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EVALUATION OF HIGHER-ORDER DERIVATIVES OF THE GAMMA FUNCTION

Junesang Choi, H. M. Srivastava

The authors present explicit formulas for the evaluation of higher-order derivatives of the familiar Gamma function. They also consider several applications of these explicit formulas. Further applications involving computation and evaluation of some families of definite integrals are also indicated.

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

The familiar Gamma function $\Gamma(z)$ is represented by the following Eulerian integral of the second kind:

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, \mathrm{d}t \quad \big(\Re(z) > 0\big) \tag{1.1}$$

and its relative, the Beta function $B(\alpha, \beta)$, by the following Eulerian integral of the first kind:

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\,\mathrm{d}t \quad \big(\Re(\alpha) > 0;\ \Re(\beta) > 0\big). \tag{1.2}$$

In view of the Weierstrass canonical product form for the Gamma function:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right)^{-1} e^{z/n} \right) \quad \big(z \in \mathbf{C} \setminus \{0, -1, -2, \dots\}\big), \tag{1.3}$$

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where γ denotes the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \cong 0.577\,215\,664\,901\,532\ \ldots, \tag{1.4}$$

the well-known relationship:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \big(\alpha,\ \beta \in \mathbf{C}\setminus\{0,\ -1,\ -2,\ \ldots\}\big) \tag{1.5}$$

can be used to continue the Beta function analytically as indicated above.

The Gamma function satisfies a simple functional relationship:

$$\Gamma(z+1) = z\Gamma(z) \quad \text{and} \quad \Gamma(n+1) = n!\ \big(n \in \mathbf{N}_0 := \mathbf{N}\cup\{0\};\ \mathbf{N} := \{1,\ 2,\ \ldots\}\big), \tag{1.6}$$

which motivated Leonhard Euler (1707–1783) to undertake the problem of interpolation of $n!$ between the positive integer values of n.

The Digamma (or Psi) function $\psi(z)$ defined by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t \tag{1.7}$$

is meromorphic on the whole complex z-plane with simple poles at $z = 0,\ -1,\ -2, \ldots$ (with residue 1). Its special values include

$$\psi(1) = -\gamma \quad \text{and} \quad \psi(1/2) = -\gamma - 2\,\log 2. \tag{1.8}$$

The Riemann Zeta function $\zeta(s)$ defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{1-2^{-s}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s} & \big(\Re(s) > 1\big), \\ \displaystyle\big(1-2^{1-s}\big)^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s} & \big(\Re(s) > 0;\ s \neq 1\big) \end{cases} \tag{1.9}$$

can indeed be continued meromorphically to the whole complex s-plane with a simple pole at $s = 1$ (with residue 1).

Euler evaluated $\zeta(2n)$ $(n \in \mathbf{N})$ explicitly, and we have

$$\begin{aligned} &\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \\ &\zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(10) = \frac{\pi^{10}}{93555}, \ldots, \end{aligned} \tag{1.10}$$

from which follow some relationships among them; these relationships are recalled here for later use:

$$\begin{aligned} &\big(\zeta(2)\big)^2 = \frac{5}{2}\,\zeta(4), \quad \zeta(2)\zeta(4) = \frac{7}{4}\,\zeta(6), \quad \zeta(2)\zeta(6) = \frac{5}{3}\,\zeta(8), \\ &\big(\zeta(4)\big)^2 = \frac{7}{6}\,\zeta(8), \quad \zeta(2)\zeta(8) = \frac{33}{20}\,\zeta(10), \quad \zeta(4)\zeta(6) = \frac{11}{10}\,\zeta(10). \end{aligned} \tag{1.11}$$

In fact, the evaluation of $\zeta(2n)$ $(n \in \mathbf{N})$ can be carried out by means of the known formula:

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2\,(2n)!} \quad \big(n \in \mathbf{N}_0\big), \tag{1.12}$$

where B_n denotes the BERNOULLI numbers defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\, z^n \quad \big(|z| < 2\pi\big), \tag{1.13}$$

for which we have the recursion formula:

