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SOME PROPERTIES OF INCREASING FUNCTIONS, ESPECIALLY THOSE RELATED TO RECURRENTLY DEFINED SEQUENCES

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In this paper one studies some properties of increasing real functions defined on real intervals, especially in connection with sets S_x and T_x, and also monotony and convergence of sequences defined recurrently by such functions.

This text comprises several results; those forming the first group and included in Theorem 1 refer to different properties of an increasing function $f : I \to \mathbf{R}$, while the results of the second group, colected in the formulation of Theorem 2, treat exhaustively the question of convergence and monotony of a sequence (x_n) defined recurrently by means of such a function:

$$x_1 = x, \quad x_{n+1} = f(x_n) \quad (n = 1, \ldots).$$

Here and in that which follows, $\mathbf{R}$ denotes the set of all real numbers and $I \subseteq \mathbf{R}$ an interval which is neither empty nor singleton. As usually, $\mathbf{N}$ is the set of natural numbers and $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$.

1. RESULTS

Theorem 1. *Let*

$$(1) \qquad \begin{cases} I \subseteq \mathbf{R} \textit{ be an interval which is neither empty nor singleton} \\ \textit{and } f : I \to \mathbf{R} \textit{ an increasing function } (\textit{non necessarily strictly}). \end{cases}$$

Under this condition:

1° *If* $x < \sup I$ *and* $f(x) > x$, *then the set*

$$S_x := \{y : x < y \in \mathbf{R} \wedge (x, y) \subseteq I \wedge f(t) > t (x \le t < y)\}$$

is not empty and

$$(x, \min\{f(x), \sup I\}) \subseteq S_x, \quad (x, \sup S_x) \subseteq S_x;$$

1991 Mathematics Subject Classification: 26A48 *moreover,* $\sup S_x = \max S_x$ *if* $\sup S_x < +\infty$. *The corresponding symmetrical assertions are also true.* (*The exact meaning of the last sentence is*: *retaining the supposition that* f *is increasing-strictly or not, the assertions which differ from the previous assertions ony in the fact that right and left sides have changed places, i.e. all inequalites have been repaced by opposite inequalites, -remain true.*- The same sentence should be repeated at several places; at any of them it will be replaced by the sign $\bowtie$.)

2° *If* $z := \sup S_x$ $(= \max S_x) \in I$, *then* $z \leq f(z)$. *If moreover* $z < \sup I$, *we have*

$$z = f(z) = f(z-0) = \min F_x^+,$$

where

$$F_x^+ := \{t : x < t \wedge f(t) = t\}$$

(*so* F_x^+ *is the set of all fixed points of* f *greater than* x). *If the set*

$$F_x^- := \{t : t < x \wedge f(t) = t\}$$

(*i.e. the set of all fixed points of* f *less than* x) *is not empty, its maximum need not exist.*

3° *A sufficient, but not a necessary, condition for the existence of a fixed point of* f *is the existence of numbers* $x, y \in I$ *such that* $x < y$, $f(x) > x$ *and* $f(y) < y$. *If this condition is satisfied, at least one fixed point of* f *lies in the interval* (x, y).

4° *Again, let*

$$\begin{cases} x < \sup I, f(x) > x, \\ \textit{and let } T_x := \{t : x < t \in \mathbf{R} \quad \wedge \quad f(t-0) = t\} \neq \emptyset. \end{cases} \tag{2}$$

Then there exists the minimum y *of* T_x. *This minimum can be an accumulation point of the set* T_x. *A necessary, but not a sufficient, condition for this is* $y \in F_x^+$. *A suficient, but not a necessary, condition for an* $x \in I$ *with properties* $x < \sup I$ *and* $f(x) > x$ *to be* $T_x \neq \emptyset$ *is* $F_x^+ \neq \emptyset$ (*i.e. that there exists at least one fixed point of* f *greater than* x).

