UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 10 (1999), 31–36.

AN EXTREMAL PROBLEM FOR THE LENGTH OF ALGEBRAIC POLYNOMIALS

Gradimir V. Milovanović, Dobrilo Đ. Tošić

An extremal problem for the length $L(P) = |a_0| + |a_1| + \cdots + |a_n|$ of polynomials $P(x) = \sum_{k=0}^{n} a_k x^k$ with the uniform norm $\|P\| \le 1$ on $[0,1]$ is considered. We prove that $L(P) \le T_n(3)$, where T_n is the Chebyshev polynomial of the first kind and degree n. The equality is attained for the shifted Chebyshev polynomial $\pm T_n(2x-1)$. We use this result to give an application to the condition of polynomials on $[0,1]$ in power form.

1. INTRODUCTION

Let $\mathcal{P}_n$ denotes the set of algebraic polynomials of degree at most n. In this note we consider an extremal problem for the length

$$L(P) = |a_0| + |a_1| + \cdots + |a_n| \tag{1.1}$$

of polynomials $P(x) = \sum\limits_{k=0}^{n} a_k x^k$ with the uniform norm on $[0,1]$ less than or equal to 1, i.e.,

$$\|P\| = \max_{0 \le x \le 1} |P(x)| \le 1. \tag{1.2}$$

Such a class of polynomials we denote by $\hat{\mathcal{P}}_n$.

It is known (cf. [**5**, p. 131]) that for two arbitrary polynomials P, Q we have

$$L(PQ) \le L(P)L(Q), \qquad L(P \mp Q) \le L(P) + L(Q).$$

We will prove that

$$L(P) \le T_n(3) = \frac{1}{2}\left((3+\sqrt{8})^n + (3-\sqrt{8})^n\right), \tag{1.3}$$

where $T_n(x)$ is the CHEBYSHEV polynomials of the first kind.

1991 Mathematics Subject Classification: Primary 33A65; Secondary 41A55.
This work was supported in part by the Serbian Scientific Foundation, grant number #04M03.

Otherwise, the CHEBYSHEV polynomials are appeared in many extremal problems with polynomials (see, for example, the monographs written by RIVLIN [**7**], PASZKOWSKI [**6**], FOX and PARKER [**2**], and MILOVANOVIĆ, MITRINOVIĆ and RASSIAS [**5**], as well as the papers of DEVORE [**1**], MICCHELLI and RIVLIN [**4**] and ROULIER and VARGA [**8**]). Several maximizing linear functionals on $\mathcal{P}_n$ were considered in [**6**, Ch. 2B].

The proof of (1.3) can be done using the general theorem on the CHEBYSHEV alternation. However, our proof here is based on some interpolation facts. Also, we give an application of this estimate to the condition of algebraic polynomials on $[0,1]$ in power form.

2. MAIN RESULT

The extremal points of $T_n(t)$ on $[-1,1]$ we denote by

$$\eta_k := -\cos\frac{k\pi}{n},\qquad k=0,1,\ldots,n, \tag{2.1}$$

where $T_n(\eta_k)=(-1)^{n-k}$, $k=0,1,\ldots,n$.

We prove the following result:

Theorem 2.1. *Let* $P(x)=\sum\limits_{k=0}^{n} a_kx^k$ *be an arbitrary polynomials in the class* $\hat{\mathcal{P}}_n$ *and* $L(P)$ *be its length defined by* (1.1). *Then*

$$L(P)\le T_n(3)=\frac{1}{2}\left((3+\sqrt{8})^n+(3-\sqrt{8})^n\right),$$

with equality if $P(x)=\pm T_n(2x-1)$.

Proof. Regarding to (2.1), we take the extremal points of the shifted CHEBYSHEV polynomial $T_n^*(x)=T_n(2x-1)$ as

$$x_\nu := \frac{1}{2}(1+\eta_\nu)=\sin^2\frac{\nu\pi}{2n},\qquad \nu=0,1,\ldots,n.$$

Let $P(x)=\sum\limits_{k=0}^{n} a_kx^k\in\hat{\mathcal{P}}_n$ and put

$$P(x_\nu)=\sum_{k=0}^{n} a_k{x_\nu}^k=p_\nu,\qquad \nu=0,1,\ldots,n. \tag{2.2}$$