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k \quad \big(n \in \mathbf{N} \setminus \{1\}\big) \tag{1.14}$$

and the following *numerical* values:

$$\begin{aligned} &B_0 = 1,\ \ B_1 = -\frac{1}{2}\,,\ \ B_2 = \frac{1}{6}\,,\ \ B_4 = -\frac{1}{30}\,,\ \ B_6 = \frac{1}{42}\,,\\ &B_8 = -\frac{1}{30}\,,\ \ B_{10} = \frac{5}{66}\,, \ldots, \text{ and } \quad B_{2n+1} = 0 \quad (n \in \mathbf{N}). \end{aligned} \tag{1.15}$$

We may recall here a known recursion formula for $\zeta(2n)$:

$$\zeta(2n) = \frac{2}{2n+1} \sum_{k=1}^{n-1} \zeta(2k)\zeta(2n-2k) \quad \big(n \in \mathbf{N} \setminus \{1\}\big), \tag{1.16}$$

which can also be used to evaluate $\zeta(2n)$ $\big(n \in \mathbf{N} \setminus \{1\}\big)$.

The Polygamma functions are defined by

$$\psi^{(n)}(z) := \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\big(\log \Gamma(z)\big) \ \ (n \in \mathbf{N}) \quad \text{and} \quad \psi^{(0)}(z) = \psi(z), \tag{1.17}$$

from which it is easy to derive the relationship:

$$\psi^{(n)}(z) = (-1)^{n+1} n!\, \zeta(n+1, z) \quad (n \in \mathbf{N}), \tag{1.18}$$

with the generalized (or HURWITZ) Zeta function $\zeta(s, a)$ defined by

$$\zeta(s, a) := \sum_{k=0}^{\infty} (k + a)^{-s} \quad \big(\Re\,(s) > 1;\ \ a \neq 0, -1, -2, \ldots\big). \tag{1.19}$$

Clearly, we have

$$\zeta(s, 1) = \zeta(s) = (2^s - 1)^{-1}\, \zeta\left(s, \frac{1}{2}\right). \tag{1.20}$$

It is not difficult to derive the following identity from the definition (1.19):

$$\zeta(s,a) = \zeta(s,n+a) + \sum_{k=0}^{n-1}(k+a)^{-s} \quad (n \in \mathbf{N}). \tag{1.21}$$

The function $\log \Gamma(1+z)$ is known to have the following Maclaurin series expansion:

$$\log \Gamma(1+z) = -\gamma z + \sum_{n=2}^{\infty} (-1)^n \zeta(n)\, \frac{z^n}{n} \quad (|z| < 1). \tag{1.22}$$

For the function $\Gamma(1+z)$ itself, it is also known that (*cf.*, *e.g.*, [**2**, p. 27])

$$\Gamma(1+z) = \sum_{n=0}^{\infty} a_n\, z^n \quad \big(|z| < 1\big), \tag{1.23}$$

where

$$a_0 = 1 \quad \text{and} \quad n\, a_n = -\gamma\, a_{n-1} + \sum_{k=2}^{n} (-1)^k\, a_{n-k}\, \zeta(k), \tag{1.24}$$

an empty sum being interpreted (as usual) to be nil.

The main object of this paper is to present explicit formulas for some higher-order derivatives of the Gamma function $\Gamma(z)$ at $z = 1$ and $z = 1/2$. We then show how these explicit formulas can be applied in order to evaluate several families of definite integrals.