Theorem 2. *Under the suppositions* (1), *the folowing assertions hold*:

1° *If the condition* (2) *is satisfied, so that there exists* $y := \min T_x$ (*assertion* 4° *of Theorem* 1), *and also the condition*

$$f(t) < y \quad (x \leq t < y) \quad \vee\, y \in F_x^+ \tag{3}$$

then by

$$x_1 = x, \quad x_{n+1} = f(x_n) \quad (n = 1, \ldots) \tag{4}$$

is defined an infinite sequence (x_n)*,which increases strictly and converges to* y *if the first part in* (3) *holds, and if only the second part in* (3) *is satisfied, it increases stricty for* $n \in \{1, \ldots, m\}$*, with an* $m \in \mathbf{N}$*, and for* $n \geq m$ *it takes constantly the value* y*.*

(It is not dificult to see that the first part of disjunction (3) can be formulated as follows:a left neighbourhood of the point y in which the function f should be constant-does not exist).

1°.1 *If* $x < \sup I$*,* $f(x) > x$ *and* $T_x = \emptyset$*, then* (4) *defines an infinite sequence* (x_n) *which increases strictly and tends to* $+\infty$*, or* (4) *defines only a finite sequence which is strictly increasing on its domain-depending on whether* $\sup I = +\infty$ *or* $\sup I < +\infty$*.*

2° *Suppose that* (2) *is satisfied and that* (3) *is not. Then there exists a strictly increasing sequence*

$$y_\ell \quad (\ell = 0, 1, \ldots), \tag{5}$$

finite or infinite, such that

$$\begin{cases} y_0 = x; \quad \text{with } y_{\ell-1} \text{ instead of } x \text{ condition (2) is satisfied and} \\ \text{condition (3) is not, and } y_\ell = \min T_{y_{\ell-1}} \quad (\ell = 1, \ldots), \end{cases} \tag{6}$$

and moreover

$$x_{p_\ell} = y_\ell \ (\ell = 0, 1, \ldots),\ p_0 = 1,\ p_{\ell-1} < p_\ell \quad (\ell = 1, \ldots),$$

where $x_n (n = 1, \ldots)$ *is the sequence defined by* (4)*. Further :*

2°.1 *The sequence* (5) *is finite if for some* $\ell_0 \in \mathbf{N}$:

$$y_{\ell_0} = \sup I, \ \ or \tag{7}$$

$$y_{\ell_0} < \sup I \quad \wedge \quad T_{y_{\ell_0}} = \emptyset, \ or \tag{8}$$

$$\text{with } y_{\ell_0} \text{ instead of } x \text{ both conditions (2) and (3) are satisfied.} \tag{9}$$

2°.1.1 *In the case* (7)*, sequence* (4) *is defined and stricty increasing on the set* $\{1, \ldots, p_{\ell_0} + 1\}$ *or on the set* $\{1, \ldots, p_{\ell_0}\}$ *, –depending on whether* $\sup I \in I$ *or* $\sup I \notin I$*.*

2°.1.2 *In the case* (8)*, sequence* (4) *is infinite and strictly increasingly tends to* $+\infty$*, or it is finite and strictly increasing on its domain, –depending on whether* $\sup I = +\infty$ *or* $\sup I < +\infty$*.*

$2^\circ.1.3$ *In the case* (9), *sequence* (4) *is infinite and strictly increasingly converges to* $z := \min T_{y_{\ell_0}}$, *or for some* $m\,(\geq p_{\ell_0})$ *is strictly increasing on the set* $\{1,\dots,m\}$ *and for* $n \geq m$ *constantly takes the value* z, *–depending on whether, with* y_{ℓ_0}, *instead of* y, *the first member of the disjunction* (3) *is satisfied or only its second member holds.*

$2^\circ.2$ *If for none* $\ell_0 \in \mathbf{N}$ *any of conditions* (7), (8) *and* (9) *is satisfied, then sequence* (5), *and consecuently sequence* (4), *is infinite, and moreover sequence* (4) *is strictly increasing and converges to a number* $u \in \mathbf{R}$ *with the property* $f(u-0) = u$ (*i.e. with the property* $u \in T_x$), *or it is strictly increasing and tends to* $+\infty$, *– depending on whether sequence* (y_ℓ) *is bounded or unbounded.*

All previous cases are effectively possible. $\bowtie$ (*This sign refers to the whole statement, i.e. to all assertions in* 2).

