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MODULI OF CONTINUITY AND SPACES OF FUNCTIONS

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By introducing a natural generalization of the usual modulus of continuity, we prove the closedness of various spaces of bounded and continuous functions in a unified way. Moreover, we prove an extension of the theorem on the uniform continuity of continuous functions on compact sets. We give some sufficient conditions for the uniform convergence of the composition of uniformly convergent sequences. The latter results can be applied to easily prove a Leibniz-type rule for the differentiation of parametric Riemann–Stieltjes integrals, and a Helly-type convergence theorem for path integrals.

1. THE UNIFORM METRIC

Throughout in the sequel, X and Y will denote metric spaces and Y^X will denote the family of all functions from X into Y.

Since each nonvoid set is a metric space with its discrete metric, the following definition would not be generalized by assuming only that X is a nonvoid set.

Definition 1.1. *The extended real valued function d defined by*

$$d(f,g) = \sup\big\{ d\big(f(x), g(x)\big) \,:\, x \in X \big\}$$

for all $f, g \in Y^X$ will be called the uniform metric on Y^X.

By introducing some obvious modification of the corresponding metric space definitions, one can easily prove the following two basic theorems.

Theorem 1.2. *The uniform metric on Y^X is an extended real valued metric on Y^X.*

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REMARK 1.3. Note that if X is infinite, then the uniform metric on Y^X is a metric if and only if Y is bounded.

Theorem 1.4. *The generalized metric space Y^X is complete if and only if Y is complete.*

Corollary 1.5. *A closed subspace of Y^X is complete whenever Y is complete.*

REMARK 1.6. Note that if the constant members of Y^X are contained in some complete subspace of Y^X, then Y is also complete.

In the sequel, we shall also use the following more particular

Definition 1.7. *If A is a subset of a metric or generalized metric space X and $0 < r < \infty$, then the set $A^{-r} = \{x \in X : \exists a \in A : d(a,x) < r\}$ will be called the r-closure of A in X.*

Moreover, the set A will be called r-closure in X if $A^{-r} = A$. The set A is called strongly closed in X if it is r-closed in X for all $0 < r < \infty$.

REMARK 1.8. Note that the closure of A in X can now be expressed by

$$A^{-} = \bigcap_{0<r<\infty} A^{-r}.$$

Therefore, the r-closed sets, and hence the strongly closed sets, are closed.

2. MODULI OF CONTINUITY

Definition 2.1. *If $f \in Y^X$ and $\emptyset \neq A \subset X$, then the extended real valued function ω_f^A defined by*

$$\omega_f^A(r) = \sup\{d(f(a), f(x)) : a \in A,\ x \in X,\ d(a,x) \leq r\}$$

for all $0 \leq r \leq \infty$, will be called the modulus of continuity of f with respect to A.

REMARK 2.2. Note that $\omega_f^a = \omega_f^{\{a\}}$, where $a \in X$, is a local modulus of continuity of f and $\omega_f = \omega_f^X$ is a global modulus of continuity of f.

REMARK 2.3. Moreover, it is also worth noticing that $\omega_f(\infty) = \omega_f^X(\infty) = \operatorname{diam}(f(X))$.

The definition of $\omega_f^A(\infty)$ can also be justified by the next theorem which shows that the function ω_f^A is continuous at the point ∞.

Theorem 2.4. *If $f \in Y^X$ and $\emptyset \neq A \subset X$, then ω_f^A is a nondecreasing function on $[0,\infty]$ such that $\omega_f^A(0) = 0$ and $\omega_f^A(\infty) = \lim_{r\to\infty} \omega_f^A(r)$.*

Proof. To prove the above limit property, note that for each $\beta < \omega_f^A(\infty)$ there exist $a_0 \in A$ and $x_0 \in X$ such that $\beta < d(f(a_0), f(x_0))$. Therefore, by defining $\alpha = d(a_0, x_0)$ we evindently have

$$\beta < d(f(a_0), f(x_0)) \leq \omega_f^A(\alpha) \leq \omega_f^A(r) \leq \omega_f^A(\infty)$$

for all $\alpha < r \leq \infty$.

REMARK 2.5. A particular case of Theorem 2.4 gives $\operatorname{diam}\big(f(X)\big) = \lim_{r\to\infty} \omega_f(r)$ for all $f \in Y^X$.