2. EVALUATIONS OF $\Gamma^{(n)}(1)$ AND $\Gamma^{(n)}(1/2)$

Upon replacing z by $z-1$ in (1.23), we readily obtain

$$a_n = \frac{\Gamma^{(n)}(1)}{n!} \quad (n \in \mathbf{N}_0). \tag{2.1}$$

Formula (2.1) and (1.24), together, immediately yield the recursion formula:

$$\Gamma^{(n+1)}(1) = -\gamma\, \Gamma^{(n)}(1) + n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(n-k)!}\, \zeta(k+1)\, \Gamma^{(n-k)}(1) \quad (n \in \mathbf{N}_0), \tag{2.2}$$

which can alternatively be derived directly from the definition (1.17) with $n = 1$ (and z replaced by $z+1$) by applying such results of Section 1 as the relationship (1.18) with $z = 1$.

Similarly, we obtain another recursion formula:

$$\Gamma^{(n+1)}(1/2) = -\delta\,\Gamma^{(n)}(1/2) + n! \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(n-k)!}\left(2^{k+1}-1\right)\zeta(k+1)\,\Gamma^{(n-k)}(1/2) \qquad (n \in \mathbf{N}_0;\ \delta := \gamma + 2\log 2). \tag{2.3}$$

Now, using the formulas in Section 1, we compute $\Gamma^{(n)}(1)$ explicitly for the first ten derivatives:

$$\Gamma'(1) = -\gamma = \psi(1); \quad \Gamma^{(2)}(1) = \gamma^2 + \zeta(2); \quad \Gamma^{(3)}(1) = -\gamma^3 - 3\gamma\zeta(2) - 2\zeta(3),$$

which is the *corrected* version of a formula given by Campbell [**1**, p. 25];

$$\Gamma^{(4)}(1) = \gamma^4 + 6\,\gamma^2\,\zeta(2) + 8\gamma\zeta(3) + \frac{27}{2}\,\zeta(4);$$

$$\Gamma^{(5)}(1) = -\gamma^5 - 10\,\gamma^3\,\zeta(2) - 20\,\gamma^2\,\zeta(3) - \frac{135}{2}\,\gamma\zeta(4) - 20\,\zeta(2)\,\zeta(3) - 24\,\zeta(5);$$

$$\begin{aligned}\Gamma^{(6)}(1) = {} & \gamma^6 + 15\,\gamma^4\,\zeta(2) + 40\,\gamma^3\,\zeta(3) + \frac{405}{2}\,\gamma^2\zeta(4) \\ & +120\,\gamma\,\zeta(2)\,\zeta(3) + 144\,\gamma\,\zeta(5) + 40\big(\zeta(3)\big)^2 + \frac{2745}{8}\,\zeta(6);\end{aligned}$$

$$\begin{aligned}\Gamma^{(7)}(1) = {} & -\gamma^7 - 21\,\gamma^5\,\zeta(2) - 70\,\gamma^4\,\zeta(3) - \frac{945}{2}\,\gamma^3\zeta(4) \\ & -420\,\gamma^2\,\zeta(2)\,\zeta(3) - 504\,\gamma^2\,\zeta(5) - 280\,\gamma\,\big(\zeta(3)\big)^2 - \frac{19215}{8}\,\gamma\zeta(6) \\ & -945\,\zeta(3)\,\zeta(4) - 504\,\zeta(2)\,\zeta(5) - 720\,\zeta(7);\end{aligned}$$

$$\begin{aligned}\Gamma^{(8)}(1) = {} & \gamma^8 + 28\,\gamma^6\zeta(2) + 112\,\gamma^5\,\zeta(3) + 945\,\gamma^4\zeta(4) \\ & +1120\,\gamma^3\,\zeta(2)\,\zeta(3) + 1344\,\gamma^3\,\zeta(5) + 1120\,\gamma^2\,\big(\zeta(3)\big)^2 + \frac{19215}{2}\,\gamma^2\,\zeta(6) \\ & +7560\,\gamma\,\zeta(3)\,\zeta(4) + 4032\,\gamma\,\zeta(2)\,\zeta(5) + 5760\,\gamma\,\zeta(7) + 1120\,\zeta(2)\big(\zeta(3)\big)^2 \\ & +2688\,\zeta(3)\,\zeta(5) + \frac{132405}{8}\,\zeta(8);\end{aligned}$$

