Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 7 (1996), 68–71.

ORDINAL DECOMPOSITIONS OF SEMIGROUPS

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In this paper we characterize the complete lattice of ordinal decompositions of an arbitrary semigroup by complete 1-sublattices of the lattice of its strongly prime ideals. We also give another proof of the Lyapin's theorem concerning indecomposability of the components of the greatest ordinal decomposition of a semigroup.

Ordinal decompositions of semigroups were first defined and studied by A. M. Kaufman [**5**], in connection with studying of linearly ordered groups, where they were called "successively-annihilating sums". After that, they were studied by a number of authors, in connection with various important problems of the Theory of semigroups, and they were obtained the name "ordinal sums" (for more informations we refer to [**6**], [**8**], [**9**] and [**10**]).

A fundamental result concerning ordinal decompositions was obtained by E. S. Lyapin [**7**]. By this result, ordinal decompositions of any semigroup S form a complete sublattice of the partion lattice of S, and the components of the greatest ordinal decomposition of S are ordinally indecomposable. The purpose if this paper is to give a characterization of this lattice. We also give another proof of indecomposability of components in the greatest ordinal decomposition of a semigroup. The methodology applied here is based on the idea of T. Tamura [**12**] that semilattice decompositions can be studied through quasi-orders satisfying certain conditions, which is developed by the authors in [**3**], where they established connections between semilattice and chain decompositions and certain sublattices of the lattice of ideals of a semigroup.

If ξ is a binary relation on a set A, ξ^{-1} will denote the relation defined by: $a\,\xi^{-1}\,b \Leftrightarrow b\,\xi\,a$, for $a \in A$, $a\xi = \{x \in A \mid a\,\xi\,a\}$, $\xi a = \{x \in A \mid x\,\xi\,a\}$, and for $X \subseteq A$, $X\xi = \bigcup_{x\in X} x\xi$ and $\xi X = \bigcup_{x\in X} \xi x$. By a *quasi-order* we mean a reflexive

[0]1991 Mathematics Subject Classification: Primary 20M10, Secondary 06F05.
Supported by Grant 042B of RFNS through Math. Inst. SANU.

and transitive binary relation. For a quasi-order ξ on a set A, $\widetilde{\xi}$ will denote the *natural equivalence* of ξ defined by: $\widetilde{\xi} = \xi \cap \xi^{-1}$. A binary relation ξ on a semigroup S is: *positive*, if $a\,\xi\,ab$ and $b\,\xi\,ab$, for all $a, b \in S$, *linear*, if for all $a, b \in S$, $a\,\xi\,b$ or $b\,\xi\,a$, and it satisfies the *cm-property*, if for $a, b, c \in S$, $a\,\xi\,c$ and $b\,\xi\,c$ implies $ab\,\xi\,c$, [**12**].

A subset K of a lattice L is *complete for meets* (*joins*) if for any non-empty subset X of K, K contains the meet (join) of X in L, whenever it exists, and K is a *complete subset* of L if it is complete both for meets and joins. Clearly, any complete subset of L is its sublattice. If L is a lattice with unity 1, then any sublattice of L containing 1 will be called a 1-*sublattice* of L.

An ideal I of a semigroup S is *completely prime* if for $a, b \in S$, $ab \in I$ implies $a \in I$ or $b \in I$. A subset A of a semigroup S is *consistent* if for $x, y \in S$, $xy \in A$ implies $x, y \in A$. A consistent subsemigroup of S will be called a *filter*. It is well-known that a subset A of a semigroup S is a filter of S if and only if $S - A$ is completely prime ideal of S. By $\mathcal{I}d(S)$ we will denote the lattice of all ideals of a semigroup S, which is a conditionally complete lattice with the unity. Let K be a subset of $\mathcal{I}d(S)$ complete for meets, containing the unity of $\mathcal{I}d(S)$. Then for any $a \in S$, there exists a smallest element of K containing a, in notation $K(a)$, called the *principal element* of K generated by a.

A congruence ϱ on a semigroup S is a *semilattice* (*chain*) *congruence* if S/ϱ is a semilattice (chain). A semigroup S is an *ordinal sum* of semigroups S_α, $\alpha \in Y$, if Y is a chain and for any $a \in S_\alpha$, $b \in S_\beta$, $\alpha < \beta$ implies $ab = ba = a$. In this case, the related chain congruence will be called an *ordinal sum congruence* on S, the related partition will be called an *ordinal decomposition* of S, and the components $S_\alpha, \alpha \in Y$, will be called *ordinal components* of S. A semigroup S is *ordinally indecomposable* if it has no an ordinal decomposition with more than one component. Note that the sum of any two components in an ordinal decomposition of a semigroup can be considered as the ordinal sum of posets (see G. Birkhoff [**1**, p. 198]), with respect to its partial orders defined by: $a \leq b \Leftrightarrow a = b$ or $ab = ba = a$.

For undefined notions and notations we refer to [**1**], [**2**], [**4**], [**8**] and [**11**].

To characterize ordinal decompositions of a semigroup we introduce the following notion:

Definition 1. *An ideal P of a semigroup S is strongly prime if for $x, y \in S$, $xy = p \in P$ implies $x = p$ or $y = p$ or $x, y \in P$.*

It is easy to check the following:

Lemma 1. *The set of strongly prime ideals of a semigroup S is a complete* 1-*sublattice of $\mathcal{I}d(S)$.*

The lattice of strongly prime ideals of a semigroup S will be denoted by $\mathcal{I}d^{\,\mathbf{sp}}(S)$. Now we are ready to prove the main theorem of this paper.

