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EXAMPLES OF CANONICAL AND WEAK CANONICAL REPRESENTATION OF STOCHASTIC PROCESSES

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In this paper we consider the second-order real-valued stochastic processes $x(t), t \in (a,b) \subset \mathbf{R}$, with $Ex(t) = 0$, for each t, and analyze some characteristic examples of such processes and the difference between their canonical representation in a Cramer sense, and their representation which satisfies a weaker condition. Also, we consider a spectral multiplicity of these processes.

Let $x(t)$, $t \in (a,b) \subset \mathbf{R}$ be a second-order real-valued process with $Ex(t) = 0$ for each t. Let $H(x,t)$ be the linear closure generated by $x(s), s \in (a,t]$ in the HILBERT space H of all random variables with finite variance ($Ex^2(t) < \infty$). We will suppose that $x(t), t \in (a,b)$ is continuous left and purely nondeterministic (i.e. $\bigcap_{t>a} H(x,t) = 0$). It is well known (see [**1**]) that there is a representation :

$$x(t) = \sum_{n=1}^{N} \int_a^t g_n(t,u)\, \mathrm{d}z_n(u), \quad u = t,\ t \in (a,b), \tag{1}$$

where:

1. The processes $z_n(u), n = 1, \dots, N$ are mutually orthogonal with orthogonal increments such that $Ez_n(u) = 0$ and $Ez_n^2(u) = F_n(u)$, where $F_n(u), n = 1, \dots, N$ are non decreasing functions left continuous everywhere on (a,b).
2. The non-random functions $g_n(t,u), u \le t$, are such that:

$$Ex^2(t) = \sum_{n=1}^{N} \int_a^t g_n^2(t,u)\, \mathrm{d}F_n(u) < \infty, \text{ for each } t \in (a,b).$$

3. $dF_1 > dF_2 > \cdots > dF_n$, where the relation $>$ means absolute continuity between measures.
4. $H(x,t) = \sum_{n=1}^{N} \oplus H(z_n,t),\ t \in (a,b)$.

The expansion (1) satisfying the conditions 1, 2, 3 and 4 is the *canonical representation* or CRAMER representation for the process $x(t)$. The number N

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(finite or infinite) is called the *multiplicity* of $x(t)$, and N is uniquely determined by the process $x(t)$. But, the processes $z_n(u)$ and the functions $g_n(t,u)$ are not uniquely determined.

For finite N, the representation (1) is canonical if and only if the family of functions $\{g_n(t,u)\}_{n=1,\dots,N}$ is complete in the space $L^2(dF(u))$, $dF = \{dF_n\}_{n=1,\dots,N}$ (see Lemma 3.1 of [**1**]).

If condition 4 in the representation (1) is replaced by a weaker condition :

$$P_{H(x,s)}x(t) = \sum_{n=1}^{N} \int_a^s g_n(t,u)\,\mathrm{d}z_n(u), \text{ for all } s \le t, s,t \in (a,b),$$

where $P_{H(x,s)}$ is the projection operator on $H(x,s)$, then (1) is said to be a *weak-canonical representation* of $x(t)$.

It is clear that the kernel $\{g_n(t,u)\}_{n=1,\dots,N}$ of the weak-canonical representation need not be complete in the space $L^2(dF(u))$. Moreover, the following statement is valid:

Every canonical representation is the weak-canonical one (see [**1**], p. 10) *and the converse need not hold.* This fact is shown in the next simple example.