3° *If* $x = \max I$ *and* $f(x) > x$, *sequence* (4) *is defined and strictly increasing on the set* $\{1,2\}$. $\bowtie$.

4° *In order that a sequence* (4), *with* $x \in I$, *is strictly increasing and convergent to* $u \in \mathbf{R}$ *it is nessesary that* $f(u-0) = u$. *If* $u \in \mathbf{R}$, $\inf I < u \leq \sup I$ *and* $f(u-0) = u$, *a sequence* (x_n) *defined by* (4), *strictly increasing and tending to* u *need not exist.*

1.1. In particular, if the function f is *strictly increasing*, one can omit from previous formuation any mention of the condition (3), all parts of the text under 1° referring to this condition and also whole text under 2°. Therefore, in this case Theorem 2 can be repaced by the following simpler statement.

Theorem 2.1. *Let the function* $f : I \to \mathbf{R}$ *be strictly increasing. Then, if* (2) *holds,* (4) *defines an infinite sequence* (x_n) *which is strictly increasing and converges to* $y = \min T_x$. *–If* $x < \sup I$, $f(x) > x$ *and* $T_x = \emptyset$, *then by* (4) *is defined an infinite sequence which is strictly increasing and tends to* $+\infty$, *or only a finite sequence is defined and this sequence is strictly increasing on its domain, –depending on whether* $\sup I = +\infty$ *or* $\sup I < +\infty$. *–If* $x = \max I$ *and* $f(x) > x$, *sequence* (4) *is defined and strictly increasing on the set* $\{1,2\}$. *–For the convergence to* $y \in \mathbf{R}$ *of a sequence* (x_n) *defined by* (4) *and strictly increasing, it is nessesary that* $f(y-0) = y$. *–If* $y \in \mathbf{R}$, $\inf I < y \leq \sup I$ *and* $f(y-0) = y$, *then a sequence* (x_n) *defined by* (4) *which strictly increases and converges to* y *–need not exist.* $\bowtie$ (*Refers to all preceding assertions*).

1.2. If the function $f : I \to \mathbf{R}$ is *continuous*, in preceding statements the points $y \in I$ with the property $f(y-0) = y$ or $f(y+0) = y$ ought to be replaced by fixed points of f, since then, obviously, any such point is fixed and every fixed point has both previous properties, excluding one of them at the endpoint of I. Therefore, the statement which in this case comprises all assertions of Theorems 1 and 2 could be as follows:

Theorem 2.2. *Let the function $f : I \to \mathbf{R}$ be increasing and continuous. Then :*

1° *For any $x \in I$ such that $f(x) > x$, the set F_x^+ of all fixed points of f greater than x, provided that it is not empty, has its minimum, and the set F_X^- of all fixed points of f less than x, if not empty, has its maximum.*

2° *Suppose that $x < \sup I$ and $f(x) > x$. Then the sequence $x_n (n = 1, \ldots)$ defined by (4) : if $F_x^+ \neq \emptyset$, increases strictly and converges to $y := \min F_x^+$, or increases strictly on the set of indices $\{1, \ldots, m\}$ and for $n \geq m$ takes constantly the value y, –depending on whether we have $f(t) < y$ $(x < t < y)$ or this condition is not satisfied; if $F_x^+ = \emptyset$, the sequence (4) is strictly increasing and tends to $+\infty$, or increases strictly and converges to $\sup I$, or is finite and strictly increasing on its domain, –depending on whether we have $\sup I = +\infty$ or $z := \sup I < +\infty$ and $f(x) < z = f(z-0)$ or $z < +\infty$ and $f(x) \geq z \vee z < f(z-0)$.* $\bowtie$

3° *If $x = \sup I$ and $f(x) > x$, the sequence defined by (4) is defined and increases stricty on the set $\{1, 2\}$.* $\bowtie$

4° *For the existence of a sequence defined by (4), strictly monotone and convergent to $y \in I$ –it is nessesary, but not sufficient, that y be a fixed point of f.* $\bowtie$

