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ON VECTOR FIELDS ON AFFINE HYPERSURFACES

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In this paper we use the special vector fields and the main vector fields to characterize the affine hyperquadrics in $\mathbf{R}^{n+1}$.

0. INTRODUCTION

Let M be a smooth, oriented and connected manifold of dimension n and let $\mathcal{F}(M)$ be the algebra of all differentiable functions on M. We denote by $\mathcal{T}_s^r(M)$ the $\mathcal{F}(M)$-module of the tensor fields of type (r,s). In particular, for $\mathcal{T}_0^1(M)$, resp. $\mathcal{T}_1^0(M)$, we shall use the notation $\mathcal{X}(M)$, resp. $\wedge^1(M)$. By $\delta \in \mathcal{T}_1^1(M)$, we shall denote the Kronecker tensor. If $T \in \mathcal{T}_s^r(M)$, then we denote by $c_q^p T$ the tensor field c_q^p-contracted of the tensor field T.

Consider $\mathbf{R}^{n+1}$ as an affine space endowed with the standard connection D and the standard orientation form $d = \det$. If $f: M \to \mathbf{R}^{n+1}$ is an immersion and ξ is a transversal vector field for f, then we can write the following formulas [**7**]:

$$D_X f_* Y = f_*(\nabla_X Y) + h(X,Y)\cdot \xi, \tag{1}$$

$$D_X \xi = -f_*(SX) + \tau(X)\cdot \xi. \tag{2}$$

It is clear that ∇ is a torsion-free connection, h is a symmetric bilinear form and S a $(1,1)$-tensor field on M. The objects ∇, h and S are called the induced connection, the second fundamental form and the shape operator for (f,ξ).

An immersion f called nondegenerate when h is a nondegenerate form. In our paper we assume this condition for f.

The volume element $v_\xi = f^*(i_\xi d)$ is connected with ξ. The connection ∇ is called equiaffine if there exists $\xi \neq 0$ for which $\tau = 0$ (which means $\nabla v_\xi = 0$) and then ξ is called an equiaffine normal field. An equiaffine structure (∇, ξ) on a hypersurface (M,f) is called the Blaschke structure when $v_\xi = v_h$, where v_h

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is the volume element of the nondegenerate metric h and then ξ is called an affine normal field.

The conormal field $\xi^{\vee} : M \to \mathbf{R}_{n+1}$, where $\mathbf{R}_{n+1}$ is the vector space dual to the vector space $\mathbf{R}^{n+1}$, underlying the affine space $\mathbf{R}^{n+1}$, is defined by the conditions [**6**]:

$$(3) \qquad \langle \xi^{\vee}, f_*(TM)\rangle = 0, \quad \langle \xi^{\vee}, \xi\rangle = 1,$$

where $\langle\, ,\, \rangle : \mathbf{R}_{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}$ is a standard bilinear form $\langle \phi, v\rangle := \phi(v)$. $\xi^{\vee}$ is an immersion when f is a nondegenerate immersion and defines the connection $\nabla^{\vee}$ by the equation:

$$(4) \qquad D_X(\xi^{\vee}_* Y) = (\xi^{\vee})_*(\nabla^{\vee}_X Y) - B(X,Y) \cdot \xi^{\vee},$$

where $B(X,Y) = h(X, SY),\ X, Y \in \mathcal{X}(M)$.