Example 1. Let $x(t), a \le t \le b$, be represented by $x(t) = \int\limits_a^s g(u)\,\mathrm{d}z(u)$, where $z(u)$ is a process satisfying the condition 1, and $g(u)$ is a function from $L^2(dF(u))$, such that $g(u) = 0$ on an arbitrary set A^C of positive dF measure from (a,b). This representation is weak-canonical because $x(t) - x(s)$ is orthogonal on $x(s)$ for all $s < t$, and $P_{H(x,s)}x(t) = x(s)$. But this representation is not canonical because $H(z,t) \not\subset H(x,t)$. Instead of $x(t) = \int\limits_a^t g(u)\,\mathrm{d}z(u)$, we may consider the process

$$x^*(t) = \int_a^t g(u)\,\mathrm{d}z^*(u),$$

with a canonical representation, where $z^*(t) = \int\limits_a^t \chi_A(u)\,\mathrm{d}z(u)$, and $\chi_A(u)$ is a characteristic function of A.

It is interesting that there exist many processes $x(t) = \int\limits_a^t g(t,u)\,\mathrm{d}z(u)$, given by one integral representation not satisfying property to be canonical or weak-canonical.

Example 2. All processes given by:

$$x_k(t) = \int_0^t [-kt + (k+1)u]\,\mathrm{d}z(u),\ u \le t,\ u,t \in [0,b], k \in \mathbf{N},$$

$E[z^2(u)] = u$, have no complete kernels $g_k(t,u) = -k \cdot t + (k+1) \cdot u$, in $L^2(du)$. For each process $x_k(t)$, there exists a process $y_k(t) = \int_0^t u^{k-1}\,\mathrm{d}z(u) \in H(z,t)$, such that

$$E[y_k \cdot x_k(s)] = \int_0^s g_k(s,u)u^{k-1}\,\mathrm{d}u = 0, \text{ for all } s \in (0,t].$$

So these representations are not canonical. They are not weak-canonical too. Even if $x_k(t) - \int_0^s g_k(t,u)\,\mathrm{d}z(u)$ is orthogonal on $x_k(s)$ for all $s < t, k \in \mathbf{N}$, it is easy to see that $y_k(s)$ is not orthogonal to $\int_0^s g_k(t,u)\,\mathrm{d}z(u)$:

$$E[y_k(s) \cdot P_{H(x,s)}x_k(t)] = \int_0^s g_k(t,u)u^{k-1}\,\mathrm{d}u = -ts^k + s^{k+1} \neq 0, \text{ for all } s \in (0,t),$$

so $P_{H(x,s)}x_k(t) = \int_0^s g_k(t,u)\,\mathrm{d}z(u) \notin H(x,s)$.

Example 3. The similar holds for processes given in the form

$$x(t) = \int_0^t \left[p_0 + p_1\frac{u}{t} + \cdots + p_n\frac{u^n}{t^n}\right]\mathrm{d}z(u),\ u \leq t,\ u,t \in [0,b], n \in \mathbf{N},$$

$E[z^2(u)] = u, p_i = \text{const.}, i = 0,1,\ldots,n$. For certain p_i and n we may find $y_k = \int_0^b u^k\,\mathrm{d}z(u) \in H(z)$, orthogonal on $x(t)$ for all $t \in [0,b]$, solving the equation

$$\int_0^t \left[p_0 + p_1\frac{u}{t} + \cdots + p_n\frac{u^n}{t^n}\right]u^k\,\mathrm{d}u = 0,$$

by unknown $k \in \mathbf{N}$. For example $x(t) = \int_0^t \left[3 - 12\frac{u}{t} + 10\frac{u^2}{t^2}\right]\mathrm{d}z(u),\ u \leq t,\ u,t \in [0,b], E[z^2(u)] = u$, has representation non canonical because there exist $y_1 = \int_0^b u\,\mathrm{d}z(u) \in H(z)$, and $y_2 = \int_0^b u^2 dz(u) \in H(z)$, which are orthogonal on $x(t)$ for all $t \in [0,b]$. By the same reason like above we may prove that this representation is not weak-canonical too. However the multiplicity of this process is equal to one because its covariance function $B(s,t) = \min\{s,t\}$, so $x(t)$ is again a WIENER process as $x(t)$.

