3.2 and convexity criterion [**6**, Theorem 3.2].

4. In this section we discuss the notion of log-convexity in the real BANACH space $\mathcal{X} = \mathcal{C}(\mathcal{M})$, the space of continuous real-valued functions on a compact HAUSDORFF space $\mathcal{M}$. Let $\mathcal{C}$ denotes the cone of positive functions in $\mathcal{X}$ and let $\mathcal{C}^*$ be the set of non-negative regular BOREL measures on $\mathcal{M}$. A function $f : \mathcal{C} \to \mathcal{C}$ satisfying the inequality

$$f\big((1-\theta)u + \theta v\big) \leq \big(f(u)\big)^{1-\theta}\big(f(v)\big)^{\theta}$$

for all $u, v \in \mathcal{C}$ and for all θ, $0 \leq \theta \leq 1$, is said to be log-convex. The following proposition is helpful in constructing examples of functions which satisfy the above said inequality. The motivation for the statement and the proof is the Proposition 3.1 [**15**].

Proposition 4.1. (i) *Let $f : \mathcal{C} \to \mathcal{C}$ be a mapping. Then f is log-convex iff for every $w^* \in \mathcal{C}^*$ and for every pair $u, v \in \mathcal{C}$, the map $\theta \to w^*\Big(f\big((1-\theta)u + \theta v\big)\Big)$ is log-convex.*

(ii) *If $f : \mathcal{C} \to \mathcal{C}$ and $g : \mathcal{C} \to \mathcal{C}$ are log-convex then so is $f + g$; and if, in addition, f is order preserving, $f \circ g$ is also log-convex.*

Proof. (i) Suppose f is log-convex. For $u, v \in \mathcal{C}$, $0 \leq \theta \leq 1$, consider the function $h : [0,1] \to \mathbf{R}^+$, where $\mathbf{R}^+ = \{x \in \mathbf{R} : x \geq 0\}$, defined by

$$h(\theta) = w^*\Big(f\big((1-\theta)u + \theta v\big)\Big).$$

We wish to show that $h(\theta)$ is log-convex. Indeed, for $0 \leq \theta_0 \leq \theta_1$, $0 \leq t \leq 1$, and $\theta_t = (1-t)\theta_0 + t\theta_1$,

$$\begin{aligned} h(\theta_t) &= w^*\Big(f\big(\big(1 - ((1-t)\theta_0 + t\theta_1)\big)u + ((1-t)\theta_0 + t\theta_1)v\big)\Big) \\ &= w^*\Big(f\big((1-t)\big((1-\theta_0)u + \theta_0 v\big) + t\big((1-\theta_1)u + \theta_1 v\big)\big)\Big) \\ &\leq w^*\Big(\big(f\big((1-\theta_0)u + \theta_0 v\big)\big)^{1-t}\big(f\big((1-\theta_1)u + \theta_1 v\big)\big)^{t}\Big) \\ &\leq \Big(w^*\big(f\big((1-\theta_0)u + \theta_0 v\big)\big)\Big)^{1-t}\Big(w^*\big(f\big((1-\theta_1)u + \theta_1 v\big)\big)\Big)^{t}, \end{aligned}$$

using the fact that w^* is a non-negative functional and HÖLDER's inequality [**10**, page 140].

Conversely, suppose that $h(\theta)$ defined above is log-convex for all choices of $u, v \in \mathcal{C}$ and $w^* \in \mathcal{C}^*$. Choose $w^*(z) = z(m)$ for a fixed $m \in \mathcal{M}$, one finds that

$$\Big(f\big((1-\theta)u + \theta v\big)\Big)(m) \leq \big(f(u)\big)^{1-\theta}(m)\big(f(v)\big)^{\theta}(m).$$

Since $m \in \mathcal{M}$ is arbitrary, the result follows.

(ii) Let $h(z) = (f+g)(z)$, $z \in \mathcal{C}$. Then

$$\begin{aligned} h\big((1-\theta)u+\theta v\big) &= f\big((1-\theta)u+\theta v\big) + g\big((1-\theta)u+\theta v\big) \\ &\le \big(f(u)\big)^{1-\theta}\big(f(v)\big)^{\theta} + \big(g(u)\big)^{1-\theta}\big(g(v)\big)^{\theta} \\ &\le \big(f(u)+g(u)\big)^{1-\theta}\big(f(v)+g(v)\big)^{\theta} \end{aligned}$$

for all pairs $u, v \in \mathcal{C}$ and $0 \le \theta \le 1$.

If, in addition, f is order preserving, then

$$\begin{aligned} f\Big(g\big((1-\theta)u+\theta v\big)\Big) &\le f\Big(\big(g(u)\big)^{1-\theta}\big(g(v)\big)^{\theta}\Big) \\ &\le f\big((1-\theta)g(u)+\theta g(v)\big) \\ &\le \Big(f\big(g(u)\big)\Big)^{1-\theta}\Big(f\big(g(v)\big)\Big)^{\theta} \end{aligned}$$

for all pairs $u, v \in \mathcal{C}$ and $0 \le \theta \le 1$.

Theorem 4.2. *Let $f : \mathcal{C} \to \mathcal{C}$ be a twice differentiable map. Then f is log-convex iff*

$$f(u)\mathrm{D}^2 f(u)(v_1, v_2) - \mathrm{D}f(u)(v_1)\mathrm{D}f(u)(v_2) \ge 0.$$

Proof. As in the proof of Theorem 3.3, we have

$$\log f(u) = \int_{-\infty}^{0} \left(\frac{1}{\lambda - f(u)} - \frac{\lambda}{\lambda^2+1}\right) \mathrm{d}\lambda.$$

Then

$$\mathrm{D}\log f(u)(v) = \int_{-\infty}^{0} \big(\lambda - f(u)\big)^{-1}\mathrm{D}f(u)(v_1)\big(\lambda - f(u)\big)^{-1}\mathrm{d}\lambda,$$

and

$$\begin{aligned} \mathrm{D}^2 \log f(u)(v_1, v_2) = & \int_{-\infty}^{0} \big(\lambda - f(u)\big)^{-1}\mathrm{D}^2 f(u)(v_1, v_2)\big(\lambda - f(u)\big)^{-1}\mathrm{d}\lambda \\ & + \int_{-\infty}^{0} \big(\lambda - f(u)\big)^{-1}\Big(\mathrm{D}f(u)(v_2)\big(\lambda - f(u)\big)^{-1}\mathrm{D}f(u)(v_1) \\ & + \mathrm{D}f(u)(v_1)\big(\lambda - f(u)\big)^{-1}\mathrm{D}f(u)(v_2)\Big)\big(\lambda - f(u)\big)^{-1}\mathrm{d}\lambda, \end{aligned}$$

on evaluating the integrals as in the case of real variable, since the constituents of the integrands commute. The result now follows as in Theorem 3.3.

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