Theorem 2.6. *If $f \in Y^X$ and $\emptyset \neq A \subset X$, then $\omega_f^A(\infty) \leq \omega_f(\infty) \leq 2\omega_f^A(\infty)$.*

Proof. By corresponding definitions it is clear that $\omega_f^A(\infty) \leq \omega_f^X(\infty) = \omega_f(\infty)$. Moreover, if $x, y \in X$ and $a \in A$, then we evidently have

$$d\big(f(x), f(y)\big) \leq d\big(f(x), f(a)\big) + d\big(f(a), f(y)\big) \leq \omega_f^A(\infty) + \omega_f^A(\infty),$$

whence the inequality $\omega_f(\infty) \leq 2\omega_f^A(\infty)$ is also immediate.

Theorem 2.7. *If $f, g \in Y^X$, then $d(f,g) \leq \omega_f(\infty) + \omega_g(\infty) + d\big(f(X), g(X)\big)$.*

Proof. For each $x, a, b \in X$, we have

$$\begin{aligned} d\big(f(x), g(x)\big) &\leq d\big(f(x), f(a)\big) + d\big(f(a), g(b)\big) + d\big(g(b), g(x)\big) \\ &\leq \omega_f(\infty) + d\big(f(a), g(b)\big) + \omega_g(\infty), \end{aligned}$$

and hence

$$d(f,g) \leq \omega_f(\infty) + d\big(f(a), g(b)\big) + \omega_g(\infty)$$

Therefore, since

$$d\big(f(X), g(X)\big) = \inf\big\{d\big(f(a), g(b)\big) : \ a, b \in X\big\},$$

the stated inequality is also true.

REMARK 2.8. Note that the inequalities in Theorems 2.6 and 2.7 can be rewritten by using Remark 2.3.

Theorem 2.9. *If $f, g \in Y^X$, $\emptyset \neq A \subset X$ and $0 \leq r \leq \infty$, then*

$$\omega_f^A(r) \leq \omega_g^A(r) + 2d(f,g).$$

Proof. For each $a \in A$ and $x \in X$, with $d(a,x) \leq r$, we have

$$\begin{aligned} d\big(f(a), f(x)\big) &\leq d\big(f(a), g(a)\big) + d\big(g(a), g(x)\big) + d\big(g(x), f(x)\big) \\ &\leq d(f,g) + \omega_g^A(r) + d(f,g), \end{aligned}$$

whence the stated inequality is immediate.

REMARK 2.10. A paricular case of Theorem 2.9 gives

$$\operatorname{diam}\big(f(X)\big) \leq \operatorname{diam}\big(g(X)\big) + 2d(f,g)$$

for all $f, g \in Y^X$.

3. SPACES OF BOUNDED FUNCTIONS

Definition 3.1. *For each* $\emptyset \neq A \subset X$ *and* $0 \leq r \leq \infty$ *we define*

$$\mathcal{B}_A^r(X,Y) = \{ f \in Y^X : \omega_f^A(r) < \infty \}.$$

Note that for each $f \in Y^X$ we have $f \in \mathcal{B}_A^r(X,Y)$ if and only if there exists an $M < \infty$ such that

$$a \in A,\ x \in X,\ d(a,x) \leq r \Rightarrow d\big(f(a), f(x)\big) \leq M.$$

REMARK 3.3. Therefore $\mathcal{B}(X,Y) = \mathcal{B}_X^\infty(X,Y)$ is just the family of all bounded members of Y^X.

By the corresponding definitions and Theorems 2.4 and 2.6, it is clear that we also have

Theorem 3.4. *If* $\emptyset \neq A \in X$ *and* $0 \leq r \leq s \leq \infty$, *then*

$$\mathcal{B}(X,Y) = \mathcal{B}_A^\infty(X,Y) \subset \mathcal{B}_A^s(X,Y) \subset \mathcal{B}_A^r(X,Y) \subset \mathcal{B}_A^0(X,Y) = Y^X.$$

Moreover, as an immediate consequence of Theorems 1.2 and 2.7, we can also state

Theorem 3.5. *The family* $\mathcal{B}(X,Y)$, *with the corresponding restriction of the uniform metric, is a metric space.*

On the other hand, by using Theorem 2.9, we can also easily prove

Theorem 3.6. *The family* $\mathcal{B}_A^r(X,Y)$ *is a strongly closed subset of* Y^X.