For each $(p_0,p_1,\ldots,p_n)\in\mathbf{R}^{n+1}$, the coefficients a_k, $k=0,1,\ldots,n$, are determined uniquely from the VANDERMONDE system (2.2). Introducing the elementary LAGRANGE interpolation polynomials

$$\ell_\nu(x)=\prod_{\substack{j=0\\ j\ne\nu}}\frac{x-x_j}{x_\nu-x_j}=b_{n,\nu}x^n+b_{n-1,\nu}x^{n-1}+\cdots+b_{0,\nu},\qquad \nu=0,1,\ldots,n,$$

the polynomial $P(x)$ can be expressed in the form

$$P(x) = \sum_{\nu=0}^{n} p_\nu \ell_\nu(x),$$

i.e.,

$$\sum_{k=0}^{n} a_k x^k = \sum_{\nu=0}^{n} p_\nu \sum_{k=0}^{n} b_{k,\nu} x^k = \sum_{k=0}^{n} \Big(\sum_{\nu=0}^{n} b_{k,\nu} p_\nu \Big) x^k,$$

from which, we conclude that

$$a_k = \sum_{\nu=0}^{n} b_{k,\nu} p_\nu, \qquad k = 0, 1, \ldots, n.$$

Since $P \in \hat{\mathcal{P}}_n$, it has to be $|p_\nu| \le 1$, $\nu = 0, 1, \ldots, n$ (see (1.2)), and we have that

$$|a_k| = \Big| \sum_{\nu=0}^{n} b_{k,\nu} p_\nu \Big| \le \sum_{\nu=0}^{n} |b_{k,\nu}||p_\nu| \le \sum_{\nu=0}^{n} |b_{k,\nu}|,$$

and

$$L(P) = \sum_{k=0}^{n} |a_k| \le \sum_{k=0}^{n} \sum_{\nu=0}^{n} |b_{k,\nu}|.$$

We denote by $\sigma_m^{(\nu)} = \sigma_m(x_0, x_1, \ldots, x_{\nu-1}, x_{\nu+1}, \ldots, x_n)$ the m-th elementary symmetric function in n nonnegative variables $x_0, x_1, \ldots, x_{\nu-1}, x_{\nu+1}, \ldots, x_n$, where $0 \le \nu \le n$, $1 \le m \le n$, and $\sigma_0^{(\nu)} \equiv 1$.

Since

$$b_{k,\nu} = \sigma_{n-k}^{(\nu)} \frac{(-1)^{n-k}}{\prod\limits_{\substack{j=0\\ j\ne\nu}}^{n} (x_\nu - x_j)} \quad \text{and} \quad \operatorname{sgn}\Big(\prod_{\substack{j=0\\ j\ne\nu}}^{n} (x_\nu - x_j) \Big) = (-1)^{n-\nu},$$

we have $|b_{k,\nu}| = (-1)^{k+\nu} b_{k,\nu}$, so that

$$\sum_{k=0}^{n} |b_{k,\nu}| = \sum_{k=0}^{n} (-1)^{k+\nu} b_{k,\nu} = (-1)^\nu \sum_{k=0}^{n} (-1)^k b_{k,\nu} = (-1)^\nu \ell_\nu(-1).$$

Thus,

$$A_n = \sum_{k=0}^{n} \sum_{\nu=0}^{n} |b_{k,\nu}| = \sum_{\nu=0}^{n} (-1)^\nu \ell_\nu(-1).$$

Notice that $\operatorname{sgn} \ell_\nu(x) = (-1)^\nu$ for $x \le 0$. Therefore, we consider now the LEBESGUE function $\Lambda_n(x)$ for $x \le 0$, i.e.,

$$\Lambda_n(x) = \sum_{\nu=0}^{n} (-1)^\nu \ell_\nu(x).$$

Since

$$\ell_\nu(x) = \prod_{\substack{j=0 \\ j\neq\nu}} \frac{x-x_j}{x_\nu - x_j} = \prod_{\substack{j=0 \\ j\neq\nu}} \frac{t-\eta_j}{\eta_\nu - \eta_j} = \frac{1}{t-\eta_\nu} \cdot \frac{(1-t^2)T_n'(t)}{[(1-t^2)T_n'(t)]'_{t=\eta_\nu}},$$

where η_ν are given by (2.1), using the Chebyshev differential equation

$$(1-t^2)T_n''(t) - tT_n'(t) + n^2 T_n(t) = 0,$$

we find

$$\ell_\nu(x) = \frac{1}{t-\eta_\nu} \cdot \frac{(t^2-1)T_n'(t)}{\eta_\nu T_n'(\eta_\nu) + n^2 T_n(\eta_\nu)}, \qquad t = 2x-1.$$