Let $\nabla^{\wedge}$ be the Levi-Civita connection for h and let (∇, ξ) be an equiaffine structure. Then the following equations hold

$$(5) \qquad \nabla^{\wedge} = \frac{1}{2}(\nabla + \nabla^{\vee}), \quad \nabla = \nabla^{\wedge} + K, \quad \nabla^{\vee} = \nabla^{\wedge} - K,$$

where K is a symmetric tensor. We also have

$$(6) \qquad C(X,Y,Z) := \nabla h(X,Y,Z) = -2h(K(X,Y),Z).$$

C is symmetric form called the cubic form. Let us denote by R and $R^{\vee}$ the curvature tensors for ∇ and $\nabla^{\vee}$ respectively:

$$(7) \qquad R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY,$$

$$(8) \qquad R^{\vee}(X,Y)Z = B(Y,Z)X - B(X,Z)Y.$$

Take a coordinate neighborhood U with the coordinate system $\{x^1, \ldots, x^n\}$ and let Γ^k_{ij}, $\overset{\vee}{\Gamma}{}^k_{ij}$ be the connection coefficients with respect to $\{x^1, \ldots, x^n\}$:

$$(9) \qquad \nabla_{\partial_j}\partial_j = \sum_k \Gamma^k_{ij}\partial_k, \qquad \nabla^{\vee}_{\partial_i}\partial_j = \sum_k \overset{\vee}{\Gamma}{}^k_{ij}\partial_k$$

and by $C^{\vee}$ the form $\nabla^{\vee} B$.

Let K^i_{jk} and h_{ij} be the coefficients of K, respectively h, in the system of locally coordinates $(U, x^1, \ldots, x^n)$:

$$(10) \qquad K^i_{jk} = -\frac{1}{2} h^{is}\nabla_s h_{jk}, \text{ where } h^{is}h_{sj} = \delta^i_j.$$

If one defines the product of two vector fields $X, Y \in \mathcal{X}(M)$ by the formula

$$(11) \qquad X \circ Y = K(X,Y),$$

then the $\mathcal{F}(M)$-module $\mathcal{X}(M)$ becomes an algebra over $\mathcal{F}(M)$ [**9**]. This algebra is called the associated algebra to K and will be denoted by $U(M,K)$.

Definition 1. *An element $X \in U(M,K)$ is called almost main vector field if there is an 1-form $\alpha \in \wedge^1(M)$ and a function $f \in \mathcal{F}(M)$ so that* [**3**]

$$K(Z,X) = fZ + \alpha(Z)X \qquad (Z \in \mathcal{X}(M)). \tag{12}$$

If $f = 0$ and $\alpha = 0$, then X is caled special vector field in the algebra $U(M,K)$. If $\alpha = 0$, then X is called almost special vector field [**2**] *and if $f = 0$, then X is called main vector field* [**4**].

The trajectories of a regular special vector field are called special curves. The differential equations of the special curves associated to the algebra $U(M,K)$ are

$$K^i_{jk}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0. \tag{13}$$

The trajectories of a regular almost special field are called almost special curves. The differential equations of the almost special curves associated to the algebra $U(M,K)$ are

$$\left(K^i_{jk}\delta^s_r - K^s_{rk}\delta^i_j\right)\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0. \tag{13'}$$

The trajectories of a regular main vector field (resp. almost main vector field) are called main curves (resp. almost main curves) associated to the algebra $U(M,K)$.

The differential equations of the main curves are

$$\left(K^i_{jk}\delta^s_r - K^s_{jk}\delta^i_r\right)\frac{\mathrm{d}x^k}{\mathrm{d}t}\frac{\mathrm{d}x^r}{\mathrm{d}t} = 0, \tag{13''}$$

and the differential equations of the almost main curves associated to the algebra $U(M,K)$ are

$$H^{ihpr}_{jqkms}\frac{\mathrm{d}x^k}{\mathrm{d}t}\frac{\mathrm{d}x^m}{\mathrm{d}t}\frac{\mathrm{d}x^s}{\mathrm{d}t} = 0, \tag{13'''}$$

where we used the notation

$$H^{ihpr}_{jqkms} = \left(K^i_{jk}\delta^h_m - K^h_{jk}\delta^i_m\right)\left(\delta^p_q\delta^r_s - \delta^r_q\delta^p_s\right) - \left(K^p_{qk}\delta^r_m - K^r_{qk}\delta^p_m\right)\left(\delta^i_j\delta^h_s - \delta^h_j\delta^i_s\right).$$

