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ON THE KURATOWSKI MEASURE OF NONCOMPACTNESS IN METRIC LINEAR SPACES

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Dedicated to professor Dušan Adamović, on the ocassion of his 70th birthday

In this note we present some properties of the Kuratowski's measure of noncompactness in metric linear spaces.

1. INTRODUCTION

Proposition 0.1 (see ROLEWICZ [**4**]).*Let X be a metric linear space. Then there exists metric d on X which is equivalent with original metric on X such that function $|\cdot\|$: $X \to [0,+\infty)$ defined by $|x\| = d(x,0)$ has following properties:*

1) $|x\| = 0$ *if and only if* $x = 0$*;*
2) $|x\| = |-x\|$*;*
3) $|x+y\| \leq |x\| + |y\|$*;*
4) $0 < \alpha < \beta$ *implies* $|\alpha x\| < |\beta x\|$.

The mapping $|\cdot\|$ is said to be an F - *norm* or *paranorm.* If there exists a number p, $0 < p \leq 1$, such that $|tx\| = |t|^p|x\|$ for any scalar t and $x \in X$ it is said that $|\cdot\|$ is a p-*norm* and X is a p-*normed* space.

Let X be a HAUSDORFF topological vector space. A set $A \subseteq X$ is *bounded* if for each neighborhood of zero U there is a scalar α such that $A \subseteq \alpha U$. The space X is *locally bounded* if it contains a bounded neghborhood of zero. X is a locally bounded space if and only if X is metrizable and p-normable.

The theory of measures of noncompactness has many applications in Functional analysis and Operator theory (see [**2**], [**5**]). If Q is a bounded subset of metric space X, then the KURATOWSKI measure of noncompactness of Q is defined by

$$\alpha(Q) = \inf\{r > 0 \,|\, Q \subseteq \bigcup_{i=1}^{n} S_i, S_i \subseteq X, \operatorname{diam}(S_i) < r, 1 \leq i \leq n, n \in \mathbf{N}\},$$

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where diam(.) denotes the diameter.

In the proofs of our results we need the following well-known properties of the KURATOWSKI measure of noncompactness.

Proposition 0.2 (see BANÁS and GOEBEL [**2**], RAKOČEVIĆ [**5**]).*If Q, Q_1 amd Q_2 are bounded subsets of a metric spaces (X, d) then*

1) $\alpha(Q) = 0$ *if and only if* Q *is a totally bounded set;*
2) $Q_1 \subseteq Q_2$ *implies* $\alpha(Q_1) \leq \alpha(Q_2)$*;*
3) $\alpha(Q_1 \cup Q_2) = \max\{\alpha(Q_1), \alpha(Q_2)\}$*;*

In this paper we investigate some properties of the KURATOWSKI measure of noncompactness on arbitrary metric linear space. Corresponding results for the HAUSDORFF measure of noncompactness are obtained by I. JOVANOVIĆ, V. RAKOČEVIĆ [4] and I. ARANDJELOVIĆ, M. MILOVANOVIĆ-ARANDJELOVIĆ [**1**].

2. RESULTS

Let (X, d) be a metric space, $x \in X$ and $r > 0$. By $B(x, r)$ we denote $\{y \in X : d(x, y) \leq r\}$.

Proposition 1. *If Q, Q_1 and Q_2 are bounded subsets of arbitrary metric linear space X and $x \in X$, then*

1) $\alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2)$*;*
2) $\alpha(x + Q) = \alpha(Q)$*;*

Proof. Let $Q_1 \subseteq \overset{m}{\underset{i=1}{\cup}} A_i$; $\mathrm{diam}(A_i) \leq \alpha(Q_1)$, $i = 1, 2, \ldots, m$ and $Q_2 \subseteq \overset{n}{\underset{j=1}{\cup}} B_j$; $\mathrm{diam}(B_j) \leq \alpha(Q_2)$, $j = 1, 2, \ldots, n$. From

$$Q_1 + Q_2 \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} [A_i + B_j]$$

and $\mathrm{diam}(A_i + B_j) \leq \mathrm{diam}(A_i) + \mathrm{diam}(B_j)$ follows $\alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2)$.