$$\begin{aligned}\Gamma^{(9)}(1) = {} & -\gamma^9 - 36\,\gamma^7\,\zeta(2) - 168\,\gamma^6\,\zeta(3) - 1701\,\gamma^5\,\zeta(4) \\ & -2520\,\gamma^4\,\zeta(2)\,\zeta(3) - 3024\,\gamma^4\,\zeta(5) - 3360\,\gamma^3\,\big(\zeta(3)\big)^2 - \frac{57645}{2}\,\gamma^3\,\zeta(6) \\ & -34020\,\gamma^2\,\zeta(3)\,\zeta(4) - 18144\,\gamma^2\zeta(2)\,\zeta(5) - 25920\,\gamma^2\,\zeta(7) \\ & -10080\,\gamma\,\zeta(2)\big(\zeta(3)\big)^2 - 24192\,\gamma\,\zeta(3)\,\zeta(5) - \frac{1191645}{8}\,\gamma\zeta(8) \\ & -57645\,\zeta(3)\,\zeta(6) - 40824\,\zeta(4)\,\zeta(5) - 25920\,\zeta(2)\,\zeta(7) \\ & -2240\,\big(\zeta(3)\big)^3 - 40320\,\zeta(9);\end{aligned}$$

$$\Gamma^{(10)}(1) = \gamma^{10} + 45\,\gamma^8\,\zeta(2) + 240\,\gamma^7\,\zeta(3) + 2835\,\gamma^6\,\zeta(4) + 5040\,\gamma^5\,\zeta(2)\,\zeta(3)$$
$$+6048\,\gamma^5\,\zeta(5) + 8400\,\gamma^4\,\big(\zeta(3)\big)^2 + \frac{288225}{4}\,\gamma^4\,\zeta(6) + 113400\,\gamma^3\,\zeta(3)\,\zeta(4)$$
$$+60480\,\gamma^3\,\zeta(2)\,\zeta(5) + 86400\,\gamma^3\,\zeta(7) + 50400\,\gamma^2\,\zeta(2)\big(\zeta(3)\big)^2 + 120960\,\gamma^2\,\zeta(3)\,\zeta(5)$$
$$+\frac{5958225}{8}\,\gamma^2\zeta(8) + 576450\,\gamma\,\zeta(3)\,\zeta(6) + 408240\,\gamma\,\zeta(4)\,\zeta(5) + 259200\,\gamma\,\zeta(2)\,\zeta(7)$$
$$+22400\,\gamma\big(\zeta(3)\big)^3 + 403200\,\gamma\,\zeta(9) + 113400\,\big(\zeta(3)\big)^2\zeta(4) + 120960\,\zeta(2)\,\zeta(3)\,\zeta(5)$$
$$+72576\big(\zeta(5)\big)^2 + 172800\,\zeta(3)\,\zeta(7) + \frac{42329385}{32}\,\zeta(10).$$

The corresponding problem for $\Gamma^{(n)}(1/2)$ $(n = 1, \ldots, 10)$ yields

$$\Gamma'(1/2) = -\,\delta\,\sqrt{\pi}\,, \qquad \frac{1}{\sqrt{\pi}}\,\Gamma^{(2)}(1/2) = \delta^2 + 3\,\zeta(2),$$
$$\frac{1}{\sqrt{\pi}}\,\Gamma^{(3)}(1/2) = -\,\delta^3 - 9\,\delta\,\zeta(2) - 14\,\zeta(3),$$

all three of which are also recorded by Campbell [**1**, p. 25];

$$\frac{1}{\sqrt{\pi}}\,\Gamma^{(4)}(1/2) = \delta^4 + 18\,\delta^2\zeta(2) + 56\,\delta\,\zeta(3) + \frac{315}{2}\,\zeta(4);$$

