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ON SHARP BOUNDS OF THE SPECTRAL RADIUS OF GRAPHS

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The spectral radius of a graph is the spectral radius of its adjacency matrix. In this paper, some sharp bounds of the spectral radius of graphs that depend only on vertex degrees are obtained.

1. INTRODUCTION

Let D be a digraph without loops and with vertex set $\{v_1, v_2, \ldots, v_n\}$. Its adjacency matrix $A(D)$ is defined to be the $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ if there is an arc from v_i to v_j, and $a_{ij} = 0$ otherwise. Let r_i (s_i) be the out-degree (resp. in-degree) of v_i, $i = 1, 2, \ldots, n$. Clearly, r_i is i-th row sum of $A(D)$, while s_i is the i-th column sum of $A(D)$. A digraph is said to be "k-balanced" if $|r_i - s_i| \leq k$ for $i = 1, 2, \ldots, n$. A 0-balanced digraph with $r_i = s_i = r$ for $i = 1, 2, \ldots, n$ is called strong balanced digraph. It follows immediately that if D is a simple graph (undirected graph without loop and multiline), then $A(D)$ is a symmetric $(0, 1)$ matrix with zero trace. We shall denote the characteristic polynomail of D by

$$p(D) = \det(xI - A(D)) = \sum_{i=0}^{n} a_i x^{n-i}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of $p(D)$. $\rho(D) = \max(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|)$ is the spectral radius of D. Since $A(D)$ is a symmetric matrix, $\rho(D)$ is an eigenvalue of $\det(XI - A(D))$, say λ_1. Since $A(D)$ is a symmetric matrix, its eigenvalues are real, and may be ordered as

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$

Hence we shall denote the spectral radius of D by λ_1.

The following result on bounds of spectral radius have been known (see[**1**]).

Theorem A (HONG). *Let D be a 0-balanced strongly connected digraph with n vertices and m arcs. Then*

$$\lambda_1 \leq \sqrt{m - n + 1}$$

with equality iff D is the star $K_{1,n-1}$ or the complete graph K_n.

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Theorem B (Hong). *Let G be a connected graph with n vertices and e edges. Then*

$$\lambda_1 \leq \sqrt{2e - n + 1}$$

with equality iff G is the star $K_{1,n-1}$or the complete graph K_n.

In this paper we generalized the above result as follows.

Theorem 1. *Let D be a k-balanced digraph with n vertices and m arcs, $r = \min\limits_{1 \leq i \leq n} r_i$, $S = \max\limits_{1 \leq i \leq n} s_i$. Then*

$$\lambda_1 \leq \sqrt{m - r(n-1) + (r-1)S + k}$$

with equality if and only if D is the star $K_{1,n-1}$ or a strongly balanced digraph.

Theorem 2. *Let G be a simple graph with n vertices and e edges, and $r = \min\limits_{1 \leq i \leq n} r_i, R = \max\limits_{1 \leq i \leq n} r_i$. Then*

$$\lambda_1 \leq \sqrt{2e - r(n-1) + (r-1)R}$$

with equality if and if G is the star $K_{1,n-1}$or the complete graph K_n.

2. MAIN RESULT

The proof of Theorem 1.

Let A_i denote the i-th row of $A(D)$. Since D is "k-balanced" we have

$$|r_i - s_i| \leq k.$$

Let $X = (x_1, x_2, \ldots, x_n)^T$ be a unit positive eigenvector of A corresponding to the eigenvalue λ_1. For $i = 1, 2, \ldots, n$, let $X(i)$ denote the vector obtained from X by replacing with 0 those components x_j for which $a_{ij} = 0$.

Since $AX = \lambda_1 X$, we have

$$A_i X(i) = A_i X = \lambda_1 x_i.$$

By the Cauchy-Schwartz inequality, for $i = 1, 2, \ldots, n$, we have

$$\lambda_1^2 x_1^2 = |A_i X(i)|^2 \leq |A_i|^2 |X(i)|^2 = r_i \left(1 - \sum_{j:a_{ij}=0} x_j^2\right).$$

Summing the above inequalities, we have

$$\lambda_1^2 = \lambda_1^2 \sum_{j=1}^{n} x_j^2 \leq \sum_{i=1}^{n} r_i \left(1 - \sum_{j:a_{ij}=0} x_j^2\right) = m - \sum_{i=1}^{n} r_i \sum_{j:a_{ij}=0} x_j^2,$$

$$
\begin{aligned}
\sum_{i=1}^{n} r_i \sum_{j:a_{ij}=0} x_j^2 &= \sum_{i=1}^{n} r_i x_i^2 + \sum_{i=1}^{n} r_i \sum_{j:a_{ij}=0} x_j^2 \\
&\geq \sum_{i=1}^{n} r_i x_i^2 + r \sum_{i=1}^{n} \sum_{j:a_{ij}=0, j\neq i} x_j^2 \\
&= \sum_{i=1}^{n} r_i x_i^2 + r \sum_{i=1}^{n} (n-1-s_i) x_i^2 \\
&= \sum_{i=1}^{n} (r_i - s_i) x_i^2 - (r-1) \sum_{i=1}^{n} s_i x_i^2 + r(n-1) \\
&\geq -\sum_{i=1}^{n} |r_i - s_i| x_i^2 - (r-1) \sum_{i=1}^{n} S x_i^2 + r(n-1) \\
&\geq -k - (r-1)S + r(n-1).
\end{aligned} \tag{1}
$$

Therefore, we have $\lambda_1 \leq \sqrt{m-(n-1)+(r-1)S+k}$.