1.3. A conclusive comment. Taking into consideration the fact that, if x is a fixed point of the mapping f, then sequence (4) constantly takes the value x, all assertions of Theorem 2 (and partially and implicitely of Theorem 1) can be resumed as follows:

Under the hypothesis that the function $f : I \to \mathbf{R}$ is increasing, sequence (4), with $x \in I$, is increasing or decreasing, depending on whether $f(x) \geq x$ or $f(x) \leq x$; in fact strictly on its whole domain, which can be finite or infinite, or strictly up to a certain index $m \in \mathbf{N}$ and further being constant; if this sequence is infinite and strictly monotone, it can converge to a number y such that $f(y-0) = y$ in the case of increase and $f(y+0) = y$ in the case of decrease, or tend in the first case to $+\infty$ and in the second to $-\infty$; the conditions of realization of any previous possibilities are precisely determined by more extensive statements of Theorems 1 and 2.

2. PROOFS

Proof of Theorem 1. 1° Let $x < \sup I$ and $f(x) > x$. Suppose that the inclusion $(x, \min\{f(x), \sup I\}) \subseteq S_x$ is not true. Then there exist $y \in (x, \min\{f(x), \sup I\})$ and $t \in (x, y)$ such that $f(t) \leq t$, and so $f(t) \leq t < y < f(x)$ and at the same time $x < t$, which contradicts the supposition (1) on the function f. Hence $(x, \min\{f(x), \sup I\}) \subseteq S_x$ and consequently $S_x \neq \emptyset$. Further, if $y \in (x, \sup S_x)$, then exists $t \in (y, \sup S_x) \cap S_x$, and because $[x, y) \subseteq [x, t)$ we have $f(u) > u (x \leq u < y)$, which implies $y \in S_x$. Hence $(x, \sup S_x) \subseteq S_x$. It follows that $f(y) > y (x \leq y < \sup S_x)$; therefore, if $\sup S_x < +\infty$ and consequently $\sup S_x \in \mathbf{R}$, we have $\sup S_x \in S_x$, i.e. $\sup S_x = \max S_x$.

$2°$ Let us suppose that

$$z := \sup S_x \in I. \tag{10}$$

Then we have, by the last assertion in $1°$,

$$f(t) > t \quad (x \leq t < z). \tag{11}$$

Further, in this case $f(z) < z$ would imply, with an $u \in (f(z), z)$, $f(u) > u > f(z)$, that is $f(u) > f(z)$ and simultaneously $u < z$, in contradiction with our starting supposition. Therefore,

$$z \leq f(z). \tag{12}$$

In particular, if

$$z < \sup I \tag{13}$$

then $f(z) > z$ would imply, by 1^0, the existence of some $u > z$ such that $f(t) > t \quad (x \leq t < u)$, i.e. such that $z < u \in S_x$, in contradiction with (10). Hence and by (12), $f(z) = z$. This and (11) imply $z = \min F_x^+$. Finally, taking into consideration (11), we conclude that $z = f(z) \geq f(z-0) \geq z$, i.e. $f(z-0) = z$.

The last assertion is proved by the example: $f(t) = t - \frac{1}{4} \sin 1 (t \leq -1)$, $f(t) = t + \frac{1}{4} t^2 \sin \frac{1}{t} \ (-1 < t < 0)$, $f(t) = t + 1 \ (t \geq 0)$, $I = \mathbf{R}$ and $x = 0$.

$3°$ One can obtain both assertions, except the detail expressed by the words "but not a necessary", by a direct application of the known theorem on fixed point of A. Tarski [**1**]. Namely, under the accepted conditions, $f|[x, y]$ is an increasing mapping of the complete lattice $[x, y]$ into $[x, y]$. The non necessity of this condition is proved by the example of the function $f(t) = 2t \quad (t \in \mathbf{R} = I)$, or by the function $f(t) = t, \quad (t \in \mathbf{R} = I)$.

