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NON-ISOMORPHIC 4-(48,5,λ) DESIGNS FROM PSL(2,47)

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Non-isomorphic 4-(48,5,λ) designs with $PSL(2,47)$ as automorphism group are enumerated. It turns out that there are 12, 295, 1195 and 2368 pairwise non-isomorphic designs in this class, with λ equal to 8, 12, 16, 20, respectively. Auxiliary graphs serving as design invariants were used to derive this result.

1. DESIGNS

An n-set is a set of cardinality n. A t-(v, k, λ) design D is a collection of k-subsets (called *blocks*) of a v-set X of *points*, that satisfies the property that each t-subset of points is in exactly λ blocks. It is also required that no block is repeated. A group G acting on X is an automorphism group of D if the collection of blocks of D is a union of G-orbits of k-subsets. We also say that D *arises* from G.

1.1. Orbits

The projective special linear group $G = PSL(2, 47)$ acts 3-homogeneously on the projective line $\Omega = \{0, 1, ..., 46\} \cup \{\infty\}$. There are 33 orbits of 5-subsets by action of G; all these orbits are of size $51888 = |G|$. Orbit representatives are listed in Table 1 below. All of them contain the points 0, 1, and ∞; these points are omitted. Each representative is preceded by the ordinal number of the corresponding orbit:

1.	2	3	2.	2	5	3.	2	6	4.	2	7	5.	2	8	6.	2	10
7.	2	12	8.	2	13	9.	2	14	10.	2	16	11.	3	4	12.	3	7
13.	3	8	14.	3	11	15.	3	12	16.	3	13	17.	3	14	18.	3	15
19.	3	17	20.	3	19	21.	3	20	22.	3	22	3.	3	26	24.	3	39
25.	4	9	26.	4	13	27.	4	19	28.	4	20	29.	4	21	30.	4	27
31.	5	8	32.	6	10	33.	7	11									

Table 1. Orbits of 5-subsets

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1.2. Designs as column combinations of the orbit incidence matrix

Let $\Lambda = (\lambda_{i,j})$ denote the orbit incidence matrix for orbits of 4-subsets and 5-subsets by action of G; $\lambda_{i,j}$ is the number of 5-subsets from the j-th orbit of 5-subsets which are supersets of a fixed 4-subset that belongs to the i-th orbit of 4-subsets.

The 10×33 matrix Λ has the uniform row sum 44. To construct a 4-(48,5,λ) design with G as automorphism group, we find (following the Kramer-Mesner method [**3**]) a *proper* subset S of the columns of Λ with uniform row sum λ (this method has been used also in [**1**] and [**2**]). Thus designs correspond to appropriate *column combinations* of Λ. By complementation, it suffices to look for designs with $\lambda \leq 22$.

Applying a very fast backtracking algorithm to the column set of Λ, we have found the existence of 7740 designs (in the sense of column combinations) with $PSL(2,47)$ as automorphism group. Restricting for a moment the attention to the design parameters, we have the following result:

Theorem. *There exist* 4-$(48,5,\lambda)$ *designs, with* $\lambda \in \{8,12,16,20\}$ *and* $PSL(2,47)$ *as automorphism group. Direct action of the group* $PSL(2,47)$ *on the projective line does not give* 4-$(48,5,\lambda)$ *designs with other values of* λ.

It turns out that only ten of the constructed designs (column combinations) have also the general projective linear group $PGL(2,47)$ as automorphism group. The data on the constructed designs are summarized in the following table:

λ	8	12	16	20
with PSL(2,47)	24	590	2390	4736
also with PGL(2,47)	2	–	4	4

Table 2. The number of “successful” column combinations

We list only lexicographically the first column combination corresponding to designs with a given value of λ:

$\lambda =$ 8: `1 12 26 28 31 33`

$\lambda =$ 12: `1 2 12 15 24 28 30 32 33`

$\lambda =$ 16: `1 2 3 11 12 24 26 28 30 31 32 33`

$\lambda =$ 20: `1 2 3 4 11 14 17 19 20 24 26 27 28 32 33`

2. ISOMORPHISMS

Two designs are *isomorphic* if there is a bijection between their point-sets that preserves the collection of blocks. A *graph* is an ordered pair (V,E), where V and E are respectively the vertex-set and edge-set. The edges are some of the non-repeating 2-subsets of vertices. Two graphs are *isomorphic* if there is a bijection between their vertex-sets that preserves the collection of edges.

Isomorphism problem for derived 4-(48,5,λ) designs will be settled by reducing it to a simpler, graph isomorphism problem. The later problem is related to some auxiliary graphs that are *adjoined* to the designs. Thus 4-(48,5,20) designs have 778320 blocks each, while the adjoined graphs on 45 vertices are regular of degree 20 and have "only" 450 edges each.