$$\frac{1}{\sqrt{\pi}}\,\Gamma^{(5)}(1/2) = -\,\delta^5 - 30\,\delta^3\zeta(2) - 140\,\delta^2\zeta(3)$$
$$-\frac{1575}{2}\,\delta\,\zeta(4) - 420\,\zeta(2)\,\zeta(3) - 744\,\zeta(5);$$

$$\frac{1}{\sqrt{\pi}}\,\Gamma^{(6)}(1/2) = \delta^6 + 45\,\delta^4\zeta(2) + 280\,\delta^3\zeta(3) + \frac{4725}{2}\,\delta^2\zeta(4)$$
$$+2520\,\delta\,\zeta(2)\,\zeta(3) + 4464\,\delta\,\zeta(5) + 1960\big(\zeta(3)\big)^2 + \frac{131355}{8}\,\zeta(6);$$

$$\frac{1}{\sqrt{\pi}}\,\Gamma^{(7)}(1/2) = -\,\delta^7 - 63\,\delta^5\zeta(2) - 490\,\delta^4\zeta(3) - \frac{11025}{2}\,\delta^3\zeta(4) - 8820\,\delta^2\zeta(2)\,\zeta(3)$$
$$-15624\,\delta^2\zeta(5) - 13720\,\delta\big(\zeta(3)\big)^2 - \frac{919485}{8}\,\delta\,\zeta(6)$$
$$-77175\,\zeta(3)\,\zeta(4) - 46872\,\zeta(2)\,\zeta(5) - 91440\,\zeta(7);$$

$$\frac{1}{\sqrt{\pi}}\,\Gamma^{(8)}(1/2) = \delta^8 + 84\,\delta^6\zeta(2) + 784\,\delta^5\zeta(3) + 11025\,\delta^4\zeta(4) + 23520\,\delta^3\zeta(2)\,\zeta(3)$$
$$+41664\,\delta^3\zeta(5) + 54880\,\delta^2\big(\zeta(3)\big)^2 + 377055\,\delta^2\zeta(6)$$
$$+617400\,\delta\,\zeta(3)\,\zeta(4) + 374976\,\delta\,\zeta(2)\,\zeta(5) + 731520\,\delta\,\zeta(7)$$
$$+164640\,\zeta(2)\,\big(\zeta(3)\big)^2 + 583296\,\zeta(3)\,\zeta(5) + \frac{25859925}{8}\,\zeta(8);$$

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\,\Gamma^{(9)}(1/2) = \; & -\delta^9 - 108\,\delta^7\,\zeta(2) - 1176\,\delta^6\,\zeta(3) - 19845\,\delta^5\,\zeta(4)\\ & -52920\,\delta^4\,\zeta(2)\zeta(3) - 93744\,\delta^4\,\zeta(5) - 164640\,\delta^3\,\big(\zeta(3)\big)^2\\ & -\frac{2758455}{2}\,\delta^3\,\zeta(6) - 2778300\,\delta^2\,\zeta(3)\zeta(4) - 1687392\,\delta^2\,\zeta(2)\zeta(5)\\ & -3291840\,\delta^2\,\zeta(7) - 1481760\,\delta\,\zeta(2)\,\big(\zeta(3)\big)^2 - 5249664\,\delta\,\zeta(3)\zeta(5)\\ & -\frac{232739325}{8}\,\delta\,\zeta(8) - 19309185\,\zeta(3)\zeta(6) - 14764680\,\zeta(4)\zeta(5)\\ & -768320\,\big(\zeta(3)\big)^3 - 9875520\,\zeta(2)\zeta(7) - 20603520\,\zeta(9);\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{\pi}}\,\Gamma^{(10)}(1/2) = \; & \delta^{10} + 135\,\delta^8\,\zeta(2) + 1680\,\delta^7\,\zeta(3) + 33075\,\delta^6\,\zeta(4)\\ & +105840\,\delta^5\,\zeta(2)\,\zeta(3) + 187488\,\delta^5\,\zeta(5) + 411600\,\delta^4\,\big(\zeta(3)\big)^2\\ & +\frac{13792275}{4}\,\delta^4\,\zeta(6) + 9261000\,\delta^3\,\zeta(3)\,\zeta(4) + 5624640\,\delta^3\,\zeta(2)\zeta(5)\\ & +10972800\,\delta^3\,\zeta(7) + 7408800\,\delta^2\,\zeta(2)\,\big(\zeta(3)\big)^2\\ & +26248320\,\delta^2\,\zeta(3)\zeta(5) + \frac{1163696625}{8}\,\delta^2\,\zeta(8)\\ & +193091850\,\delta\,\zeta(3)\zeta(6) + 147646800\,\delta\,\zeta(4)\zeta(5)\\ & +7683200\,\delta\,\big(\zeta(3)\big)^3 + 98755200\,\delta\,\zeta(2)\zeta(7) + 206035200\,\delta\,\zeta(9)\\ & +64827000\,\big(\zeta(3)\big)^2\,\zeta(4) + 78744960\,\zeta(2)\zeta(3)\zeta(5)\\ & +153619200\,\zeta(3)\zeta(7) + 69745536\,\big(\zeta(5)\big)^2 + \frac{33466588155}{32}\,\zeta(10),\end{aligned}$$