$4°$ Let condition (2) be satisfied. Then $y := \inf T_x$ exists and $x \leq y$. If y is not an accumulation point of the set T_x, we have certainly $y = \min T_x$. Let y be an accumulation point of T_x. Then there exists to y convergent sequence (y_n) of numbers greater than y and such that $f(y_n - 0) = y_n, (n \in \mathbf{N})$. Therefore,

$$f(y) \leq f(y_n - 0) = y_n \to y \quad (n \to \infty)$$

and so

$$f(y) \leq y. \tag{14}$$

This and $f(x) > x$ imply that the equality $y = x$ is not possible, i.e. that

$$y > x. \tag{15}$$

The supposition that $f(t) \le t$ for some $t \in (x, y)$ would imply, on account of 2°, the existence of $z := \max S_x$ and the relations $z \in (x, t] \subset (x, y)$ and $z \in T_x$, in contradiction with $y = \inf T_x$. Hence

$$f(t) > t \quad (x \le t < y). \tag{16}$$

It follows that

$$f(y-0) = \lim_{t \to y-0} f(t) \ge \lim_{t \to y-0} t = y. \tag{17}$$

From (14) and (17) follows $y \ge f(y) \ge f(y-0) \ge y$, and so $y = f(y-0)$ and $f(y) = y$. The first of these equalites means that in this sedond case we also have $y = \min T_x$, which proves the first statement in this point. The second equality proves the necessity of the condition from the second statement. –The non sufficiency of this condition is proved by the example: $I = R, f(t) = \frac{1}{2}t \quad (t \in \mathbf{R})$; in this case we have $T_{-1} = \{0\}, \min T_{-1} = 0 \in F^+_{-1}$ and 0 is not an accumulation point of the set T_{-1}. –On the other hand, $\min T_x$ can indeed be an accumulation point of T_x as proved by the following case ($I \in \mathbf{R}$): $f(t) = \frac{1}{2}t \quad (t \le 0), f(t) = t + \frac{1}{4}t^2 \sin \frac{1}{t} \quad (0 < t < 1), f(t) = t + \frac{1}{4}\sin 1 \quad (t \ge 1)$; this function is obviously increasing on the intervals $(-\infty, 0]$ and $[1, +\infty)$, and on $[0, 1]$ too, because $f'(t) = 1 + \frac{1}{2}t \sin t^{-1} - \frac{1}{4}\cos t^{-1} > 1 - \frac{1}{2} - \frac{1}{4} > 0 \quad (0 < t < 1), f(+0) = 0, \quad f(1-0) = 1 + \frac{1}{4}\sin 1$. In this case $\min T_{-1} = 0$ and $T_{-1} = \{0\} \cup \{\frac{1}{k\pi} : k \in \mathbf{N}\}$, which means that $\min T_{-1}$ is an accumulation point of the set T_{-1}. –Further, when $f(x) > x$ and $x < \sup I$, the set S_x, by 1°, is not empty and then the supposition $F^+_x \ne \emptyset$ implies the relations $x < z := \sup S_x \in I$. Under the same supposition: if we have $z (= \max S_x) < \sup I$, it will be, on account of 2°, $f(z-0) = z$, and if $z = \sup I$, then we have first, by (11), $f(z) = z$, and (11) also implies $f(z-0) \ge z$, so that we obtain $z = f(z) \ge f(z-0) \ge z$, that is $f(z-0) = z$; therefore, in both cases $z \in T_x$. This means that under the cited conditions $F^+_x \ne \emptyset$ implies $T_x \ne \emptyset$. –Finally, the example of the function $f(t) = \frac{1}{2}t \quad (t < 0), \quad f(t) = t + 1 \quad (t \ge 0)$, for which $T_{-1} = \{0\}$ and $F^+_{-1} = \emptyset$, proves that $F^+_x \ne \emptyset$ is not a necessary condition for $T_x \ne \emptyset$.