2.1. Auxiliary graphs as design invariants

Let D denote a 4-$(v,5,\lambda)$ design which is a union of orbits of G-orbits of 5-subsets, for a 3-homogeneous group G. Let $Graph(D,\{a,b,c\})$ denote a graph adjoined to the design D and to a *fixed* 3-subset $\{a,b,c\}$ of its point-set Ω in the following manner:

- vertices x are associated to 4-subsets of the form $\{a,b,c,x\}$, where $x \in \Omega\backslash\{a,b,c\}$
- edges $\{x,y\}$ are associated to those blocks of D, which are of the form $\{a,b,c,x,y\}$, where $x,y \in \Omega\backslash\{a,b,c\}$.

Lemma 1. *Let $\{a,b,c\}$ and $\{a',b',c'\}$ denote two 3-subsets of the point-set of a 4-$(v,5,\lambda)$ design D arising from a 3-homogeneous group G. Then the graphs $Graph(D,\{a,b,c\})$ and $Graph(D,\{a',b',c'\})$ are isomorphic.*

Proof. Let g denote an element of G which maps the set $\{a,b,c\}$ onto $\{a',b',c'\}$; the existence of such an element g is guaranteed by 3-homogenicity.

The element g maps 4- and 5-supersets of $\{a,b,c\}$ onto 4- and 5-supersets of $\{a',b',c'\}$. Thus the set $\Omega \setminus \{a,b,c\}$ is mapped onto $\Omega \setminus \{a',b',c'\}$. These two sets are vertex-sets of $Graph(D,\{a,b,c\})$ and $Graph(D,\{a',b',c'\})$, respectively. The element g induces an isomorphism between $Graph(D,\{a,b,c\})$ and $Graph(D,\{a',b',c'\})$, since it maps each edge $\{x,y\}$ of the former graph to the corresponding edge of the later. $\square$

Remark. Lemma 1 justifies the notion "graph adjoined to design" and enables the denotation $Graph(D,\{a,b,c\})$ to be shortened to $Graph(D)$.

Lemma 2. *If $Graph(D_1)$ and $Graph(D_2)$ are non-isomorphic for 4-$(v,5,\lambda)$ designs D_1 and D_2 arising from a 3-homogeneous group G, then these two designs are also non-isomorphic.*

Proof. Suppose, on the contrary, that there exists an isomorphism ϕ mapping D_1 onto D_2. Let $\{a,b,c\}$ denote an arbitrary 3-subset of the point-set of D_1. Its 4-supersets and blocks of D_1 containing $\{a,b,c\}$ are mapped by ϕ onto those 4-sets of points and blocks of D_2, which contain $\{\phi(a),\phi(b),\phi(c)\}$. The mapping ϕ preserves set-incidencies. Thus ϕ maps the vertices and edges of $Graph(D_1,\{a,b,c\}) = Graph(D_1)$ onto the vertices and edges of $Graph(D_2,\{\phi(a),\phi(b),\phi(c)\}) = Graph(D_2)$. This proves that $Graph(D_1)$ and $Graph(D_2)$ are isomorphic, a contradiction. $\square$

Consequence: The adjoined graphs can be used as *design invariants*.

2.2. Graph isomorphisms

Isomorphism testing was performed within the classes of 24, 590, 2390, 4736 associated regular graphs on 45 vertices, with degrees 8, 12, 16, 20, respectively. The common vertex-set of all these graphs was $\{2, 3, ..., 46\}$; the points 0, 1 and ∞ were temporarily removed from the consideration.

The following three graph functions [**5**] of "Mathematica$^{\text{TM}}$" software package were applied to the adjoined graphs:

1. "NumberOfSpanningTrees($Graph$)" – calculates number of spanning trees of $Graph$.
2. "Isomorphism($Graph1, Graph2$)" – finds an isomorphism between $Graph1$ and $Graph2$, if it exists.
3. "IsomorphismQ($Graph1, Graph2, \phi$)" – tests whether the mapping ϕ establishes an isomorphism between the graphs $Graph1$ and $Graph2$.

Using a polynomial algorithm, Function 1. finds the (huge) number of spanning trees (a graph invariant!) very efficiently (this is a 70-digit number for λ = 20). It occured that each existing number of spanning trees has appeared exactly twice; this implies that 12, 295, 1195 and 2368 pairwise different numbers of spanning trees have appeared in the four considered classes of graphs, respectively.