From 1) we have $\alpha(x + Q) \leq \alpha(\{x\}) + \alpha(Q) = \alpha(Q)$, which implies $\alpha(Q) = \alpha(-x + x + Q) \leq \alpha(x + Q)$. So $\alpha(x + Q) = \alpha(Q)$. □

Proposition 2. *If X is locally bounded* HAUSDORFF *topological vector space, $Q \subseteq X$ its bounded subset, $|\cdot||$ is a p-norm on X and β arbitrary scalar then*

$$\alpha(\beta Q) = |\beta|^p \alpha(Q).$$

Proof. Let $\beta \neq 0$. From $Q \subseteq \overset{n}{\underset{i=1}{\cup}} S_i$, it follows that $\beta Q \subseteq \overset{n}{\underset{i=1}{\cup}} \beta S_i$. Now

$$\mathrm{diam}(\beta S_i) = \sup_{x,y \in S_i} |\beta(x - y)|| = |\beta|^p \sup_{x,y \in S_i} |(x - y)|| = |\beta|^p \mathrm{diam}(S_i)$$

which implies $\alpha(\beta Q) \leq |\beta|^p \alpha(Q)$. Since $\beta^{-1}\beta Q = Q$, we have $\alpha(Q) \leq |\beta|^{-p}\alpha(\beta Q)$, and so $|\beta|^p\alpha(Q) \leq \alpha(\beta Q)$. It follows $\alpha(\beta Q) = |\beta|^p\alpha(Q)$. $\square$

Next topological lemma is an extension of well known result of Borsuk, Lusternik and Schnirelman.

Lemma. *Let E_n be n-dimensional metric linear space, $F_1, \ldots, F_n$ its closed subsets and $S_*^{n-1} = \{x \in E_n \,|\, |x\| = 1\}$. If $S_*^{n-1} \subseteq \bigcup_{i=1}^{n} F_i$ then there exists $x \in S_*^{n-1}$ and $i \in \{1, \ldots n\}$ such that $x \in F_i$ and $-x \in F_i$.*

Proof. Let $\|\cdot\|$ be an Eucledean norm on E_n and $S^{n-1} = \{x \in E_n \,|\, \|x\| = 1\}$. Mapping $h : S_*^{n-1} \to S^{n-1}$ defined by $h(x) = \dfrac{x}{\|x\|}$ is the homeomorphysm S_*^{n-1} to S^{n-1} such that $h(-x) = -h(x)$. From $S_*^{n-1} \subseteq \bigcup_{i=1}^{n} F_i$ it follows $S^{n-1} \subseteq \bigcup_{i=1}^{n} h(F_i)$. By classical result of Borsuk, Lusternik and Schnirelman there exists $y \in S^{n-1}$ and $i \in \{1, \ldots, n\}$ such that $y \in h(F_i)$ and $-y \in h(F_i)$. Let $x = h^{-1}(y) = \dfrac{y}{|y\|}$. Then $x \in F_i$ and $-x \in F_i$. $\square$

Next proposition is an extension of well known result of M. Furi and A. Vignoli [**3**].

Proposition 3. *If X is an infinite-dimensional metric linear space, and $B(0,1)$ is its closed unit ball, then*

$$\operatorname{diam}(B(0,1)) \geq \alpha(B(0,1)) \geq \inf_{|x\|=1} |2x\|.$$

Proof. Let us remark that clearly $\operatorname{diam}(B(0,1)) \geq \alpha(B(0,1))$. If $\alpha(B(0,1)) < s = \inf_{|x\|=1} |2x\|$ then there exists closed sets $F_1, \ldots, F_n \subseteq X$ suche that $B(0,1) \subseteq \bigcup_{i=1}^{n} F_i$, and $\operatorname{diam}(F_i) < s, 1 \leq i \leq n$. Let $\{x_1, \ldots x_n\}$ be a linearly independent subset of X. Then $E_n = \operatorname{lin}\{x_1, \ldots x_n\}$ is a finite-dimensional subspace of X and

$$S_*^{n-1} = \{x \in E_n \,|\, |x\| = 1\} \subseteq \bigcup_{i=1}^{n} F_i \cap S_*^{n-1}.$$

By *lemma* it follows that there exists $x \in S_*^{n-1}$ and $i \in \{1, \ldots, n\}$ such that $\{x, -x\} \subseteq F_i \cap S_*^{n-1}$. From $d(x,-x) = |2x\|$ follows $\operatorname{diam}(F_i) \geq s$ which is a contradiction.

Corollary. *If X is an infinite-dimensional locally bounded* Hausdorff *topological vector space, $x_0 \in X$, $|\cdot\|$ is a p-norm on X and $r > 0$ then*

$$r2^p \leq \alpha(B(x_0, r)) \leq r \operatorname{diam}(B(0,1)).$$

Proof. For $r > 0$ conditions $|x\| \leq 1$ and $|r^{\frac{1}{p}}x\| \leq r$ are equivalent, which implies $B(0,r) = r^{\frac{1}{p}}B(0,1)$. So $\alpha(B(x_0,r)) = \alpha(x_0 + B(0,r)) = \alpha(B(0,r)) =$

$\alpha(r^{\frac{1}{p}}B(0,r)) = r\alpha(B(0,1)) \geq r2^p$ and $\mathrm{diam}(B(x_0,r)) = \mathrm{diam}(B(0,r)) = \mathrm{diam}(r^{\frac{1}{p}}B(0,1)) = r\mathrm{diam}(B(0,1))$. □

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