Proof of Theorem 2. 1° Let conditions (2) and (3) be satisfied. If the first member of the disjunction (3) is satisfied in this case, then, as can be established by a simple consideration which uses the inequality (16), sequence (4) is infinite and strictly increasing, and we also have $x < x_n < y (n \in \mathbf{N})$. Hence $u = \lim_{n \to \infty} x_n$ exists and $x < u \le y$. Since

$$u = \lim_{n \to \infty} x_n = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} f(x_n) = f(u-0), \tag{18}$$

i.e. $u \in T_x$, the inequality $u < y$ is not possible, and so $u = y$. Therefore, in this case sequence (4) is strictly increasing and converges to $y = \min T_x$. If only the second member of the disjunction (3) is satisfied, then $f(t) \le f(y) = y \quad (x \le t < y)$ and consequently sequence (4) is again infinite, bounded from above by the number y, and, on account of (16), increasing. Hence $u = \lim\limits_{n\to\infty} x_n$ exists and $x < u \le y$ (first inequality is strict because $x_1 < x_2$). The inequality $u < y$ is again impossible. Namely, in the consideration expressed by the statement comprising formula (18) we have in fact established the following:

$$\text{(19)} \qquad \left\{ \begin{array}{l} \text{if some sequence defined by (4) is strictly increasing} \\ \text{and converges to } u \in \mathbf{R} \text{ then } x < u \text{ and } f(u-0) = u. \end{array} \right.$$

Hence the strict increase of sequence (4) in this case implies $u \in T_x$, which excludes the possibility $u < y$. If sequence (4) is not strictly increasing, there exists $m \in \mathbf{N}$ such that $x_m = x_{m+1} = f(x_m)$, and this implies $x_n = x_m = u \quad (n \ge m)$; it follows $f(u) = f(x_m) = x_{m+1} = u$, and this excludes the possibility of the relation $u < y$. So we have also in this second case

$$\lim_{n\to\infty} x_n = y. \tag{20}$$

However, since now the first member of disjunction (3) does not hold, there exists $u \in [x, y)$ such that $f(t) = y \quad (u < t \le y)$. On account of (20), it is not possible that $x_n \le u \quad (n \in \mathbf{N})$ and consequently there exists $k \in \mathbf{N}$ such that $u < x_k \le y$, and this implies that we have, with $\ell = k+1, x_n = y \quad (n \ge \ell)$. Denoting by m the smallest of such numbers ℓ, we have $m \ge 2$ (because $x_1 < x_2 \le y$) and $x_n < y \quad (n = 1, \ldots, m-1), x_n = y \quad (n \ge m)$. It is clear that sequence (x_n) is strictly increasing on the set $\{1, \ldots, m\}$.

1°.1 Suppose that $x < \sup I, f(x) > x$ and $T_x = \emptyset$. By the statements 3° and 4° of Theorem 1, we have $f(t) > t \quad (x \le t \in I)$, which implies the strict increase of sequence (4) on its whole domain. If $\sup I = +\infty$, sequence (4) is infinite, as can simply be established by induction, and the inequality $u := \lim\limits_{n\to\infty} x_n < +\infty$ is not possible, because this inequality would imply, by (19), $x < u = f(u-0)$, i.e. $u \in T_x$. Therefore, in this case $\lim\limits_{n\to\infty} x_n = +\infty$. If $v := \sup I < +\infty$, the supposition $x_n \le v \quad (n \in \mathbf{N})$ would imply $w := \lim\limits_{n\to\infty} x_n \le v$ and further, by (19) again, $x < w = f(w-0)$, that is $w \in T_x$. Consequently, in this case exists an $m \in \mathbf{N}$ such that $m \ge 2$, $x_n \in I \quad (n = 1, \ldots, m-1)$ and $x_m \notin I$, which means that sequence (4) is defined on the set $\{1, \ldots, m\}$ only and that it strictly increases on this set.