Proof. If f is in the s-closure of $\mathcal{B}_A^r(X,Y)$ in Y^X for some $0 < s < \infty$, then there exists a $g \in \mathcal{B}_A^r(X,Y)$ such that $d(f,g) < s$. Hence, by Theorem 2.9, it is clear that

$$\omega_f^A(r) \leq \omega_g^A(r) + 2d(f,g) \leq \omega_g^A(r) + 2s < \infty,$$

and thus $f \in \mathcal{B}_A^r(X,Y)$ also holds.

REMARK 3.7. Thus, in particular, the family $\mathcal{B}(X,Y)$ is strongly closed in Y^X.

From Theorem 3.6, by Corollary 1.5 and Remark 1.6, it is clear that we also have

Corollary 3.8. *The space* $\mathcal{B}_A^r(X,Y)$ *is complete if and only if* Y *is complete.*

REMARK 3.9. Thus, in particular, the space $\mathcal{B}(X,Y)$ is complete if and only if Y is complete.

An even more straightforward application of Definition 3.1 and Theorem 2.9 yields

Theorem 3.10. *If* $f \in Y^X$ *and* $g \in \mathcal{B}_A^r(X,Y)$, *then* $|\omega_f^A(r) - \omega_g^A(r)| \leq 2d(f,g)$.

Remark 3.11. A particular case of this theorem gives

$$\left|\operatorname{diam}\big(f(X)\big) - \operatorname{diam}\big(g(X)\big)\right| \le 2d(f,g)$$

for all $f \in Y^X$ and $g \in \mathcal{B}(X,Y)$.

Moreover, as an immediate consequence of Theorem 3.11, we can also state

Corollary 3.12. *The function* $f \mapsto \omega_f^A(r) \quad \big(f \in \mathcal{B}_A^r(X,Y)\big)$, *is uniformly continuous.*

Remark 3.13. Thus, in particular, the function $f \mapsto \operatorname{diam}\big(f(X)\big) \quad \big(f \in \mathcal{B}(X,Y)\big)$, is uniformly continuous.

From Theorem 3.11, we can also at once get

Theorem 3.14. *If* $f \in Y^X$ *and* $g \in \mathcal{B}(X,Y)$, *then* $d(\omega_F^A, \omega_g^A) \ge 2d(f,g)$ *for all* $\emptyset \ne A \subset X$.

Hence it is clear that we also have

Corollary 3.15. *The family of the functions* $f \mapsto \omega_f^A \quad \big(f \in \mathcal{B}(X,Y)\big)$, *where* $\emptyset \ne A \in X$, *is equi-uniformly continuous.*

4. SPACES OF CONTINUOUS FUNCTIONS

Definition 4.1. *For each* $\emptyset \ne A \subset X$, *we define*

$$\mathcal{C}_A(X,Y) = \big\{ f \in Y^X : \lim_{r \to 0} \omega_f^A(r) = 0 \big\}.$$

Remark 4.2. Note that, for each $f \in Y^X$, we have $f \in \mathcal{C}_A(X,Y)$ if and only if for each $\varepsilon > 0$ the exists a $\delta > 0$ such that

$$a \in A,\ x \in X,\ d(a,x) < \delta \ \Rightarrow\ d\big(f(a), f(x)\big) < \varepsilon.$$

Remark 4.3. Therefore $\mathcal{C}_a(X,Y) = \mathcal{C}_{\{a\}}(X,Y)$, where $a \in X$, is the family of all members of Y^X which are continuous at a and $\mathcal{UC}(X,Y) = \mathcal{C}_X(X,Y)$ is the family of all uniformly continuous members of Y^X.