Theorem 1. *The lattice of ordinal decompositions of a semigroup S is isomophic to the lattice of complete* 1-*sublattices of $\mathcal{I}d^{\,\mathbf{sp}}(S)$.*

Proof. Clearly, it is enough to establish an order isomorphism between the poset od ordinal decompositions of S and the poset of complete 1-sublattices of $\mathcal{I}d^{\,\mathbf{sp}}(S)$.

This isomorphism will be established through quasy-orders on S satysfying certain conditions.

By Proposition 2 [**3**], the mapping $\xi \mapsto \widetilde{\xi}$ is an isomorphism of the poset of linear positive quasi-orders on S satisfying the cm-property onto the poset of chain congruences on S. On the other hand, by Theorems 3 and 5 [**3**], the mapping $\xi \mapsto K_\xi$, where $K_\xi = \{I \in \mathcal{I}d(S) \mid I\xi = I\}$, is an isomorphism of the poset of linear positive quasi-orders on S satisfying the cm-property and the poset of complete 1-sublattices of $\mathcal{I}d^{\mathbf{cs}}(S))$ consisting of completely prime ideals, and also, the principal elements of K_ξ are characterized by $K_\xi(a) = a\xi$, for any $a \in S$.

Thus, it remains to prove that $\widetilde{\xi}$ is an ordinal sum congruence on S if and only if K_ξ is a sublattice of $\mathcal{I}d^{\mathbf{sp}}(S)$, for any linear positive quasi-order ξ on S satisfying the cm-prooperty.

Let $\widetilde{\xi}$ be an ordinal sum congruence on S. Assume $a, x, y, p \in S$ such that $xy = p \in a\xi$. As we noted above, $a\xi$ is a completely prime ideal of S, so $x, y \in a\xi$, or $x \in a\xi$, $y \notin a\xi$ or $x \notin a\xi$, $y \in a\xi$. If $x \in a\xi$, $y \notin a\xi$ or $x \notin a\xi$, $y \in a\xi$, then $x\xi \subseteq a\xi$, and so $y \notin x\xi$, or $y\xi \subseteq a\xi$, and so $x \notin y\xi$. Therefore, $(x, y) \notin \widetilde{\xi}$, whence $xy = yx = x$ or $xy = yx = y$, i.e. $x = p$ or $y = p$. Hence, $a\xi \in \mathcal{I}d^{\mathbf{sp}}(S)$, for any $a \in S$, whence K_ξ is a sublattice of $\mathcal{I}d^{\mathbf{sp}}(S)$, since it is a complete sublattice of $\mathcal{I}d(S)$.

Conversely, let K_ξ be a sublattice of $\mathcal{I}d^{\mathbf{sp}}(S)$. Assume $a, b \in S$ such that $a\widetilde{\xi} < b\widetilde{\xi}$. Then $(b, a) \in \xi$ and $(a, b) \notin \xi$, i.e. $b \notin a\xi$, and also $ab = p \in a\xi$, $ba = q \in a\xi$, whence $a = p = q$, i.e. $ab = ba = a$, since $a\xi$ is strongly prime. Therefore, $\widetilde{\xi}$ is an ordinal sum congruence on S. □

Corollary 1. *A semigroup S is ordinally indecomposable if and only if it has no proper strongly prime ideals.*

Now we give another proof on indencomposability of the components of the greatest ordinal decomposition of a semigroup:

Theorem 2. *The components of the greatest ordinal decomposition of a semigroup S are ordinally indecomposable.*

Proof. Let $\{S_\alpha \mid \alpha \in Y\}$ be the greatest ordinal decomposition of S, let θ be related ordinal sum congruence on S and let ξ be a linear positive quasi-order on S satisfying the cm-property such that $\widetilde{\xi} = \theta$. Let $\alpha \in Y$, let P be a strongly prime ideal of S_α and let $a \in P$. Then $a\xi \cap \xi a = S_\alpha$, and since ξ is linear, then $a\xi \cup \xi a = S$. Let $P_a = S - \xi a = a\xi - S_\alpha$. By Lemma 2 [**3**], ξ is positive and it satisfies the cm-property if and only if $u\xi$ is a filter of S, for any $u \in S$. By this it follows that P_a is a completely prime ideal of S, whence $P_a = \bigcup_{x \in P_a} x\xi$, and hence, P_a is strongly prime, by Theorem 1 and Lemma 1.

Let $x \in P$, $y \in S$. If $y \in S_\alpha$, then $xy, yx \in P$, since P is an ideal of S_α. Let $y \notin S_\alpha$. Then $xy = yx = x$, and so $xy, yx \in P$, or $xy = yx = y$, and so $y = xy = yx \in x\xi \subseteq a\xi$, whence $y = xy = yx \in a\xi - S_\alpha = P_a$. Thus, $P' = P \cup P_a$ is an ideal of S. Further, let $x, y \in S$ such that $xy = p \in P'$. If $p \in P_a$, then clearly $x = p$ or $y = p$ or $x, y \in P_a \subseteq P'$. Let $p \in P$. Then $x = p$ or $y = p$ or $x, y \in a\xi$, and in the last case, $x, y \in P_a \subseteq P'$, or $x, y \in S_\alpha$, and then $x = p$ or $y = p$ or

$x, y \in P \subseteq P'$, since P is a strongly prime ideal of S_α. Therefore, P' is a strongly prime ideal of S, and since $a\xi$ is the smallest strongly prime ideal of S containing a and $a \in P' \subseteq a\xi$, then $P' = a\xi$, so $P = S_\alpha$. Hence, by Corollary 1, S_α is ordinally indecomposable. □

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(Received September 25, 1995)

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