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NOTE ON THE BEHAVIOR OF SOLUTIONS OF DIFFERENCE EQUATIONS OF ARBITRARY ORDER

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The nonlinear difference equation of the form

(1) $$\Delta^m(u_n + p_n u_{n-k}) + q_n f(u_{\tau_n}) = 0 \quad (m \geq 1,\ n = 0, 1, 2, \ldots)$$

is considered, where Δ^m is the m-order forward difference operator; (p_n) and (q_n) are sequences of real numbers with $q_n \geq 0$ eventually, (τ_n) is a sequence of integers with $\tau_n \leq n$ and $\tau_n \to \infty$ as $n \to \infty$, k is a positive integer. The function f is real valued function satisfying $uf(u) > 0$ for $u \neq 0$. The asymptotic properties of nonoscillatory solutions of (1) are studied. Sufficient conditions are also given to insure that all solutions of (1) when m is even, are oscillatory.

1. INTRODUCTION

In this paper we study the asymptotic behavior of the solutions of the nonlinear difference equation of the form

(1) $$\Delta^m(u_n + p_n u_{n-k}) + q_n f(u_{\tau_n}) = 0 \quad (m \geq 1,\ n = 0, 1, 2, \ldots),$$

where Δ is the forward difference operator, i.e.

$$\Delta v_n = v_{n+1} - v_n \ \text{ and } \ \Delta^i v_n = \Delta(\Delta^{i-1} v_n) \quad (i = 1, \ldots, m,\ \ \Delta^0 v_n = v_n),$$

(p_n) and (q_n) are sequences of real numbers with $q_n \geq 0$ eventually, (τ_n) is a sequence of integers with $\tau_n \leq n$ and $\tau_n \to \infty$ as $n \to \infty$, k is a positive integer. The function f is real valued function satisfying $uf(u) > 0$ for $u \neq 0$.

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By a solution of (1) we mean a sequence (u_n) which is defined for all $n \geq \min_{i\geq 0}\{i-k, \tau_i\}$ and satisfies (1) for n sufficiently large. We consider only such solutions which are nontrivial for all large n. A solution (u_n) of (1) is said to be nonoscillatory if the terms u_n of the sequence are eventually positive or eventually negative. Otherwise it is called oscillatory.

Recently, there has been an increasing interest in the study of oscillation and asymptotic behavior of solutions "delay" and "neutral delay" type difference equations. Most of the known results are related to the equations of type (1) in the case $m = 1$ or $m = 2$; see for example [**4**], [**5**], [**7**–**12**], [**14**], [**16**], [**17**]. Some results concerning oscillatory and asymptotic behavior of solutions of difference equations of higher order have been established in papers [**1**–**3**], [**6**], [**13**], [**15**].

Our purpose in this paper is to study the asymptotic behavior of nonoscillatory solutions of equation (1). Also, we give oscillation theorem for (1) when m is even. The obtained results extend some of those contained in [**11**].

2. MAIN RESULTS

Troughout this paper we assume that the following assumptions are satisfied:

(2) $f(u)$ is bounded away from zero if u is bounded away from zero,

(3) $\sum_{n=0}^{\infty} q_n = \infty.$

Let (u_n) be a solution of (1). Set

$$z_n = u_n + p_n u_{n-k}. \tag{4}$$

We begin with two lemmas that are useful in proving a number of our asymptotic results. All proofs in this note will be done only for the case when a nonoscillatory solution of (1) is eventually positive, since the proof for an eventually negative solution is similar.

Lemma 1. *If (u_n) is an eventually positive (negative) solution of (1), then*

(a) *$(\Delta^{m-1} z_n)$ is eventually nonincreasing (nondecreasing) and $\Delta^{m-1} z_n \to L < \infty$ $(> -\infty)$ as $n \to \infty$,*

(b) *if $L > -\infty$ $(< \infty)$, then $\liminf_{n\to\infty} |u_n| = 0$,*

(c) *if $z_n \to 0$ as $n \to \infty$, then $(\Delta^i z_n)$ is monotonic and*

$$\Delta^i z_n \to 0 \ \text{ as } n \to \infty \ \text{ and } \Delta^i z_n \Delta^{i+1} z_n < 0 \tag{5}$$

for $i = 0, 1, \ldots, m-1$ with $\Delta^m z_n \leq 0$.