Therefore,

$$\begin{aligned} \ell_0(x) &= \frac{1}{t+1} \cdot \frac{(t^2-1)T_n'(t)}{(-1)^n n^2 + n^2(-1)^n} = \frac{(-1)^n}{2n^2}(t-1)T_n'(t), \\ \ell_\nu(x) &= \frac{1}{t-\eta_\nu} \cdot \frac{(t^2-1)T_n'(t)}{n^2(-1)^{n-\nu}}, \qquad 1 \le \nu \le n-1, \\ \ell_n(x) &= \frac{1}{t-1} \cdot \frac{(t^2-1)T_n'(t)}{n^2+n^2} = \frac{1}{2n^2}(t+1)T_n'(t), \end{aligned}$$

and

$$\Lambda_n(x) = \frac{(-1)^n T_n'(t)}{n^2} \left\{ \frac{1}{2}(t-1) + (t^2-1)\sum_{\nu=1}^{n-1} \frac{1}{t-\eta_\nu} + \frac{1}{2}(t+1) \right\}.$$

Since,

$$\frac{T_n''(t)}{T_n'(t)} = \sum_{\nu=1}^{n-1} \frac{1}{t-\eta_\nu},$$

we obtain

$$\Lambda_n(x) = \frac{(-1)^n T_n'(t)}{n^2} \left[t + (t^2-1)\frac{T_n''(t)}{T_n'(t)} \right] = \frac{(-1)^n}{n^2} \left[tT_n'(t) + (t^2-1)T_n''(t) \right],$$

i.e., $\Lambda_n(x) = (-1)^n T_n(t) = T_n(-t) = T_n(1-2x)$. Thus, $A_n = \Lambda_n(-1) = T_n(3)$.

Solving the difference equation $T_{n+1}(3) = 6T_n(3) - T_{n-1}(3)$, with starting values $T_0(3) = 1$ and $T_1(3) = 3$, we find

$$T_n(3) = \frac{1}{2}\left((3+\sqrt{8})^n + (3-\sqrt{8})^n\right).$$

Since $T_n^*(x) = T_n(2x-1) = T_{2n}(\sqrt{x})$, $x \ge 0$, we have that

$$T_n^*(x) = n\sum_{k=0}^{n} \frac{(-1)^{n-k}}{n+k} \binom{n+k}{n-k} (4x)^k.$$

Evidently that for $\pm T_n(2x-1)$ we have

$$L(\pm T_n^*(\,\cdot\,)) = n\sum_{k=0}^{n} \frac{4^k}{n+k}\binom{n+k}{n-k} = T_n(3).$$

Theorem 2.1 can be also interpreted in the following form:

Theorem 2.2. *Let* $P(x) = \sum_{k=0}^{n} a_k x^k$ *be an arbitrary polynomials in the class* $\mathcal{P}_n$ *and* $L(P)$ *be its length defined by* (1.1). *Then the best constant in*

$$A_n = \sup_{P\in\mathcal{P}_n\setminus\{0\}} \frac{L(P)}{\|P\|}$$

is $A_n = T_n(3)$. *The extremal polynomial is* $\gamma T_n^*(x)$, *where* γ *is an arbitrary constant different from zero.*

In Table 2.1 we give the best constant A_n and the extremal polynomial $T_n^*(x)$ for $0 \le n \le 7$.