Remark 4.4. By Remark 4.2, it is also clear that $\mathcal{C}_A(X,Y) \subset \mathcal{C}_B(X,Y)$ whenever $\emptyset \ne B \subset A \subset X$. Therefore,

$$\mathcal{C}_A(X,Y) \subset \bigcap_{a \in A} \mathcal{C}_a(X,Y)$$

and thus in particular

$$\mathcal{UC}(X,Y) \subset \bigcap_{a \in X} \mathcal{C}_a(X,Y) = \mathcal{C}(X,Y).$$

Moreover, by using Theorem 2.9, we can also easily prove

Theorem 4.5. *The family $\mathcal{C}_A(X,Y)$ is a closed subset of Y^X.*

Proof. If f is in closure of $\mathcal{C}_A(X,Y)$ in Y^X, then for each $\varepsilon>0$ there exists a $g\in\mathcal{C}_A(X,Y)$ such that $d(f,g)<\varepsilon/4$. Moreover, since $\lim\limits_{r\to 0}\omega_g^A(r)=0$, there exists a $\delta>0$ such that $\omega_g^A(r)<\varepsilon/2$ for all $0\le r<\delta$. Hence, by Theorem 2.9, it is clear that

$$\omega_f^A(r)\le\omega_g^A(r)+2d(f,g)<\varepsilon$$

for all $0\le r<\delta$. Therefore, $\lim\limits_{r\to 0}\omega_f^A(r)=0$, and hence $f\in\mathcal{C}_A(X,Y)$ is also true.

Remark 4.6. Useful particular cases of Theorem 4.4 give the families $\mathcal{C}_a(X,Y)$, where $a\in X$, $\mathcal{C}(X,Y)$ and $\mathcal{UC}(X,Y)$ are closed in Y^X.

Moreover, from Theorem 4.5, by 1.5 and Remark 1.6, it is clear that we also have

Corollary 4.7. *The space $\mathcal{C}_A(X,Y)$ is complete if and only if Y is complete.*

Remark 4.8. Thus, in particular, the spaces $\mathcal{C}_a(X,Y)$, where $a\in X$, $\mathcal{C}(X,Y)$ and $\mathcal{UC}(X,Y)$ are complete if and only if Y is complete.

A simple application of Remark 4.2 also gives

Theorem 4.9. *If $\emptyset\ne A\subset X$ and $0<r<\infty$, then*

(1) $f\in\mathcal{C}_A(X,Y)\ \Rightarrow\ f\,|\,A\in\mathcal{UC}(A,Y)$;

(2) $f\,|\,A^{-r}\in\mathcal{UC}(A^{-r},Y)\ \Rightarrow\ f\in\mathcal{C}_A(X,Y)$.

Moreover, by using a standard compactness argument, we can also easily prove the next important

Theorem 4.10. *If A is nonvoid compact subset of X, then*

$$\mathcal{C}_A(X,Y)=\bigcap_{a\in A}\mathcal{C}_a(X,Y).$$

Proof. If this is not the case, then by Remark 4.4 there exists an $f\in Y^X$ such that $f\in\mathcal{C}_a(X,Y)$ for all $a\in A$, but $f\notin\mathcal{C}_A(X,Y)$. Therefore, by Remark 4.2, there exists an $\varepsilon>0$ such that for each $\delta>0$ there exist $a\in A$ and $x\in X$ such that $d(a,x)<\delta$ and $d\big(f(a),f(x)\big)\ge\varepsilon$. Hence, by induction, we can define sequences (a_n) and (x_n) in A and X, respectively, such that $d(a_n,x_n)<1/n$ and $d\big(f(a_n),f(x_n)\big)\ge\varepsilon$ for all $n\in\mathbf{N}$. Moreover, since A is compact, there exist a subsequence (a_{k_n}) of (a_n) and a point $a\in A$ such that $a=\lim\limits_{n\to\infty}a_{k_n}$. Hence, since

$$d(a,x_n)\le d(a,a_{k_n})+d(a_{k_n},x_{k_n})\le d(a,a_{k_n})+1/k_n$$

for all $n\in\mathbf{N}$, it is clear that we also have $a=\lim\limits_{n\to\infty}x_{k_n}$. Hence, by using that f is continuous at the point a, we can infer that $f(a)=\lim\limits_{n\to\infty}f(a_{k_n})$ and $f(a)=\lim\limits_{n\to\infty}f(x_{k_n})$. Therefore, because of the continuity of the metric in Y, we also have

$$\lim_{n\to\infty}d\big(f(a_{k_n}),f(x_{k_n})\big)=d\big(f(a),f(a)\big)=0.$$

Hence, since $\varepsilon \leq d\big(f(a_{k_n}), f(x_{k_n})\big)$ for all $n \in \mathbf{N}$, it follows that

$$\varepsilon \leq \lim_{n\to\infty} d\big(f(a_{k_n}), f(x_{k_n})\big) = 0 < \varepsilon$$

and this contradiction proves the theorem.