Function 2. is a time consuming one, since it is based on an (exponential) backtracking algorithm. Several pairs of the adjoined graphs with the same numbers of spanning trees were treated by using this function. It occured that the same mapping, denoted by π, has established isomorphisms with all these pairs. After several succesful attempts, we conjectured that the mapping π establishes isomorphisms within all the remaining pairs of the adjoined graphs with the same numbers of spanning trees. To check this, we continued the attempts with the very fast Function 3., which had the argument ϕ replaced by π. It turned out that our conjecture was true.

In this way, we have checked that the two graphs within each pair of adjoined graphs with the same numbers of spanning trees are actually isomorphic. Thus we have completed a computational derivation of the following result:

Theorem 2. *There are* 12, 295, 1195 *and* 2368 *pairwise non-isomorphic graphs of the form* $Graph(D)$*, where* D *is a* 4-$(48, 5, \lambda)$ *design with* $PSL(2, 47)$ *as automorphism group, for* λ *equal to* 8, 12, 16, 20*, respectively. Each one of these non-isomorphic graphs appears exactly twice within the class of all found designs* D*.*

Remark. Lemma 2 implies that the values mentioned in Theorem 2 are *lower bounds* for the number of non-isomorphic 4-(48,5,λ) designs.

2.3. Design isomorphisms

Isomorphism between some two adjoined graphs is just a necessary condition for isomorphism between the corresponding designs. However, it will be proved that the graph isomorphisms established by the above mapping π can be extended into design isomorphisms:

Theorem 3. *Let D_1 and D_2 be two 4-$(48,5,\lambda)$ designs with $PSL(2,47)$ as automorphism group, which satisfy that $Graph(D_1)$ and $Graph(D_2)$ are isomorphic under the mapping π. Then the mapping π establishes an isomorphism between the designs D_1 and D_2.*

Throughout the proof, "PGL-orbits" and "PSL-orbits" will refer to orbits of 5-subsets under action of 5-subsets under action of $PGL(2,47)$ and $PSL(2,47)$. We shall primarily prove the following lemma:

Lemma 3. *The mapping π maps a collection of* PSL-*orbits onto another collection containing the same number of* PSL-*orbits.*

Proof of lemma. The mapping π can be written in the form:

$$\pi : x \longrightarrow \frac{x}{x+46}.$$

In this way, π is a hyperbolic involution of the projective line Ω. Fixed points of this involution are 0 and 2.

Since the determinant $\begin{vmatrix} 1 & 0 \\ 1 & 46 \end{vmatrix}$ is not a square over the field $GF(47)$, the mapping π belongs to $PGL(2,47) \setminus PSL(2,47)$. Such a mapping preserves PGL-orbits; each element of group G maps a G-orbit onto itself.

The group $PSL(2,47)$ is the subgroup of index 2 of $PGL(2,47)$, which contains the mappings with square determinants. Let be given a 5-subset X of Ω. If there exists a mapping from the coset of mappings with non-square determinants, which maps X onto the same PSL-orbit, then the PGL-orbit and PSL-orbit determined by X coincide. Otherwise the images of X under the mappings of the coset constitute another PSL-orbit of the same cardinality within the same PGL-orbit. In the first case, the mapping π maps the PSL-orbit determined by X onto itself; in the second, it maps the two PSL-orbits within the PGL-orbit determined by X onto each other. $\square$

Remark. The mapping π induces an *involution of* PSL-*orbits.*

Proof of Theorem 3. Note that the mapping π fixes the set $F = \{0, 1, \infty\}$. Together with the assumption that π establishes an isomorphism between $Graph(D_1)$ and $Graph(D_2)$, we have that π maps the subcollection of blocks of D_1 containing F onto the subcollection of blocks of D_2 containing F. Both of these subcollections are spread over the same number of PSL-orbits. Namely, 3-homogenicity of the group $PSL(2,47)$ implies that each PSL-orbit has a non-empty intersection with the collection of 5-subsets containing F; this fact has already been used with the choice of orbit representatives.

We recall that both D_1 and D_2 are unions of the same number of PSL-orbits. All these orbits are completely determined by their blocks containing F. Lemma 3 gives that $\pi(D_1)$ is a union of PSL-orbits. We conclude that $\pi(D_1)$ and D_2 are the same one union of PSL-orbits, which completes the proof. □

Theorems 2 and 3 settle the enumeration problem for the non-isomorphic designs in the considered class. Our main result reads:

Theorem 4. *There exist* 12, 295, 1195 *and* 2368 *pairwise non-isomorphic* 4-$(48, 5, \lambda)$ *designs with* $PSL(2, 47)$ *as automorphism group, and with* λ *equal to* $8, 12, 16, 20$*, respectively.*

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