TABLE 2.1

n	A_n	$T_n^*(x)$
0	1	1
1	3	$2x-1$
2	17	$8x^2-8x+1$
3	99	$32x^3-48x^2+18x-1$
4	577	$128x^4-256x^3+160x^2-32x+1$
5	3363	$512x^5-1280x^4+1120x^3-400x^2+50x-1$
6	19601	$2048x^6-6144x^5+6912x^4-3584x^3+840x^2-72x+1$
7	114243	$8192x^7-28672x^6+39424x^5-26880x^4+9408x^3-1568x^2+98x-1$

Following GAUTSCHI [**3**] we can give an application of Theorem 2.2 to the condition of polynomials on $[0,1]$ in power form. Let $M_{n+1} : \mathbf{R}^{n+1} \to \mathcal{P}_n$ be the linear map associating to each vector $\boldsymbol{a} = (a_0, a_1, \dots, a_n) \in \mathbf{R}^{n+1}$ the polynomial

$$P(x) = \sum_{k=0}^{n} a_k x^k \in \mathcal{P}_n.$$

Denoting the inverse map of M_{n+1} by M_{n+1}^{-1} we have that $\boldsymbol{a} = M_{n+1}^{-1}P$ for any $P \in \mathcal{P}_n$. Taking the uniform norms in $\mathbf{R}^{n+1}$ and $\mathcal{P}_n$, Gautschi [**3**] considered the condition of the map M_{n+1}, relative to the compact interval $[a,b]$, defined by

$$\text{cond}\, M_{n+1} = \|M_{n+1}\| \cdot \|M_{n+1}^{-1}\|, \tag{2.3}$$

and studied the growth rate of $\text{cond}\, M_{n+1}$ as $n \to \infty$, as well as how this growth depends on the particular interval $[a,b]$ chosen.

We take here the uniform norm in $\mathcal{P}_n$, $\|P\| = \max_{0\le x\le 1} |P(x)|$, and l_1-norm in $\mathbf{R}^{n+1}$, $\|\boldsymbol{a}\| = \sum_{k=0}^{n} |a_k| = L(P)$. Then, evidently, for each $P \in \mathcal{P}_n$,

$$\|P\| = \max_{0\le x\le 1} \left|\sum_{k=0}^{n} a_k x^k\right| \le L(P),$$

with equality if $P(x)$ is a constant. Therefore, $\|M_{n+1}\| = 1$. On the other hand, Theorem 2.2 gives

$$\|M_{n+1}^{-1}\| = \sup_{P \in \mathcal{P}_n \setminus \{0\}} \frac{\|\boldsymbol{a}\|_1}{\|P\|} = A_n,$$

so that (2.3) becomes

$$\operatorname{cond} M_{n+1} = A_n = \frac{1}{2}\Big((3+\sqrt{8})^n + (3-\sqrt{8})^n\Big).$$

Thus, $\operatorname{cond} M_{n+1} \sim (3+\sqrt{8})^n/2$ as $n \to \infty$.

ACKNOWLEDGMENTS

The authors are indebted to Professor Walter Gautschi for the reference [**3**] and for an application of Theorem 2.2.

REFERENCES

1. R. A. DeVore: *A property of Chebyshev polynomials,* J. Approx. Theory **12** (1974), 418–419.
2. L. Fox, I. B. Parker: *Chebyshev Polynomials in Numerical Analysis,* Oxford University Press, London, 1968.
3. W. Gautschi: *The condition of polynomials in power form,* Math. Comp. **33** (1978), 343–352.
4. C. A. Micchelli, T. J. Rivlin: *Some new characterizations of the Chebyshev polynomials,* J. Approx. Theory **12** (1974), 420–424.
5. G. V. Milovanović, D. S. Mitrinović, Th. M. Rassias: *Topics in Polynomials: Extremal Problems, Inequalities, Zeros,* World Scientific, Singapore – New Jersey – London – Hong Kong, 1994.
6. S. Paszkowski: *Numerical Applications of Chebyshev Polynomials and Serie* (Russian), Nauka, Moscow, 1983.
7. T. J. Rivlin: *The Chebyshev Polynomials,* Wiley, New York, 1974.
8. J. A. Roulier, R. S. Varga: *Another property of Chebyshev polynomials,* J. Approx. Theory **22** (1978), 233–242.

University of Niš,
Faculty of Electronic Engineering,
Department of Mathematics,
P.O.Box 73, 18 001 Niš,
Yugoslavia
grade@elfak.ni.ac.yu

(Received September 9, 1997)
(Revised April 10, 1998)

University of Belgrade,
Faculty of Electrical Engineering,
P.O.Box 35-54, 11 120 Belgrade,
Yugoslavia
etosicdo@ubbg.etf.bg.ac.yu