2° Suppose that condition (2) is satisfied and that condition (3) is not. In this case, on account of $f(t) \le f(y-0) = y \quad (x \le t < y)$, there exists $u \in [x, y)$ such that $f(t) = y \quad (u < t < y)$. Denoting by v the infimum of the set of all such numbers u, we have $x \leq v < y$, $f(t) < y \quad (t \in J)$ and $f(t) = y \quad (t \in [x, y) \setminus J)$, where J denotes the interval $[x, v]$ or the interval $[x, v)$ (this second interval being empty if $v = x$), –depending on whether $f(v) < y$ or $f(v) = y$. Then we cannot have $x_n \in J \quad (n \in \mathbf{N})$; for, this supposition would imply, in virtue of (16), the strict increase of sequence (4) and further $x < \lim\limits_{n\to\infty} x_n \leq v < y$, which, by (19), is impossible. Therefore, there exists $m := \max(\{1\} \bigcup \{k : k \in \mathbf{N} \wedge x_k \in J\})$. Sequence (4) is obviously strictly increasing on the set $\{1, \ldots, m\}$. If $J = \emptyset$, we have $m = 1$, $x_1 < y$, $x_2 = f(x_1) = y$, and when $J \neq \emptyset$, then $x_m \in J, \quad x_m < f(x_m) = x_{m+1} \in (x, y) \setminus J, \quad x_{m+1} < f(x_{m+1}) = x_{m+2} = y = y_1$. Hence, if we put $p_0 = 1$ and in the first case $p_1 = 2 \quad (= m + 1)$, and in the second one $p_1 = m + 2$, it will be $p_1 > 1 = p_0, x_{p_1} = y_1$ and sequence (4) will increase strictly on the set $\{1, \ldots, p_1\}$. It is clear, further, that either one of conditions (7), (8) and (9), with 1 instead of ℓ_0, is satisfied, or, with y_1 instead of x, condition (2) is satisfied and condition (3) is not. In the case (7) and (8), obviously, the continuation of the forming of sequence (y_ℓ) with demanded properties is not possible. In the case (9) it is possible to make at most one step yet in this forming, because, if we put in this case $y_2 = \min T_{y_1}$, we will have by the result under 1^0, $x_n \leq y_2 \quad (n \in \mathbf{N})$. In the last of mentioned cases, however, we have $y_1 \in I, \quad f(y_1) > y_1$ and $T_{y_1} \neq \emptyset$; hence, putting $y_2 = \min T_{y_1}$, one concludes, on the basis of the above analysis concerning the same situation with the point $y_0 = x$ instead of the point y_1, that there exists $p_2 > p_1$ such that $y_2 = x_{p_2}$ and that sequence (x_n) increases strictly on the set $\{p_1, \ldots, p_2\}$.

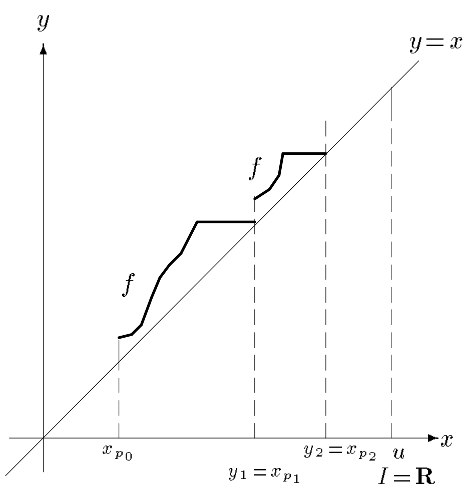

Fig. 1

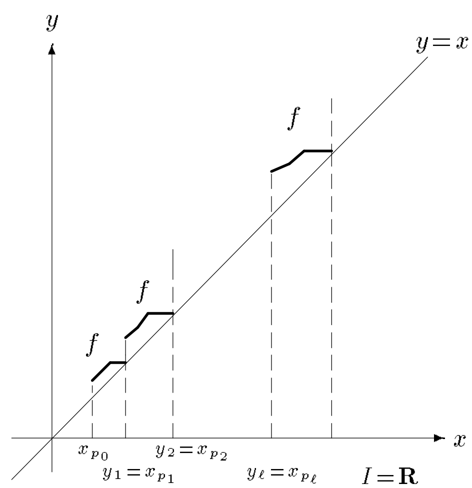

Fig. 2

Continuing in this way, one can establish the correctness of all statements in 2° concerning the existing possibilities. It is not difficult to prove by examples that each of them can effectively take place. Namely, the examples respectively represented by figures 1 and 2 prove effective realizability of both possibilities in the case 2°.2.