If the process $x(t)$ given by a representation (1):

$$x(t) = \sum_{i=1}^{N}\int_a^t g_n(t,u)\,\mathrm{d}z_n(u),\ u \leq t,\ t \in (a,b),$$

where $N \geq 2$, then, it is known (see [**3**]) that the next statement is valid:
If the process $x(t)$ is given by a canonical (weak-canonical) representation (1), *then each process*

$$x_n(t) = \int_a^t g_n(t,u)\, dz_n(u),\ u \leq t,\ t \in (a,b),$$

$n = 1, \ldots, N$, *has a canonical (weak-canonical) representation, and the converse need not hold.*

Example 4. If we have two mutually orthogonal stationary processes given by canonical representations:

$$x_1(t) = \int_{-\infty}^t g_1(t-u)\,\mathrm{d}z_1(u),\ x_2(t) = \int_{-\infty}^t g_2(t-u)\,\mathrm{d}z_2(u),\ u \leq t,\ u,t \in (-\infty,\infty),$$

then the representation of the sum $x(t) = x_1(t)+x_2(t)$, is not canonical (multiplicity of $x(t)$ is equal to 1). Moreover, it is weak-canonical if and only if $f_1(u) = a \cdot f_2(u)$, where $f_1(u), f_2(u)$ are spectral densities, $a =$ const. (see [**3**]).

Example 5. Let

$$x(t) = \int_{-\infty}^t e^{-c(t-u)}\,\mathrm{d}z_1(u) + \int_{-\infty}^t d \cdot e^{-c(t-u)}\,\mathrm{d}z_2(u),\ u \leq t,\ u,t \in \mathbf{R},$$

be a process, where $z_1(u)$ and $z_2(u)$ are the mutually orthogonal processes with orthogonal increments such that $Ez_n(u) = 0$, $Ez_n^2(u) = f_n(u)\,\mathrm{d}u$, $n = 1,2$; $c, d =$ const., $f_1(u) = 2c$, $f_2(u) = 2cd^2$. According to example 4, $x(t)$ has a weak-canonical representation.

Example 6. Let

$$x(t) = \int_0^t \mathrm{d}z_1(u) + f(t)\int_0^t \mathrm{d}z_2(u),\ u \leq t,\ t \geq 0,$$

be a process, where $z_1(u)$ and $z_2(u)$ are the mutually orthogonal Wiener processes and $f(t)$ is absolutely continuous and $f^{'}(t) \in L^2([0,t])$. Both processes $\int\limits_0^t \mathrm{d}z_1(u)$ and $f(t)\int\limits_0^t \mathrm{d}z_2(u)$ are given by canonical representation. According to [**2**], $x(t)$ has a multiplicity $N = 1$, so the representation of $x(t)$ is not canonical and in a general case it is not weak-canonical too, except when $f(t)$ =const. (see [**3**]).

For processes given by canonical (weak-canonical) representation, we have a criterion to check when they have multiplicity of unity. In [**5**] we proved that the certain *regularity conditions* for $g_n(t,u)$ and $F_n(u), u \leq t, u,t \in (a,b), n = 1,\ldots,N$, ensure a multiplicity of unity for a process which has a weak-canonical representation. This statement is a natural extension of Cramer's Theorem 5.1.

(see [**1**]) where the regularity conditions ensure a multiplicity of unity for a process which has a canonical representation.

There exist processes given by a non weak-canonical representation, like those from examples 2 and 3, but having multiplicity of unity. Also, it is interesting that processes like the one from example 3, do not satisfy the regularity conditions and CRAMER's statement (Theorem 5.2. in [**1**]) does not help. It means that we can use different ways to determine the multiplicity of a process. So, the problem (still open) is to find out a criterion which would be depended probably on kernel functions $g_n(t,u)$ and $z_n(u), u \leq t, t \in (a,b), n \in \mathbf{N}$, easy to check whether multiplicity is equal to one or not for a wide class of processes.

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