(d) *Let $z_n \to 0$ as $n \to \infty$. If m is even, then $z_n < 0$ $(z_n > 0)$ for $u_n > 0$ $(u_n < 0)$. If m is odd, the $z_n > 0$ $(z_n < 0)$ for $u_n > 0$ $(u_n < 0)$.*

Proof. Let (u_n) be an eventually positive solution of (1). Then there exists a positive integer n_0 such that $u_{n-k} > 0$ and $u_{\tau_n} > 0$ for $n \geq n_0$. From (1) and (4) $\Delta^m z_n = -q_n f(u_{\tau_n}) \leq 0$, so $(\Delta^{m-1} z_n)$ is nonincreasing and converges to $L < \infty$. Thus, (a) holds.

If $L > -\infty$, then summing (1) from n_0 to n and then letting $n \to \infty$, we have

$$\sum_{i=n_0}^{\infty} q_i f(u_{\tau_i}) = \Delta^{m-1} z_{n_0} - L < \infty.$$

The last inequality, together with (2) and (3) implies $\liminf_{n\to\infty} = 0$ and so (b) holds.

Now suppose $z \to 0$ as $n \to \infty$. Then we see that $\Delta^i z_n \to 0$ as $n \to \infty$ for $i = 1, 2, \ldots, m-1$. By (a), $(\Delta^{m-1} z_n)$ is nonincreasing and since $q_n \not\equiv 0$ eventually we see that $\Delta^{m-1} z_n > 0$ for $n \geq n_0$. Hence, if $m \geq 2$, then $(\Delta^{m-2} z_n)$ is increasing and so $\Delta^{m-2} z_n < 0$ for $n \geq n_0$. Continuing in this manner we obtain (c).

Part (d) follows immediately from (c).

In our next result we will ask that there exist constants P_1 and P_2 such that either

$$P_1 \leq p_n \leq 0, \tag{6}$$

$$-1 \leq p_n \leq 0, \tag{7}$$

or

$$P_2 \leq p_n \leq -1. \tag{8}$$

Lemma 2. *Let (u_n) be a nonoscillatory solution of (1). Thet the following statements are true:*

(a) *If (6) holds and (u_n) is eventually positive (negative), then $(\Delta_i z_n)$ is monotonic and either*

$$\Delta^i z_n \to -\infty \ (\Delta^i z_n \to \infty) \text{ as } n \to \infty \text{ for } i = 0, 1, \ldots, m-1 \tag{9}$$

or

$$\Delta^i z_n \to 0 \text{ as } n \to \infty \text{ and } \Delta^i z_n \Delta^{i+1} z_n < 0 \tag{10}$$

for $i = 0, 1, \ldots, m-1$ with $\Delta^m z_n \leq 0$.

(b) *Let (6) holds. If m is even, then $z_n < 0$ $(z_n > 0)$ for $u_n > 0$ $(u_n < 0)$. If m is odd and (10) holds, then $z_n > 0$ $(z_n < 0)$ for $u_n > 0$ $(u_n < 0)$.*

(c) *If (7) holds, then (10) holds.*

(d) *If (8) holds, m is odd and $u_n > 0$ $(u_n < 0)$, then $\Delta^i z_n \to -\infty$ $(\Delta^i z_n \to \infty)$ as $n \to \infty$ for $i = 0, 1, \ldots, m-1$.*

Proof. If (u_n) is an eventually positive solution of (1), then there exists n_0 such that $u_{n-k} > 0$ and $u_{\tau_n} > 0$ for $n \geq n_0$. From (a) and (b) of Lemma 1, we have that $(\Delta^{m-1} z_n)$ is nonincreasing for $n \geq n_0$, $\Delta^{m-1} z_n \to L \geq -\infty$ as $n \to \infty$, and $\liminf_{n\to\infty} u_n = 0$ if $L > -\infty$. If $L = -\infty$, then clearly (9) holds. If $-\infty < L < 0$, then a summation shows that $z_n \leq L_1$ for some constant $L_1 < 0$. But from (6) we have