[A concrete aspect of the second example gives the function (see Figure 3) $f(t) = [t]+1 \quad (t \in \mathbf{R})$; in this case, we have, for each $x \in \mathbf{R}$, $\quad y_\ell = [x]+\ell \quad (\ell \in \mathbf{N}), p_\ell = \ell+1 \quad (\ell \in \mathbf{N}_0)$ and consequently $x_\ell = x_{p_{\ell-1}} = y_{\ell-1} = [x]+\ell-1 \quad (\ell \geq 2)$, which implies $x_\ell \uparrow +\infty \ (\ell \to \infty)$.]

3° This statement is obvious.

4° The assertion formulated by the first sentence coincides with the statement (19). The assertion contained in the second sentence is proved by the example of the function cited at the end of the proof of statement 2° of Theorem 1: in this case, for $u = 0$ we have $f(u-0) = u$ and $-\infty = \inf I < u < \sup I = +\infty$, and for any real $x < u$ such that $f(x) > x$ sequence (4) converges (statement 1° of Theorem 2) to the nearest fixed point of f greater then x; this fixed point, however, is less than u.

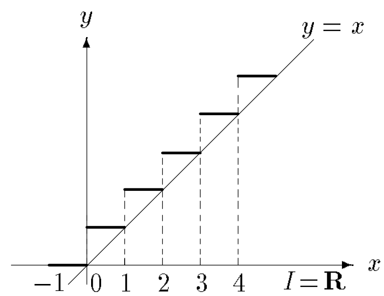

Fig. 3

Proof of Theorem 2.1. All statements of this theorem follow from Theorem 2, on account of the fact that when the function f is strictly increasing the condition (2) cannot hold without the condition (3) (more precisely, (2) cannot hold if the first part of disjunction (3) does not hold simuntanously), and consequently all assertions concerning the case when (2) $\wedge\neg$ (3) can be omitted.

All statements of Theorem 2.2., *excepting the last one, follow from Theorems* 1 *and* 2, with regard to the remarks which precede the formulation of this theorem. Its last statement is proved by the example of the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(t) = t - \frac{1}{4}\sin 1 \quad (t \leq -1), \quad f(t) = t + \frac{1}{4}t^2 \sin\frac{1}{t} \quad (-1 < t < 0), f(t) = t \quad (t \geq 0)$: number 0 is a fixed point of the mapping f and for each $x < 0$ we have $\min F_x^+ < 0$, and therefore the corresponding sequence (4) cannot converge to 0.

3. TWO SUPPLEMENTARY REMARKS

3.1. The statement 3° of Theorem 1 can be formulated (interpreted) as follows: *If the function $f : I \to \mathbf{R}$ is increasing, then the function $g(x) = f(x) - x (x \in I)$ has the following property (of a continous function): it cannot pass from a positive to a negative value without taking all midvalues between corresponding points.*

Let us remark that the statement which differs from the previous one only by the interchange of the words "*positive*" and "*negative*" does not hold.On the other hand,it is easy to see that in this formulation the words "*positive*" and "*negative*" can be changed by the words, "*greater*" and "*smaller*", respectively.

3.2. All assertions in the preceding text, excepting that in **3.1**, refer exclusively to the order structure of the system of real numbers, i.e. to the set $\mathbf{R}$ordered by the relation $\leq$, –directly or through the topology generated in the usual manner by this order (which includes convergence of sequences and limits and continuity of functions treated here). On the other hand, it is well known (see, for example, [**2**] p.151, or [**3**] p.217) that: every totally ordered and dense set which is conditionally complete (namely, in which each nonempty set bounded from above has its supremum), unbounded from both sides and separable (i.e. with a denumerable everywhere dense part)–is isomorphic with the set $\mathbf{R}$ordered by the relation $\leq$. Therefore:

All theorems and other statements in this paper, excepting that in **3.1.**, *hold for every totally ordered, dense, conditionally complete, unbounded from both sides and separabile set–provided that convergence of sequences and limit value and continuity of functions are defined in the topology generated by this order.*

We remark that this conclusion is in no way affected by the fact that the algebraic structure of the real number system was used in some details of the given proofs (in constructions of examples and counterexamples for particular and negative assertions).

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