δ being defined already with (2.3).

3. APPLICATIONS

Replacing z by $z+1$ in (1.1) and differentiating both sides of the resulting equation n times with respect to z, and then setting $z=0$, we obtain

$$\Gamma^{(n)}(1) = \int_0^\infty e^{-t}\,(\log t)^n\,\mathrm{d}t \quad (n \in \mathbf{N}_0). \tag{3.1}$$

Similarly, we have

$$\Gamma^{(n)}(1/2) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,(\log t)^n\,\mathrm{d}t \quad (n \in \mathbf{N}_0). \tag{3.2}$$

Both (3.1) and (3.2) are recorded by Campbell [**1**, p. 25].

We next recall another explicit form of the Gamma function $\Gamma(z)$ (see [**3**, p. 243]):

$$\Gamma(z) = \int_0^1 \left(\log \frac{1}{t}\right)^{z-1} \mathrm{d}t \quad \big(\Re(z) > 0\big), \tag{3.3}$$

which follows immediately from (1.1) by setting $t = \log(1/\tau)$.

Making use of (3.3) instead of (1.1) $\big($or, alternatively, by setting $t = \log(1/\tau)$ in (3.1) and (3.2)$\big)$, we obtain

$$\Gamma^{(n)}(1) = \int_0^1 \left(\log\left(\log \frac{1}{t}\right)\right)^n \mathrm{d}t \quad (n \in \mathbf{N}_0) \tag{3.4}$$

and

$$\Gamma^{(n)}(1/2) = \int_0^1 \left(\log \frac{1}{t}\right)^{-\frac{1}{2}} \left(\log\left(\log \frac{1}{t}\right)\right)^n \mathrm{d}t \quad (n \in \mathbf{N}_0). \tag{3.5}$$

By using the computations of Section 2, each of the integral formulas (3.1), (3.2), (3.4), and (3.5) can be expressed explicitly in terms of γ and $\zeta(n)$, at least for $n = 1, \ldots, 10$.