$$P_1 u_{n-k} \leq p_n u_{n-k} < z_n \leq L_1 < 0,$$

which contradicts $\liminf_{n\to\infty} = 0$. Thus $L \geq 0$. If $L > 0$, then we have $\Delta^{m-1} z_n \geq L$ and a summation shows that $z_n \to \infty$ as $n \to \infty$. Since $u_n \geq z_n$ hence $u_n \to \infty$ as $u_n \to \infty$, a contradiction. Therefore $L = 0$, i.e. $\Delta^{m-1} z_n \to 0$ as $n \to \infty$. Moreover, $\Delta^{m-1} z_n > 0$ since $(\Delta^{m-1} z_n)$ is nonincreasing and $q_n \not\equiv 0$ eventually. Hence $(\Delta^{m-2} z_n)$ is increasing. Also, $\Delta^{m-2} z_n < 0$ since otherwise $(\Delta^{m-2} z_n)$ is eventually positive and increasing, which implies (z_n) has a positive lower bound, contradicting $\liminf_{n\to\infty} u_n = 0$. Furthermore, if $\Delta^{m-2} z_n \to L_2 < 0$ as $n \to \infty$, then $\Delta^{m-2} z_n \leq L_2$ and a summation shows that eventually $z_n \leq L_3$ for some negative constant L_3. But this again contradicts $\liminf_{n\to\infty} u_n = 0$. Therefore, $(\Delta^{m-2} z_n)$ is increasing and tends to zero as $n \to \infty$. Continuing in this manner we see that (10) holds and this completes the proof of (a).

To prove (b) for $u_n > 0$ we need only observe that either (9) or (10) implies $z_n < 0$ when m is even, and (10) implies $z_n > 0$ when m is odd.

Now suppose (7) holds. If (10) does not hold, then by (a), (9) holds, so $z_n < 0$ for all large n. By (7), we have

$$u_n < -p_n u_{n-k} \leq u_{n-k}$$

for all large n. But the last inequality implies that (u_n) is bounded which contradicts (9).

For the proof of (d), again assume that (u_n) is eventually positive. If (9) does not hold, then (10) holds, which implies that $\liminf_{n\to\infty} u_n = 0$. From (b) we have $z_n > 0$ for $n \geq n_1 \geq n_0$. Thus, by (8), $u_n > -p_n u_{n-k} \geq u_{n-k}$ which condradicts $\liminf_{n\to\infty} u_n = 0$.

Theorem 1. *Let $p_n \geq 0$. Then every nonoscillatory solution (u_n) of (1) satisfies the following*:

(i) $|u_n| \leq b n^{m-1}$ *for some constant $b > 0$ and all large n,*
(ii) *if n^{m-1}/p_n is bounded, then (u_n) is bounded,*
(iii) *if $n^{m-1}/p_n \to 0$ as $n \to \infty$, then $u_n \to 0$ as $n \to \infty$.*

Proof. Let (u_n) be an eventually positive solution of (1). As before, from (1) we have $\Delta^m z_n \leq 0$ eventually, so $z_n \leq W(n)$, where $W(n)$ is a polynomial of degree less than or equal to $m-1$. Hence, there exist constants $b > 0$ and n_1 such that $z_n \leq b n^{m-1}$ for $n \geq n_1$. Clearly (i) follows since $p_n \geq 0$. Also, $p_n u_{n-k} \leq b n^{m-1}$ and hence (ii) and (iii) follow.

By imposing a stronger condition on (p_n), namely, that there exists a constant $P_3 > 0$ such that

$$0 \le p_n \le P_3 < 1, \tag{11}$$

we can obtain much sharper results on the behavior of the solution of (1), than those obtained in Theorem 1.

Theorem 2. *Assume that* (11) *holds.*

(i) *If m is even, then every solution of* (1) *is oscillatory.*

(ii) *If m is odd, then every nonoscillatory solution of* (1) *tends to zero as $n \to \infty$.*

Proof. If (u_n) is an eventually positive solution of (1), say $u_{n-k} > 0$ and $u_{\tau_n} > 0$ for $n \ge n_0$, then, by (a) of Lemma 1, we have that $(\Delta^{m-1} z_n)$ is nonincreasing and converges to $L \ge -\infty$ as $n \to \infty$. Moreover, it is easy to see that if $L < 0$, then (z_n) is eventually negative contradicting $u_n > 0$. Thus, $L \ge 0$, and from (b) of Lemma 1, we have $\liminf_{n\to\infty} u_n = 0$. It is clear since $\Delta^m z_n \le 0$, that $\Delta^i z_n$ is monotonic for $i = 0, 1, \dots, m-1$. Since (z_n) is monotonic, then $z_n \to \ell$ as $n \to \infty$. Observe that $\ell \ge 0$ since $\ell < 0$ implies $u_n < 0$. Suppose $\ell > 0$. For the case (z_n) increasing, we have