Remark. The algebra $U(M,K)$ is commutative. The algebra $U(M,K)$ is associative if and only if

$$h^{mt}\nabla_r h_{mi}\nabla_s h_{jt} = h^{mt}\nabla_s h_{mi}\nabla_r h_{jt}.$$

1. MAIN RESULTS

Proposition 1. *If a main curve $x^i = x^i(t)$ associated to the algebra $U(M,K)$ is asymptotic line of M, then it verifies*

$$\nabla_i h_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0. \tag{14}$$

Proof. By using (10) and (13″) we obtain the differential equaitions of the main curves

$$\left(h^{is} \nabla_s h_{jk} \delta^q_r - h^{qs} \nabla_s h_{jk} \delta^i_r\right) \frac{\mathrm{d}x^k}{\mathrm{d}t} \frac{\mathrm{d}x^r}{\mathrm{d}t} = 0. \tag{15}$$

If we multiply (15) by $h_{ip} h_{qc}$ and sum up in relation to i and q, we obtain

$$\left(h_{rc} \nabla_p h_{jk} - h_{rp} \nabla_c h_{jk}\right) \frac{\mathrm{d}x^k}{\mathrm{d}t} \frac{\mathrm{d}x^r}{\mathrm{d}t} = 0. \tag{16}$$

Multiplying (16) by $\frac{\mathrm{d}x^c}{\mathrm{d}t}$ and using the equation $h_{ir} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^r}{\mathrm{d}t} = 0$ and the equation of Codazzi, we obtain

$$h_{rp} \nabla_i h_{jk} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} \frac{\mathrm{d}x^r}{\mathrm{d}t} = 0. \tag{17}$$

Multiplying (17) by h^{pq} and summing we have

$$\left(\nabla_i h_{jk} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t}\right) \frac{\mathrm{d}x^q}{\mathrm{d}t} = 0. \tag{18}$$

Owing to the fact the points of the curves are regular, from (18) and from the equation of Codazzi, we obtain the conclusion of the proposition.

Proposition 2. *We suppose that $\nabla_i h_{jk} = \theta_i h_{jk}$, where θ_i are the coefficients of an 1-form θ on M. Then the differential equations of the main curves of M are*

$$\left(\theta_k \frac{\mathrm{d}x^k}{\mathrm{d}t}\right) \left(R_{rjqp} \frac{\mathrm{d}x^r}{\mathrm{d}t}\right) = 0, \tag{19}$$

where R_{rjqp} are the components of curvature tensor of M.

Proof. By using the equations of Codazzi and the relations $\nabla_i h_{jk} = \theta_i h_{jk}$, the system (16) becomes

$$\theta_k (h_{rq} h_{jp} - h_{rp} h_{jq}) \frac{\mathrm{d}x^k}{\mathrm{d}t} \frac{\mathrm{d}x^r}{\mathrm{d}t} = 0. \tag{20}$$

By using the equations of Gauss

$$R_{ijk\ell} = h_{ik} h_{j\ell} - h_{i\ell} h_{jk}$$

and the relations (20) we obtain (19).

Proposition 3. *The following affirmations are equivalent*:

(i) *All elements of the algebra $U(M,K)$ are special vector fields.*

(ii) *All the elements of the algebra $U(M,K)$ are main vector fields.*

(iii) *All the elements of the algebra $U(M,K)$ are almost special vector fields.*

(iv) $\nabla_i h_{jk} = 0 \Leftrightarrow C = 0$.

(v) *M is a hyperquadric.*

Proof. From (10) we have $K = 0$, that is

$$(10') \qquad h^{is}\nabla_s h_{jk} = 0.$$

Multiplying (10′) by h_{iq} and summing, we obtain (iv).

