UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 10 (1999), 27–29.

SOME INTEGRAL INEQUALITIES

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In this article, using Cauchy's inequality and the arithmetic mean–geometric mean inequality, some integral inequalities are proved.

1. MAIN RESULTS

Three integral inequalities below will be verified:

Theorem 1. *Let Ω be a domain in $\mathbf{R}^n$ and $f, g \in L(\Omega)$ such that $I(f) > 0$ and $I(g) > 0$, where*

$$I(f) = \int\limits_{\Omega} f(x)\,\mathrm{dx}. \tag{1}$$

Further, let $h : \Omega \to \mathbf{R}$ such that $h^2 \in L(\Omega)$. If

$$I(f)I(gh) = I(g)I(fh), \tag{2}$$

then

$$I(fh^2)I(g)^2 + I(gh^2)I(f)^2 \geq I(fh)I(gh)I(f+g), \tag{3}$$

the equality case is valid if and only if h is constant.

Corollary. *Under the conditions of Theorem 1, we have*

$$\frac{I(f)I(gh^2)}{I(gh)} + \frac{I(g)I(fh^2)}{I(fh)} \geq I\big((f+g)h\big). \tag{4}$$

The equality in (4) holds if and only if h is constant.

Theorem 2. *Let Ω be a domain in $\mathbf{R}^n$ and $f, g \in L(\Omega)$ satisfying $I(f) > 0$ and $I(g) > 0$, furthermore $h^2 \in L(\Omega)$. Then*

$$I(fh^2)[I(g)]^2 + I(gh^2)[I(f)]^2 \geq 2\big|I(fh)I(gh)\big|\sqrt{I(f)I(g)}. \tag{5}$$

1991 Mathematics Subject Classification: 26D15.

The first author was supported in part by NSF of Henan Province, The People's Republic of China.

2. PROOFS OF MAIN RESULTS

From (2), it is deduced that

$$\begin{aligned} I(f)I\big((f+g)h\big) &= I(fh)I(f+g), \\ I(g)I\big((f+g)h\big) &= I(gh)I(f+g). \end{aligned}$$

Therefore, the right side of inequality (3) can be rewritten as

$$I(fh)I(gh)I(f+g) = I(f)I(gh)I\big((f+g)h\big) = I(g)I(fh)I\big((f+g)h\big). \tag{6}$$

From (2), it is easy to see that $I(fh)$ and $I(gh)$ have the same sign. If they all equal to zero, then the inequality (3) holds clearly. Without loss of generality, suppose they are positive, then, from (6), we have

$$\begin{aligned} &\frac{[I(g)]^2 I(fh^2) + [I(f)]^2 I(gh^2)}{I(gh)I(fh)I(f+g)} \\ =\ &\frac{1}{I\big((f+g)h\big)}\left[\frac{[I(g)]^2 I(fh^2)}{I(g)I(fh)} + \frac{[I(f)]^2 I(gh^2)}{I(f)I(gh)}\right] \\ =\ &\frac{1}{I\big((f+g)h\big)}\left[\frac{I(g)I(fh^2)}{I(fh)} + \frac{I(f)I(gh^2)}{I(gh)}\right]. \end{aligned} \tag{7}$$

Using Cauchy's inequality and (2) yields

$$\frac{I(fh^2)}{I(fh)} \geq \frac{I(fh)}{I(f)} = \frac{I(gh)}{I(g)}, \tag{8}$$

$$\frac{I(gh^2)}{I(gh)} \geq \frac{I(gh)}{I(g)} = \frac{I(fh)}{I(f)}. \tag{9}$$

Substitution of (8) and (9) into (7) produces (3).

Since (8) and (9) take equality if and only if $h(x)$ is constant, so does inequality (3) if and only if $h(x)$ is constant. The proof of Theorem 1 is completed.

By Cauchy's inequality, we obtain

$$I(fh^2) \geq \frac{[I(fh)]^2}{I(f)} \quad \text{and} \quad I(gh^2) \geq \frac{[I(gh)]^2}{I(g)}.$$

Therefore

$$\begin{aligned} &I(fh^2)[I(g)]^2 + I(gh^2)[I(f)]^2 \\ \geq\ &\frac{[I(g)]^2[I(fh)]^2}{I(f)} + \frac{[I(f)]^2[I(gh)]^2}{I(g)} \\ \geq\ &2\big|I(gh)I(fh)\big|\sqrt{I(g)I(f)}, \end{aligned}$$

note that in the final line, the arithmetic mean–geometric mean inequality is used. This completes the proof of Theorem 2.

The Corollary is deduced from substitution of (6) into (3).

ACKNOWLEDGMENTS

The authors are indebted to the referees for their valuable comments and helpful suggestions simplifying the mathematical expressions of this paper.

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(Received June 10, 1997)
(Revised November 9, 1998)

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