$$z_n = u_n + p_n u_{n-k} \le u_n + p_n z_{n-k} \le u_n + P_3 z_n,$$

so $z_n(1 - P_3) \le u_n$, which, in view of (11), contradicts $\liminf_{n\to\infty} u_n = 0$. If (z_n) is decreasing, let $1 - P_3 = \varepsilon > 0$. Then $z_n \le u_n + P_3 z_{n-k}$, and since ℓ is finite

$$\frac{z_n}{z_{n-k}} \le \frac{u_n}{\ell} + P_3. \tag{12}$$

Since $P_3 + \frac{\varepsilon}{2} < 1$, so there exists $n_1 > n_0$ such that $z_n / z_{n-k} \ge P_3 + \frac{\varepsilon}{2}$ for $n \ge n_1$. Hence from (12) we get $u_n \ge \frac{\ell\varepsilon}{2}$ contradicting $\liminf_{n\to\infty} u_n = 0$. Thus, we have $z_n \to 0$ as $n \to \infty$, which implies, by (c) of Lemma 1, that (5) holds. Now observe that part (d) of Lemma 1 implies that $z_n < 0$ for m even and $z_n > 0$ for m odd. But $z_n < 0$ contradicts $u_n > 0$ and $p_n \ge 0$, so (i) holds. For m odd, $z_n > 0$, so $u_n \le z_n \to 0$ as $n \to \infty$ and (ii) holds.

In our next two theorems the sequence (p_n) is allowed to oscillate.

Theorem 3. *If (p_n) is not eventually negative, then any solution (u_n) of* (1) *is either oscillatory or satisfies $\liminf_{n\to\infty} |u_n| = 0$.*

Proof. Assume (u_n) is a solution of (1) that is eventually positive. Then as before, by (a) of Lemma 1, $\Delta^{m-1} z_n \to L < \infty$ as $n \to \infty$, and by (b) of Lemma 1, $\liminf_{n\to\infty} u_n = 0$ if $L > -\infty$. If $L = -\infty$, then clearly $z_n \to -\infty$ contradicting $u_n > 0$ since (p_n) is not eventually negative.

Theorem 4. *If there exists a constant P_4 such that*

$$p_n \ge P_4, \tag{13}$$

then any nonoscillatory solution (u_n) *of* (1) *satisfies either* $|u_n| \to \infty$ *as* $n \to \infty$ *or* $\liminf_{n\to\infty} |u_n| = 0$. *Moreover, if* $P_4 \geq -1$, *then the second conclusion holds.*

Proof. Let (u_n) be an eventually positive solution of (1). As in the proof of Theorem 3, we see that $\Delta^{m-1} z_n \to L < \infty$, and that if $L > -\infty$, then $\liminf_{n\to\infty} u_n = 0$. Furthermore, if $L = -\infty$, then $z \to -\infty$ as $n \to \infty$. Thus, it follows from (13) that

$$P_4 u_{n-k} \leq u_n + p_n u_{n-k} = z_n \to -\infty \text{ as } n \to \infty,$$

so $p_n < 0$ and $u_n \to \infty$ as $n \to \infty$.

If $P_4 \geq -1$, then clearly either $\liminf_{n\to\infty} u_n = 0$, or $u_n + p_n u_{n-k} = z_n < 0$ for all large n. Therefore, $u_n < -p_n u_{n-k} \leq u_{n-k}$ for all large n, which implies that (u_n) is bounded. But (u_n) bounded contradicts $L = -\infty$ and the proof is complete.

Theorem 5. *Assume that* (8) *holds. If* m *is odd and* (u_n) *is a nonoscillatory solution of* (1), *then* $|u_n| \to \infty$ *as* $n \to \infty$.

Proof. Let (u_n) be nonoscillatory solution of (1) and assume that (u_n) is eventually positive. By (d) of Lemma 2, $z_n \to -\infty$ as $n \to \infty$. But (8) implies that $P_2 u_{n-k} < z_n$ and, hence, $u_n \to \infty$ as $n \to \infty$.

Theorem 6. *If* m *is even and there exist constants* P_2 *and* P_5 *such that*

$$P_2 \leq p_n \leq P_5 < -1, \tag{14}$$

then every bounded nonoscillatory solution of (1) *tends to zero as* $n \to \infty$.