In order equality to hold, all inequalities in the above argument must be equalities. In particular, from (1) we must have

$$\sum_{i=1}^{n} r_i \sum_{j:a_{ij}=0} x_j^2 = r \sum_{i=1}^{n} \sum_{j:a_{ij}=0, j\neq i} x_j^2$$

and

$$\sum_{i=1}^{n} (r_i - s_i)\, x_i^2 = -\sum_{i=1}^{n} k x_i^2, \qquad (r-1)\sum_{i=1}^{n} s_i x_i^2 = (r-1)\sum_{i=1}^{n} S x_i^2.$$

Hence, for each i we have

(i) $r_i = r$ or $r_i = n-1$; (ii) $r_i - s_i = -|r_i - s_i| = -k$;

(iii) if $r \neq 1$, then $s_i = S$.

Note that $\sum r_i = \sum s_i$, we have either $k=0$, $r_i = s_i = r = S$ or $n-1$, $i = 1, 2, \ldots, n$ or $k = 0$, $r_i = s_i = 1$ or $n-1$. That implies either D is a 0-balanced digraph with $r_i = s_i = r$ or $n-1$, $i = 1, 2, \ldots, n$. Conversely, it is easy to verify that the equality $\lambda_1 = r$ holds in the strongly balanced digraph with $r_i = s_i = r$ and in $K_{1,n-1}$.

Example. A directed cycle of order n is a strongly balanced digraph, $m = n, r_i = s_i = 1$ for $i = 1, 2, \ldots, n$, $k = 0$ and

$$\lambda_1 = \sqrt{n-(n-1)} = 1.$$

Let T_5 be a strongly balanced tournarment with 5 vertices. Then $n = 5$, $m = 10$, $r_i = s_i = 2$, $k = 0$, and

$$\lambda_1 = \sqrt{10 - 2\,(5-1) + (2-1)\cdot 2 + 0} = 2$$

but by Brualdi and Hoffman's bound (see[**2**]) $\lambda_1 \leq 3$, since $m = 3^2 + 1$.

Example. For the digraph D with adjacency matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

we have $n = 4, m = 6, r = 1, S = 2, k = 1$ and thus

$$\lambda_1 \leq \sqrt{6 - 1 \cdot (4 - 1) + 1} = \sqrt{4} = 2.$$

By definition $p(D) = f(\lambda) = \lambda^4 - 4\lambda - 1$. Since $f(1.3) < 0$ and $f(\lambda) > 0$ for $\lambda \geq 1.4$, we have

$$1.3 < \lambda_1 < 1.4.$$

Corollary 1.1. *Let D be a 0-balanced strongly connected digraph with n vertices and m arcs. Then*

(1) $$\lambda_1 \leq \sqrt{m - r\,(n-1) + (r-1)\,S}.$$

Remark. If D is a 0-balanced strongly connected digraph without vertices of outdegree 0, then $r \geq 1$ and

$$\lambda_1 \leq \sqrt{m - r\,(n - 1 - S) - S} \leq \sqrt{m - n + 1 + S - S} = \sqrt{m - n + 1}.$$

This is Theorem A.

Corollary 1.2. *Let G be a simple connected graph with n vertices and e edges. Then*

(2) $$\lambda_1 \leq \sqrt{2e - r(n-1) + (r-1)S}$$

with equality if and only if G is the star $K_{1,n-1}$ or a regular graph.

Proof. G is a 0-balanced digraph. Thus $m = 2e$. Now (3) follows from (2).

The following example shows that the bound (3) improves that in Theorem B.

Example. Let

$$A(D) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then $n = 6, e = 10, r = 3, S = 4$. By (3) we have

$$\lambda_1 \leq \sqrt{2 \times 10 - 3 \times 5 + 2 \times 4} = \sqrt{13}.$$

But by Theorem B

$$\lambda_1 \leq \sqrt{2 \times 10 - 6 + 1} = \sqrt{15}.$$

Remark. If G is a simple connected graph without isolated vertices, then $r \geq 1$. By (3) we have

$$\lambda_1 \leq \sqrt{2e - n + 1}.$$

This is Theorem B.

Corollary 1.3. *Let G be a simple planar connected graph with n vertices and m edges. Then*

$$\lambda_1 \leq \sqrt{2\,(3n-6) - r\,(n-1) + (r-1)\,S}. \tag{3}$$

Proof. Note that $m \leq 3n-6$ for a planar graph, and so we have (4).

Corollary 1.4. *Let G be a simple connected graph with n vertices and e edges. Then*

$$\sum_{i=2}^{n} \lambda_i^2\,(G) = 2e - \lambda_1^2 \geq r\,(n-1) - (r-1)\,S = r\,(n-1-S) + S \tag{4}$$

with equality if and only if

(a) *G is a regular graph*;

(b) *G is the star $K_{1,n-1}$.*

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