Now, as an immediate consequence of Theorem 4.10, we can also state

Corollary 4.11. *If X is a compact, then $\mathcal{C}(X,Y) = \mathcal{UC}(X,Y)$.*

5. CONTINUITY OF THE COMPOSITION

By assuming that Ω is also a metric space, in addition to the results of Section 2, we can also easily prove

Theorem 5.1. *If $\varphi, \psi \in X^{\Omega}$ and $f, g \in Y^X$, then*

$$d(f\circ\varphi, g\circ\psi) \leq \omega_f^{\varphi(\Omega)}\big(d(\varphi,\psi)\big) + d(f,g).$$

Proof. By defining $F = f\circ\varphi$, $G = g\circ\psi$, $A = \varphi(\Omega)$, we evidently have

$$\begin{aligned} d\big(F(t), G(t)\big) &= d\Big(f\big(\varphi(t)\big), g\big(\psi(t)\big)\Big) \\ &\leq d\Big(f\big(\varphi(t)\big), f\big(\psi(t)\big)\Big) + d\Big(f\big(\psi(t)\big), g\big(\psi(t)\big)\Big) \\ &\leq \omega_f^A\big(d(\varphi,\psi)\big) + d(f,g) \end{aligned}$$

for all $t \in \Omega$. Hence, it follows that

$$d(F,G) \leq \omega_f^A\big(d(\varphi,\psi)\big) + d(f,g).$$

Now, as an immediate consequence of Theorem 5.1, we can also state

Corollary 5.2. *The function $(f,\varphi) \mapsto f\circ\varphi$ $\big(f \in Y^X,\ \varphi \in X^{\Omega}\big)$ is continuous at each point (f,φ) of $Y^X \times X^{\Omega}$ with $f \in \mathcal{C}_{\varphi(\Omega)}(X,Y)$.*

Proof. Note that if $f \in Y^X$ and $\varphi \in X^{\Omega}$ such that $f \in \mathcal{C}_{\varphi(\Omega)}(X,Y)$, then for each $\varepsilon > 0$ there exists a $\rho > 0$ such that $\omega_f^{\varphi(\Omega)}(r) < \varepsilon/2$ for all $0 \leq r < \rho$. Therefore, by defining $\delta = \min\{\rho, \varepsilon/2\}$ and using Theorem 5.1, we evidently have

$$d(f\circ\varphi, g\circ\psi) \leq \omega_f^{\varphi(\Omega)}\big(d(\varphi,\psi)\big) + d(f,g) < \varepsilon$$

for all $g \in Y^X$ and $\psi \in X^{\Omega}$ with $d(f,g) < \delta$ and $d(\varphi,\psi) < \delta$.

Remark 5.3. In particular, Corollary 5.2 shows that the function $(f,g) \mapsto f\circ\varphi$ $(f \in Y^X, \varphi \in X^{\Omega})$ is continuous at each point of $\mathcal{UC}(X,Y) \times X^{\Omega}$.

Moreover, combining Corollary 5.2 with Theorem 4.10, we can also easily prove the following more important

Theorem 5.4. *If Ω is compact, then the function*

$$(f, \varphi) \mapsto f \circ \varphi \quad \big(f \in \mathcal{C}(X, Y),\ \varphi \in \mathcal{C}(\Omega, X)\big)$$

is continuous.

Proof. Note that if $f \in \mathcal{C}(X, Y)$ and $\varphi \in \mathcal{C}(\Omega, X)$, then, by using that the continuous image of a compact set is compact, we can infer that $\varphi(\Omega)$ is a compact subset of X. Therefore, by theorem 4.10, we also have $f \in \mathcal{C}_{\varphi(\Omega)}(X, Y)$. And thus, Corollary 5.2 can be applied to get the conclusion of the theorem.

Remark 5.5. Note that if Ω is compact, then by Corollary 4.11 we have $f \circ \varphi \in \mathcal{UC}(\Omega, Y)$ for all $f \in \mathcal{C}(X, Y)$ and $\varphi \in \mathcal{C}(\Omega, X)$.

REFERENCES

1. J. Dieudonné: *Foundations of Modern Analysis.* Academic Press, New York, 1960.

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