Proof. Assume that (1) has a bounded nonoscillatory solution (u_n) and let (u_n) is eventually positive. Part (a) of Lemma 2 implies that either (9) or (10) holds. If (9) holds, then the argument used in the proof of Theorem 5 shows that $u_n \to \infty$ as $n \to \infty$ contradicting (u_n) being bounded. Therefore, (10) holds and by (b) of Lemma 2, together with (10), implies that (z_n) is negative and increases to zero as $n \to \infty$. Since (u_n) is bounded $\limsup_{n\to\infty} u_n = a$ is nonnegative and finite. If $a > 0$, then there is a increasing sequence of positive integers (n_i) such that $u_{n_i - k} \to a$ as $i \to \infty$. Let $\alpha = P_5 + 1 < 0$, $\varepsilon = -\frac{\alpha a}{8} > 0$, $\delta = \frac{\alpha a}{8P_5} > 0$ and $\lambda = \frac{-3\alpha a}{4} > 0$. Then there exists a positive integer n_0 such that $z_{n_i} > -\varepsilon$ and $u_{n_i - k} > a - \delta > 0$ for $i \geq n_0$. Thus, for each $i \geq n_0$ we have

$$-\varepsilon < z_{n_i} < u_{n_i} + P_5(a - \delta),$$

so

$$-u_{n_i} < P_5 a - P_5 \delta + \varepsilon = (\alpha - 1)a - \frac{\alpha a}{4} = -\lambda - a,$$

or $u_{n_i} > a + \lambda$ for $i \geq n_0$ contradicting $\limsup_{n\to\infty} u_n = a > 0$. Hence, we conclude that $\limsup_{n\to\infty} u_n = 0$, which implies that $u_n \to 0$ as $n \to \infty$.

Theorem 7. *Suppose that there exists a constant* P_6 *such that*

$$-1 < P_6 \leq p_n \leq 0 \tag{7'}$$

and (u_n) *is a nonoscillatory solution of* (1).

(i) *If* m *is even and* (7) *holds, then* (u_n) *is bounded.*

(ii) *If* m *is even or odd and* (7′) *holds, then* $u_n \to 0$ *as* $n \to \infty$.

Proof. Let (u_n) be a nonoscillatory solution of (1) and let (u_{n-k}) and (u_{τ_n}) are both positive for $n \geq n_0$. Then part (c) of Lemma 2 implies that (10) holds. If m is even, it follows from (7′) and (d) of Lemma 1 that $u_n \leq P_6 u_{n-k}$ for $n \geq n_0$. (If (7) holds, then $u_n \leq u_{n-k}$ for $n \geq n_0$, so (i) is proved.) Thus $u_{n+k} \leq -P_6 u_n$, $u_{n+2k} \leq (-P_6)^2 u_n$ and by induction we see that $u_{n+ik} \leq (-P_6)^i u_n$ for every positive integer i. Since $0 < -P_6 < 1$, the last inequality implies that $u_n \to 0$ as $n \to \infty$.

If m is odd, then (7′) and (d) of Lemma 1 imply that $0 < z_n < M$ for some positive constant M, so $0 < u_n < -P_6 u_{n-k} + M$. If (u_n) is unbounded, then there exists an increasing sequence of positive integres (n_i) such that $n_1 > n_0$, $u_{n_i} \to \infty$ as $i \to \infty$ and $u_{n_i} = \max\limits_{n_1 \leq n \leq n_i} u_n$. New for each i we have

$$u_{n_i} < -P_6 u_{n_i - k} + M \leq -P_6 u_{n_i} + M \ \text{ or } \ (1 + P_6) u_{n_i} \leq M,$$

which is impossible in view of (7′). Thus, (u_n) is bounded and there exists a constant $a > 0$ such that $\limsup\limits_{n\to\infty} u_n = a$. Hence, there is a subsequence if (u_n), say (u_{t_i}) such that $u_{t_i} \to a$ as $i \to \infty$. Then from (7′) we get $-P_6 u_{t_i - k} \geq u_{t_i} - z_{t_i}$. Since $a > 0$, there is a positive number ε satisfying $(1 - P_6)\varepsilon < (1 + P_6)a$ and so $0 < -P_6(a + \varepsilon) < a - \varepsilon$. But for all sufficiently large i, $u_{t_i} < a + \varepsilon$, hence we have

$$a - \varepsilon > -P_6 u_{t_i - k} \geq u_{t_i} - z_{t_i} \ \text{ for all such } \ i.$$

Letting $i \to \infty$ the last inequality contradicts $u_{t_i} \to a$ as $i \to \infty$ since $z_{t_i} \to 0$ as $i \to \infty$. Thus $u_n \to 0$ as $n \to \infty$ also in this case.

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