(ii) $\Rightarrow$ (iv). Because the algebra $U(M,K)$ is commutative, from

$$(22) \qquad K(Z,X) = \alpha(Z)X \qquad (X,Z \in \mathcal{X}(M))$$

results $\alpha(Z)X = \alpha(X)Z \;\; (X,Z \in \mathcal{X}(M))$.

In local coordinates, the last relation is written

$$\alpha_k \frac{\partial}{\partial x^i} = \alpha_i \frac{\partial}{\partial x^k}$$

or $\alpha_k\delta_i^s - \alpha_i\delta_k^s = 0$. Making here $s = i$ and summing up, results $K = 0$. Using (22) results $K = 0$ which implies $\nabla_X h = 0$, that is (iv).

(iii) $\Rightarrow$ (iv) Because all elements of the algebra $U(M,K)$ are almost special vector fields, we have

$$(23) \qquad K(Z,X) = f_X Z \qquad (X,Z \in \mathcal{X}(M)).$$

By using (23) the following relations result

$$f_{X+Y} = f_X + f_Y, \qquad f_{\lambda X} = \lambda f_X,$$

for every $X,Y \in \mathcal{X}(M)$ and $\lambda \in \mathcal{F}(M)$.

It results that $f_X = \omega(X) \;\; (X \in \mathcal{X}(M))$, where ω is an 1-form on M. So, we have $K(Z,X) = \omega(X)Z \;\; (X,Z \in \mathcal{X}(M))$, from where we obtain $K = 0$, so that $\nabla_X h = 0$, that is (iv).

(iv) $\Rightarrow$ (i), (iv) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (iii). Obvious. (iv) $\Rightarrow$ (v). It is known.

Proposition 4. *The following affirmations are equivalent*:

(i) *All the elements of the algebra $U(M,K)$ are almost main vector fields.*

(ii) *M is a hypersphere.*

Proof. (i) $\Rightarrow$ (ii). From (i) we have that for every $X \in \mathcal{X}(M)$ there is a function $f_X \in \mathcal{F}(M)$ and an 1-form ω_X on M so that

$$(24) \qquad K(Z,X) = f_X Z + \omega_X(Z)X \;\; (Z \in \mathcal{X}(M)).$$

Let $\{x^1, \dots, x^n\}$ be a system of local coordinates. Writing (24) for $X = \dfrac{\partial}{\partial x^i}$, $X = \dfrac{\partial}{\partial x^k}$, we have as follows

$$K^s_{ki} = \delta^s_k f \frac{\partial}{\partial x^i} + \omega_k \delta^s_i, \tag{24$'$}$$

where ω_k are the coefficients of the 1-form ω_X. Contracting in rapport to s and k, we come to

$$f \frac{\partial}{\partial x^i} = \frac{1}{n}\,(A_i - \omega_i),$$

where $A_i = A^s_{si}$. We reach the conclusion that $f \dfrac{\partial}{\partial x^i} = \eta_i$ are the coefficients of an 1-form on M. It results that we have

$$K(Z,X) = \eta(X)Z + \omega(Z)X \qquad (X, Z \in \mathcal{X}(M)). \tag{25}$$

Because the algebra $U(M,K)$ is commutative, we have

$$\eta(X)Z + \omega(Z)X = \eta(Z)X + \omega(X)Z,$$

or

$$\theta(X)Z = \theta(Z)X, \tag{26}$$

for every $X, Z \in \mathcal{X}(M)$, where $\theta = \eta - \omega$.

In local coordinates (26) may be written

$$\theta_i \frac{\partial}{\partial x^k} - \theta_k \frac{\partial}{\partial x^i} = 0, \tag{26$'$}$$

which implies $\theta = 0$, that is $\eta = \omega$.

Now (25) becomes

$$K(Z,X) = \omega(X)Z + \omega(Z)X \qquad (X, Z \in \mathcal{X}(M)). \tag{25$'$}$$

Taking a similar way to the one in [**8**] we will obtain $K = 0$, which means that M is a hypersphere.

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