We now recall a familiar integral formula:

$$\int_0^\infty \frac{t^{\lambda-1}}{1+t}\,\mathrm{d}t = \frac{\pi}{\sin\lambda\pi} = \Gamma(\lambda)\,\Gamma(1-\lambda) \quad \big(0 < \Re(\lambda) < 1\big), \tag{3.6}$$

which is a special case of the Beta integral (1.2) when

$$\alpha = 1 - \beta = \lambda \quad \left(\text{and} \quad t = \frac{\tau}{1+\tau}\right).$$

Differentiating both sides of (3.6) n times with respect to λ, if we employ the familiar Leibniz rule for differentiating the Γ-product and set $\lambda = 1/2$ in the resulting equation, we obtain

$$\begin{aligned} \int_0^\infty \frac{(\log t)^n}{(1+t)\sqrt{t}}\,\mathrm{d}t &:= I(n) \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \Gamma^{(n-k)}(1/2)\,\Gamma^{(k)}(1/2) \quad (n \in \mathbf{N}_0). \end{aligned} \tag{3.7}$$

It is interesting to observe that

$$\int_0^\infty \frac{(\log t)^{2n+1}}{(1+t)\sqrt{t}}\,\mathrm{d}t = 0 \quad (n \in \mathbf{N}_0), \tag{3.8}$$

which can also be shown as follows by separating the sum on the right-hand side of (3.7) into two parts:

$$\begin{aligned} I(2n+1) &= \left(\sum_{k=0}^{n} + \sum_{k=n+1}^{2n+1}\right) (-1)^k \binom{2n+1}{k} \Gamma^{(2n+1-k)}(1/2)\,\Gamma^{(k)}(1/2) \\ &= \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \Gamma^{(2n+1-k)}(1/2)\,\Gamma^{(k)}(1/2) \\ &\qquad + \sum_{k=0}^{n} (-1)^{n+1-k} \binom{2n+1}{n-k} \Gamma^{(n-k)}(1/2)\,\Gamma^{(n+1+k)}(1/2), \end{aligned}$$

which, upon reversing the order of the latter sum, yields

$$\begin{aligned} I(2n+1) &= \left(1+(-1)^{2n+1}\right) \sum_{k=0}^{n} (-1)^k \binom{2n+1}{k} \Gamma^{(2n+1-k)}(1/2)\,\Gamma^{(k)}(1/2) \\ &= 0. \end{aligned} \tag{3.9}$$

On the other hand, in the case of even integers, we have

$$\begin{aligned} I(2n) &= \int_0^\infty \frac{(\log\, t)^{2n}}{(1+t)\sqrt{t}}\, \mathrm{d}t = 2\sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \Gamma^{(2n-k)}(1/2)\,\Gamma^{(k)}(1/2) \\ &\qquad + (-1)^n \binom{2n}{n} \left(\Gamma^{(n)}(1/2)\right)^2 \quad (n \in \mathbf{N}_0), \end{aligned} \tag{3.10}$$

which, in view of the results already presented in Section 2, readily yields the following special cases:

$$\int_0^\infty \frac{(\log\, t)^2}{(1+t)\sqrt{t}}\, \mathrm{d}t = \pi^3, \tag{3.11}$$

$$\int_0^\infty \frac{(\log\, t)^4}{(1+t)\sqrt{t}}\, \mathrm{d}t = 5\,\pi^5, \tag{3.12}$$

$$\int_0^\infty \frac{(\log\, t)^6}{(1+t)\sqrt{t}}\, \mathrm{d}t = 61\,\pi^7, \tag{3.13}$$

$$\int_0^\infty \frac{(\log\, t)^8}{(1+t)\sqrt{t}}\, \mathrm{d}t = 1385\,\pi^9, \tag{3.14}$$

and

$$\int_0^\infty \frac{(\log\, t)^{10}}{(1+t)\sqrt{t}}\, \mathrm{d}t = 50521\,\pi^{11}. \tag{3.15}$$

We conclude by remarking that the evaluations presented here may find further applications involving (for example) computation and evaluation of some families of definite integrals.

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Junesang Choi
Department of Mathematics,
College of Natural Sciences.
Dongguk University,
Kyongju 780-714,
Korea
E-Mail: **junesang@mail.dongguk.ac.kr**

H. M. Srivastava
Department of Mathematics and Statistics,
University of Victoria,
Victoria, British Columbia V8W 3P4,
Canada
E-Mail: **harimsri@math.uvic.ca**